Journal of Nonlinear and Convex Analysis
Volume 8, Number 2, 2007, 211–225

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SENSITIVITY ANALYSIS FOR A SYSTEM OF PARAMETRIC MIXED QUASI-VARIATIONAL INCLUSIONS

X. P. DING* AND J. C. YAO**

Abstract. In this paper, we introduce a new system of parametric mixed quasi-variational inclusions. By using resolvent operator technique of maximal monotone mappings and the property of fixed point set of set-valued contractive mappings, we study the behavior and sensitivity analysis of solution set for the system of parametric mixed quasi-variational inclusions. In particular, we proved that the solution set of the system of parametric mixed quasi-variational inclusions is nonempty closed and established that the solution set of the system of parametric mixed quasi-variational inclusions is Lipschitz continuous with respect to the parameters under suitable conditions.

1. Introduction

The behavior of solution set of variational inequality problems as the result of changes in the problem data is always concerned. In recent years, much attention has been devoted to developing general methods for the sensitivity analysis of solution set of various variational inequalities and variational inclusions. From the mathematical and engineering points of view, sensitivity properties of various variational inequalities can provide a new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for parametric variational inequalities has been studied extensively by many authors using quite different methods. By using the projection technique, Dafermos [4], Mukherjee and Verma [11], Noor [13], Yen [19] dealt with the sensitivity analysis for variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [17] dealt with the sensitivity analysis for variational inequalities with single-valued mappings in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Noor and Noor [14], and Agarwal et al. [2] studied sensitivity analysis for quasi-variational inclusions with single-valued mappings. By using projection technique and the property of fixed point set of set-valued contractive mappings, Ding and Luo [5] studied the behavior and sensitivity analysis of solution set for generalized quasi-variational inequalities. Recently, Liu et al. [10], Salahuddin [18], Park and Jeong [15], and Ding [6]–[8] studied the behavior and sensitivity analysis of solution set of generalized nonlinear implicit quasi-variational inclusions of several type

2000 *Mathematics Subject Classification.* Primary 47H09, 47J20.

Key words and phrases. System of parametric generalized mixed implicit quasi-variational inclusions, Maximal monotone mappings, Sensitivity analysis, Resolvent operator, Hilbert space.

*The research of this author was supported by the NSF of Sichuan Education Department of China.

**The research of this authors was supported by the National Science Council of the Republic of China.

with set-valued mappings. Agarwal et al. [3] first studied sensitivity analysis for a system of nonlinear mixed quasi-variational inclusions with single-valued mappings and maximal monotone mappings.

Inspired and motivated by recent research works in this field, in this paper, we introduce a new system of parametric mixed quasi-variational inclusions involving set-valued mappings and maximal monotone mappings. By using resolvent operator technique and the property of fixed point set of set-valued contractive mappings, we prove existence of solutions and study the behavior and sensitivity analysis of solution set for the system of parametric mixed quasi-variational inclusions. In particular, we prove that the solution set of the system of parametric mixed quasi-variational inclusions is nonempty closed and established that the solution set of the system of parametric mixed quasi-variational inclusions is Lipschitz continuous with respect to the parameters under suitable conditions. Our results improve, unify and generalize the responding results in [1], [2], [4]–[15], [10]–[15], [17], [18].

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let 2^H and $C(H)$ denote the family of all nonempty subsets of H and the family of all nonempty compact subsets of H, respectively. Let $\tilde{H}(\cdot,\cdot)$ denote the Hausdorff metric on $C(H)$ defined by

$$\tilde{H}(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}, \quad \forall\ A,B \in C(H),$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$ and $d(A,b) = \inf_{a\in A}\|a-b\|$.

Let Ω be a nonempty open subset of H in which the parameter λ takes values, $A,B,C,D,E,F,G,Q,S : H\times\Omega \to C(H)$ be set-valued mappings and $m : H\times\Omega \to H$, $N_1 : H\times H\times H\times H\times\Omega \to H$ and $N_2 : H\times H\times\Omega \to H$ be single-valued mappings. Let $M_1, M_2 : H\times H\times\Omega \to 2^H$ be set-valued mappings such that for each given $(z,\lambda) \in H\times\Omega$, $M_1(\cdot,z,\lambda)$ and $M_2(\cdot,z,\lambda)$ are both maximal monotone mappings with $(G-m)(H\times\{\lambda\})\bigcap \mathrm{dom}M_1(\cdot,z,\lambda) \neq \emptyset$. Throughout this paper, unless otherwise stated, we will consider the following system of parametric mixed quasi-variational inclusion (SPMQVI): for each given $\lambda \in \Omega$ and $w_1, w_2 \in H$, find $(x,y) = (x(\lambda), y(\lambda)) \in H\times H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(y,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, $e = e(y,\lambda) \in E(y,\lambda)$, $f = f(x,\lambda) \in F(x,\lambda)$, $g = g(x,\lambda) \in G(x,\lambda)$, $q = q(x,\lambda) \in Q(x,\lambda)$, and $s = s(x,\lambda) \in S(x,\lambda)$ such that

$$\text{(2.1)} \qquad \begin{cases} w_1 \in N_1(a,b,c,d,\lambda) + M_1((g-m)(x,\lambda),e,\lambda)), \\ w_2 \in y - x + \mu N_2(f,q,\lambda) + \mu M_2(y,s,\lambda), \end{cases}$$

where $\mu > 0$ is a constant.

Special cases: (I) If $G = g$ is a single-valued mapping, then the SPMQVI (2.1) reduces to the following parametric problem: for each given $\lambda \in \Omega$ and $w_1, w_2 \in H$, find $(x,y) = (x(\lambda), y(\lambda)) \in H\times H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(y,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, $e = e(y,\lambda) \in E(y,\lambda)$, $f = f(x,\lambda) \in$

$F(x,\lambda)$, $q = q(x,\lambda) \in Q(x,\lambda)$, and $s = s(x,\lambda) \in S(x,\lambda)$ such that

$$\begin{cases} w_1 \in N_1(a,b,c,d,\lambda) + M_1((g-m)(x,\lambda),e,\lambda)), \\ w_2 \in y - x + \mu N_2(f,q,\lambda) + \mu M_2(y,,s,\lambda), \end{cases} \tag{2.2}$$

where $\mu > 0$ is a constant.

The parametric problem (2.2) appears to be a new one.

(II) If $w_2 = 0$, $M_2 \equiv N_2 \equiv 0$, then the SPMQVI (2.1) reduces to the following parametric mixed quasi-variational inclusion problem: for each given $\lambda \in \Omega$ and $w_1 \in H$, find $x = x(\lambda) \in H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(x,\lambda) \in C(x,\lambda)$, $d = d(x,\lambda) \in D(x,\lambda)$, $e = e(x,\lambda) \in E(x,\lambda)$ and $g = g(x,\lambda) \in G(x,\lambda)$ such that

$$w_1 \in N_1(a,b,c,d,\lambda) + M_1((g-m)(x,\lambda),e,\lambda)). \tag{2.3}$$

The parametric problem (2.3) also appears to be a new one.

(III) If $P, W : H \times H \times \Omega \to C(H)$ are two set-valued mappings and $N_1(a,b,c,d,\lambda) = P(a,b,\lambda) - W(c,d,\lambda)$ for all $(a,b,c,d,\lambda) \in H \times H \times H \times H \times \Omega$, then the parametric problem (2.3) collapses to the following parametric problem: for each given $\lambda \in \Omega$ and $w_1 \in H$, find $x = x(\lambda) \in H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(x,\lambda) \in C(x,\lambda)$, $d = d(x,\lambda) \in D(x,\lambda)$, $e = e(x,\lambda) \in E(x,\lambda)$ and $g = g(x,\lambda) \in G(x,\lambda)$ such that

$$w_1 \in P(a,b,\lambda) - W(c,d,\lambda) + M_1((g-m)(x,\lambda),e,\lambda)). \tag{2.4}$$

The parametric problem (2.3) with $G = g$ being a single-valued mapping was introduced and studied by Liu et al. [10].

(IV) If $W \equiv 0$, then the parametric problem (2.4) reduces to the following parametric problem: for each given $\lambda \in \Omega$ and $w_1 \in H$, find $x = x(\lambda) \in H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $e = e(x,\lambda) \in E(x,\lambda)$ and $g = g(x,\lambda) \in G(x,\lambda)$ such that

$$w_1 \in P(a,b,\lambda) + M_1((g-m)(x,\lambda),e,\lambda)). \tag{2.5}$$

The parametric problem (2.5) was introduced and studied by Ding [6].

(V) If $\varphi : H \times H \times \Omega \to \mathbf{R}\bigcup\{+\infty\}$ and $\psi : H \times H \times \Lambda \to \mathbf{R}\bigcup\{+\infty\}$ are such that for each fixed $(z,\lambda) \in H \times \Omega$, $\varphi(\cdot,z,\lambda)$ and $\psi(\cdot,z,\lambda)$ are both proper convex lower semicontinuous functional and $\partial\varphi(\cdot,z,\lambda)$ and $\partial\psi(\cdot,z,\lambda)$ are the subdifferential of $\varphi(\cdot,z,\lambda)$ and $\psi(\cdot,z,\lambda)$, respectively. By [16], $\partial\varphi(\cdot,z,\lambda)$ and $\partial\psi(\cdot,z,\lambda)$ are both maximal monotone mappings. Let $M_1(\cdot,z,\lambda) = \partial\varphi(\cdot,z,\lambda)$ and $M_2(\cdot,z,\lambda) = \partial\psi(\cdot,z,\lambda)$ such that $(G-m)(H \times \{\lambda\})\bigcap \mathrm{dom}\ \partial\varphi(\cdot,z,\lambda) \neq \emptyset$ for all $(z,\lambda) \in H \times \Omega$. It is easy to see that the SPMQVI (2.1) reduces to the following system of parametric problems: for each given $\lambda \in \Omega$ and $w_1, w_2 \in H$, find $(x,y) = (x(\lambda),y(\lambda)) \in H \times H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(y,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, $e = e(y,\lambda) \in E(y,\lambda)$, $g(x,\lambda) \in G(x,\lambda)$, $f = f(x,\lambda) \in F(x,\lambda)$, $q = q(x,\lambda) \in Q(x,\lambda)$, and $s = s(x,\lambda) \in S(x,\lambda)$ such that

$$\begin{cases} \langle N_1(a,b,c,d,\lambda) - w_1, u - (g-m)(x,\lambda)\rangle \\ \quad \geq \varphi((g-m)(x,\lambda),e,\lambda) - \varphi(u,e,\lambda),\ \forall\ u \in H, \\ \langle y - x - w_2 + \mu N_2(f,q,\lambda), u - y\rangle \geq \mu\psi(y,s,\lambda) - \mu\psi(u,s,\lambda),\ \forall\ u \in H, \end{cases} \tag{2.6}$$

where $\mu > 0$ is a constant.

(VI) If $G = g$ is a single-valued mapping, $w_1 = w_2 = m = 0$, $K_1, K_2 : H \times \Omega \to 2^H$ are two set-valued mappings such that for each $(e, \lambda), (s, \lambda) \in H \times \Omega$, $K_1(e, \lambda)$ and $K_2(s, \lambda)$ are both closed convex subsets of H, and $\varphi(\cdot, e, \lambda) = I_{K_1(e,\lambda)}(\cdot)$ and $\psi(\cdot, s, \lambda) = I_{K_2(s,\lambda)}(\cdot)$ are the indicator functions of $K_1(e, \lambda)$ and $K_2(s, \lambda)$ respectively, i.e.,

$$I_{K_1(e,\lambda)}(u) = \begin{cases} 0, & \text{if } u \in K_1(e, \lambda), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$I_{K_2(s,\lambda)}(u) = \begin{cases} 0, & \text{if } u \in K_2(s, \lambda), \\ +\infty, & \text{otherwise,} \end{cases}$$

then the parametric problem (2.6) collapses to the following system of parametric quasi-variational inequalities problems: for each given $\lambda \in \Omega$ and $w_1, w_2 \in H$, find $(x, y) = (x(\lambda), y(\lambda)) \in H \times H$, $a = a(x, \lambda) \in A(x, \lambda)$, $b = b(x, \lambda) \in B(x, \lambda)$, $c = c(y, \lambda) \in C(y, \lambda)$, $d = d(y, \lambda) \in D(y, \lambda)$, $e = e(y, \lambda) \in E(y, \lambda)$, $g = g(x, \lambda) \in G(x, \lambda)$, $f = f(x, \lambda) \in F(x, \lambda)$, $q = q(x, \lambda) \in Q(x, \lambda)$, and $s = s(x, \lambda) \in S(x, \lambda)$ such that $(g - m)(x, \lambda) \in K_1(e, \lambda)$, $y \in K_2(s, \lambda)$ and

$$\begin{cases} \langle N_1(a, b, c, d, \lambda) - w_1, u - g(x, \lambda) \rangle \geq 0, \ \forall\, u \in K_1(e, \lambda), \\ \langle y - x - w_2 + \mu N_2(f, q, \lambda), u - y \rangle \geq 0, \ \forall\, u \in K_2(s, \lambda), \end{cases} \tag{2.7}$$

where $\mu > 0$ is a constant.

In brief, for appropriate and suitable choices of m, $A, B, C, D, E, F, G, Q, S$, N_1, N_2 M_1 and M_2, it is easy to see that the SPMQVI (2.1) includes a number of systems of (parametric) mixed quasi-variational inclusions, (parametric) mixed quasi-variational inclusions, systems of (parametric) mixed quasi-variational inequalities and (parametric) mixed quasi-variational inequalities studied by many authors as special cases.

Definition 2.1. Let $M : H \times H \times \Omega \to 2^H$ be a set-valued mapping such that for each fixed $(z, \lambda) \in H \times \Omega$, $M(\cdot, z, \lambda) : H \to 2^H$ is maximal monotone. The parametric implicit resolvent operator $J_\rho^{M(\cdot,z,\lambda)} : H \to H$ is defined by

$$J_\rho^{M(\cdot,z,\lambda)}(u) = (I + \rho M(\cdot, z, \lambda))^{-1}(u), \ \forall\, u \in H,$$

where I is the identity mapping on H and $\rho > 0$ is a constant.

It is well-known that for each $(z, \lambda) \in H \times \Omega$, $J_\rho^{M(\cdot,z,\lambda)}$ is a single-valued nonexpansive mapping.

From the definition of parametric implicit resolvent operator, we have the following lemma.

Lemma 2.1. *For each fixed $\lambda \in \Omega$, $(x, y) = (x(\lambda), y(\lambda)) \in H \times H$, $a = a(x, \lambda) \in A(x, \lambda)$, $b = b(x, \lambda) \in B(x, \lambda)$, $c = c(y, \lambda) \in C(y, \lambda)$, $d = d(y, \lambda) \in D(y, \lambda)$, $e = e(y, \lambda) \in E(y, \lambda)$, $g = g(x, \lambda) \in G(x, \lambda)$, $f = f(x, \lambda) \in F(x, \lambda)$, $q = q(x, \lambda) \in Q(x, \lambda)$, and $s = s(x, \lambda) \in S(x, \lambda)$ is a solution of the SPMQVI (2.1) if and only if*

$$\begin{cases} (g - m)(x, \lambda) = J_\rho^{M_1(\cdot,e,\lambda)}[(g - m)(x, \lambda) + \rho(w_1 - N_1(a, b, c, d, \lambda)], \\ y = J_\mu^{M_2(\cdot,s,\lambda)}[x + w_2 - \mu N_2(f, q, \lambda)], \end{cases}$$

where $\rho > 0$ and $\mu > 0$ are constants.

Now, define a set-valued mapping $R : H \times \Omega \to 2^H$ by

$$R(x,\lambda) = \bigcup_{f\in F(x,\lambda),q\in Q(x,\lambda),s\in S(x,\lambda)} J_{\mu}^{M_2(\cdot,s,\lambda)}[x + w_2 - \mu N_2(f,q,\lambda)],\ \forall\ (x,\lambda) \in H\times\Omega,$$

and define a set-valued mapping $T : H \times \Omega \to 2^H$ by

$$T(x,\lambda) = \{u \in H : \exists\ y \in R(x,\lambda) \text{ such that}$$
$$u \in \bigcup_{a\in A(x,\lambda),b\in B(x,\lambda),g\in G(x,\lambda),c\in C(y,\lambda),d\in D(y,\lambda),e\in E(y,\lambda)} [x - (g-m)(x,\lambda)$$
$$+ J_{\rho}^{M_1(\cdot,e,\lambda)}((g-m)(x,\lambda) + \rho(w_1 - N_1(a,b,c,d,\lambda))]\},\ \forall\ (x,\lambda) \in H\times\Omega.$$

By the definition of set-valued mappings R and T, we have the following result.

Lemma 2.2. *For each fixed $\lambda \in \Omega$, $(x,y) = (x(\lambda),y(\lambda)) \in H \times H$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(x,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, $g = g(x,\lambda) \in G(x,\lambda)$, $e = e(y,\lambda) \in E(y,\lambda)$, $f = f(x,\lambda) \in F(x,\lambda)$, $q = q(x,\lambda) \in Q(x,\lambda)$, $s = s(x,\lambda) \in S(x,\lambda)$ is a solution of the SPMQVI (2.1) if and only if $x = x(\lambda)$ is a fixed point of $T(x,\lambda)$.*

Proof. If (x,y,a,b,c,d,g,e,f,q,s) is a solution of the SPMQVI (2.1), by Lemma 2.1, we have $y \in R(x,\lambda)$ and $x = x(\lambda) \in T(x,\lambda)$, i.e., $x = x(\lambda)$ is a fixed point of T. Conversely, if $x = x(\lambda)$ is a fixed point of $T(x,\lambda)$, then there exists $y = y(x,\lambda) \in R(x,\lambda)$, $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $c = c(y,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, $g = g(x,\lambda) \in G(x,\lambda)$, $e = e(y,\lambda) \in E(y,\lambda)$ such that

$$(g-m)(x,\lambda) = J_{\rho}^{M_1(\cdot,e,\lambda)}[(g-m)(x,\lambda) + \rho(w_1 - N_1(a,b,c,d,\lambda)].$$

From $y = y(x,\lambda) \in R(x,\lambda)$ it follows that there exist $q = q(x,\lambda) \in Q(x,\lambda)$, $f = f(x,\lambda) \in F(x,\lambda)$, and $s = s(x,\lambda) \in S(x,\lambda)$ such that

$$y = J_{\mu}^{M_2(\cdot,s,\lambda)}[x + w_2 - \mu N_2(f,q,\lambda)].$$

By Lemma 2.1, (x,y,a,b,c,d,g,e,f,q,s) is a solution of the SPMQVI (2.1). □

Remark 2.1. By Lemma 2.2, we observe that the sensitivity analysis of the solution set of the SPMQVI (2.1) with respect to the parameter $\lambda \in \Omega$ is essentially the sensitivity analysis of the fixed point set of the set-valued mapping $T(x,\lambda)$ with respect to the parameter $\lambda \in \Omega$.

Lemma 2.3 ([9])**.** *Let (X,d) be a complete metric space and $T_1,T_2 : X \to C(X)$ be two set-valued contractive mappings with same contractive constants $\theta \in (0,1)$, i.e.,*

$$\tilde{H}(T_i(x),T_i(y)) \leq \theta d(x,y),\ \forall\ x,y \in X,\ i = 1,2.$$

Then

$$\tilde{H}(F(T_1),F(T_2)) \leq \frac{1}{1-\theta}\sup_{x\in X}\tilde{H}(T_1(x),T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed-point sets of T_1 and T_2, respectively.

Definition 2.2. Let $N_1 : H \times H \times H \times H \times \Omega \to H$ and $N_2 : H \times H \times \Omega \to H$ be single-valued mappings and $A,B,F,G : H \times \Omega \to C(H)$ be set-valued mappings.

(i) G is said to be α_G-strongly monotone in first argument if there exists $\alpha_G > 0$ such that

$$\langle g-\bar{g}, x-\bar{x}\rangle \geq \alpha_G\|x-\bar{x}\|^2,\ \forall\ (x,\bar{x},\lambda)\in H\times H\times\Omega,\ g\in G(x,\lambda),\ \bar{g}\in G(\bar{x},\lambda);$$

(ii) G is said to be ℓ_G-Lipschitz continuous in first argument if there exists $\ell_G > 0$ such that

$$\tilde{H}(G(x,\lambda),G(\bar{x},\lambda)) \leq \ell_G\|x-\bar{x}\|,\ \forall\ (x,\bar{x},\lambda)\in H\times H\times\Omega;$$

(iii) N_1 is said to be $\alpha_{A,B}$-mixed strongly monotone in first and second argument with respect to A and B if there exists $\alpha_{A,B} > 0$ such that

$$\langle N_1(a,b,c,d,\lambda) - N_1(\bar{a},\bar{b},c,d,\lambda), x-\bar{x}\rangle \geq \alpha_{A,B}\|x-\bar{x}\|^2$$

for all $(x,\bar{x},c,d,\lambda)\in H\times H\times H\times H\times\Omega,,\ a\in A(x,\lambda),\ b\in B(x,\lambda),\ \bar{a}\in A(\bar{x},\lambda),\ \bar{b}\in B(\bar{x},\lambda)$;

(iv) N_1 is said to be ℓ_1-ℓ_2-mixed Lipschitz continuous in first and second arguments if there exists $\ell_1 > 0$ and $\ell_2 > 0$ such that

$$\|N_1(a,b,c,d,\lambda)-N_1(\bar{a},\bar{b},c,d,\lambda)\| \leq \ell_1\|a-\bar{a}\|+\ell_1\|b-\bar{b}\|,\ \forall\, a,b,c,d,\bar{a},\bar{b}\in H,\ \lambda\in\Omega;$$

(v) N_2 is said to be α_F-strongly monotone in first argument with respect to F if there exists $\alpha_F > 0$ such that

$$\langle N_2(f,q,\lambda) - N_2(\bar{f},q,\lambda), x-\bar{x}\rangle \geq \alpha_F\|x-\bar{x}\|^2,$$
$$\forall\ x,\bar{x},q\in H,\lambda\in\Omega, f\in F(x,\lambda),\bar{f}\in F(\bar{x},\lambda);$$

3. Existence and sensitivity analysis

Theorem 3.1. *Let $A,B,C,D,E,F,G,Q,S : H\times\Omega\to C(H)$ be set-valued mappings, and $m : H\times\Omega\to H$, $N_1 : H\times H\times H\times H\times\Omega\to H$ and $N_2 : H\times H\times\Omega\to H$ be single-valued mappings. Let $M_1, M_2 : H\times H\times\Omega\to 2^H$ be set-valued mappings such that for each $(z,\lambda)\in H\times\Omega$, $M_1(\cdot,z,\lambda), M_2(\cdot,z,\lambda) : H\to 2^H$ are both maximal monotone with $(G-m)(H\times\{\lambda\})\bigcap \mathrm{dom}\ M_1(\cdot,z,\lambda)\neq\emptyset$. Suppose the following conditions are satisfied:*

(i) *A,B,C,D,E,F,G,Q,S,m are all Lipschitz continuous in first argument with Lipschitz constants ℓ_A, ℓ_B, ℓ_C, ℓ_D, ℓ_E, ℓ_F, ℓ_G, ℓ_Q, ℓ_S, ℓ_m, respectively;*

(ii) *G is α_G-strongly monotone in first argument:*

(iii) *$N_1(\cdot,\cdot,\cdot,\cdot,\cdot,)$ is $\alpha_{A,B}$-mixed strongly monotone in first and second arguments with respect to A and B, mixed ℓ_1-ℓ_2-Lipschitz continuous in first and second arguments, and mixed ℓ_3-ℓ_4-Lipschitz continuous in third and fourth arguments;*

(iv) *$N_2(\cdot,\cdot,\cdot)$ is α_F-strongly monotone in first argument with respect to F, k_1-Lipschitz continuous in first argument, and k_2-Lipschitz continuous in second argument.*

Further assume that there exist $\sigma,\eta > 0$ satisfying

$$\|J_\rho^{M_1(\cdot,e,\lambda)}(u) - J_\rho^{M_1(\cdot,\bar{e},\lambda)}(u)\| \leq \sigma\|e-\bar{e}\|,\ \forall\ e,\bar{e},u\in H,\ \lambda\in\Omega, \tag{3.1}$$

$$\|J_\mu^{M_2(\cdot,s,\lambda)}(u) - J_\mu^{M_2(\cdot,\bar{s},\lambda)}(u)\| \leq \eta\|s-\bar{s}\|,\ \forall\ s,\bar{s},u\in H,\ \lambda\in\Omega. \tag{3.2}$$

If

(3.3)
$$\begin{cases} \Theta_1 = 2(\ell_m + \sqrt{1-2\alpha_G+\ell_G^2}),\ \Gamma_1 = \ell_1\ell_A + \ell_2\ell_B,\ \Gamma_2 = \ell_3\ell_C + \ell_4\ell_D, \\ \Theta_2 = \sqrt{1-2\mu\alpha_F + \mu^2 k_1^2\ell_F^2} + \mu k_2^2\ell_q + \eta\ell_S,\ \Theta_1 + \sigma\ell_E < 1, \\ \alpha_{A,B} > (1-\Theta_1-\sigma\ell_E)\Gamma_2 + \sqrt{(\Gamma_1^2-\Gamma_2^2)(\Theta_1+\sigma\ell_E)(2-\Theta_1-\sigma\ell_E)}, \\ |\rho - \frac{\alpha_{A,B}-\Gamma_2(1-\Theta_1-\sigma\ell_E)}{\Gamma_1^2-\Gamma_2^2}| \le \frac{\sqrt{[\alpha_{A,B}-(1-\Theta_1-\sigma\ell_E)\Gamma_2]^2-(\Gamma_1^2-\Gamma_2^2)(\Theta_1+\sigma\ell_E)(2-\Theta_1-\sigma\ell_E)}}{\Gamma_1^2-\Gamma_2^2} \\ \eta\ell_S < 1\ k_2\ell_Q < k_1\ell_F,\ \alpha_F > k_2\ell_Q(1-\eta\ell_S) + \sqrt{2\eta\ell_S(1-\eta\ell_S)(k_1^2\ell_F^2 - k_2^2\ell_Q^2)} \\ |\mu - \frac{\alpha_F - k_2\ell_Q(1-\eta\ell_S)}{k_1^2\ell_F^2-k_2^2\ell_Q^2}| \le \frac{\sqrt{[\alpha_F-k_2\ell_Q(1-\eta\ell_S)]^2-\eta\ell_S(2-\eta\ell_S)(k_1^2\ell_F^2-k_2^2\ell_Q^2)}}{k_1^2\ell_F^2-k_2^2\ell_Q^2}, \end{cases}$$

Then for each $\lambda \in \Omega$, *the solution set* $Sol(\lambda) = \{(x,y) \in H \times H : \exists\ y \in R(x,\lambda)$ such that $x \in T(x,\lambda)\}$ *of the SPMQVI (2.1) is nonempty and closed in* $H \times H$.

Proof. For each $(x,y,\lambda) \in H \times H \times \Omega$, since $A(x,\lambda)$, $B(x,\lambda)$, $C(y,\lambda)$, $D(y,\lambda)$, $E(y,\lambda)$, $F(x,\lambda)$, $G(x,\lambda), Q(x,\lambda), S(x,\lambda), \in C(H)$, and $J_\rho^{M_1(\cdot,e,\lambda)}$ and $J_\mu^{M_2(\cdot,s,\lambda)}$ are continuous, we have $R(x,\lambda) \in C(H)$ and $T(x,\lambda) \in C(H)$. Now for each fixed $\lambda \in \Omega$, we prove that $T(x,\lambda)$ is a set-valued contractive mapping. For any $x \in H$ and any $z \in T(x,\lambda)$, there exist $y \in R(x,\lambda)$ such that

$$z \in \bigcup_{a\in A(x,\lambda), b\in B(x,\lambda), g\in G(x,\lambda), c\in C(y,\lambda), d\in D(y,\lambda), e\in E(y,\lambda)} [x - (g-m)(x,\lambda) + J_\rho^{M_1(\cdot,e,\lambda)}((g-m)(x,\lambda) + \rho(w_1 - N_1(a,b,c,d,\lambda))].$$

Hence there exist $a = a(x,\lambda) \in A(x,\lambda)$, $b = b(x,\lambda) \in B(x,\lambda)$, $g = g(x,\lambda) \in G(x,\lambda)$, $c = c(y,\lambda) \in C(y,\lambda)$, $d = d(y,\lambda) \in D(y,\lambda)$, and $e = e(y,\lambda) \in E(y,\lambda)$ such that

$$z = x - (g-m)(x,\lambda) + J_\rho^{M_1(\cdot,e,\lambda)}[(g-m)(x,\lambda) + \rho(w_1 - N_1(a,b,c,d,\lambda))]. \tag{3.4}$$

From $y \in R(x,\lambda)$ it follows that there exist $f = f(x,\lambda) \in F(x,\lambda)$, $q = q(x,\lambda) \in Q(x,\lambda)$, and $s = s(x,\lambda) \in S(x,\lambda)$ such that

$$y = J_\mu^{M_2(\cdot,s,\lambda)}[x + w_2 - \mu N_2(f,q,\lambda)]. \tag{3.5}$$

For any $\bar{x} \in H$, since $F(\bar{x},\lambda), Q(\bar{x},\lambda), S(\bar{x},\lambda) \in C(H)$, there exist $\bar{f} \in F(\bar{x},\lambda)$, $\bar{q} \in Q(\bar{x},\lambda)$, $\bar{s} \in S(\bar{x},\lambda)$ such that

$$\|f-\bar{f}\| \le \tilde{H}(F(x,\lambda),F(\bar{x},\lambda)),\ \|q-\bar{q}\| \le \tilde{H}(Q(x,\lambda),Q(\bar{x},\lambda)),$$
$$\|s-\bar{s}\| \le \tilde{H}(S(x,\lambda),S(\bar{x},\lambda)).$$

Let

$$\bar{y} = J_\mu^{M_2(\cdot,\bar{s},\lambda)}[\bar{x} + w_2 - \mu N_2(\bar{f},\bar{q},\lambda)]. \tag{3.6}$$

Then we have $\bar{y} \in R(\bar{x},\lambda)$. Since $A(\bar{x},\lambda)$, $B(\bar{x},\lambda)$, $G(\bar{x},\lambda)$, $C(\bar{y},\lambda)$, $D(\bar{y},\lambda)$, $E(\bar{y},\lambda) \in C(H)$, there exist $\bar{a} = \bar{a}(\bar{x},\lambda) \in A(\bar{x},\lambda)$, $\bar{b} = \bar{b}(\bar{x},\lambda) \in B(\bar{x},\lambda)$, $\bar{g} = \bar{g}(\bar{x},\lambda) \in G(\bar{x},\lambda)$, $\bar{c} = \bar{c}(\bar{y},\lambda) \in C(\bar{y},\lambda)$, $\bar{d} = \bar{d}(\bar{y},\lambda) \in D(\bar{y},\lambda)$, $\bar{e} = \bar{e}(\bar{y},\lambda) \in D(\bar{y},\lambda)$, such that

$$\|a-\bar{a}\| \le \tilde{H}(A(x,\lambda),A(\bar{x},\lambda)),\ \|b-\bar{b}\| \le \tilde{H}(B(x,\lambda),B(\bar{x},\lambda)),$$

$$\|g-\bar{g}\| \le \tilde{H}(G(x,\lambda),G(\bar{x},\lambda)),\ \ \|c-\bar{c}\| \le \tilde{H}(C(y,\lambda),C(\bar{y},\lambda)),$$
$$\|d-\bar{d}\| \le \tilde{H}(D(y,\lambda),D(\bar{y},\lambda)),\ \ \|e-\bar{e}\| \le \tilde{H}(E(y,\lambda),E(\bar{y},\lambda)).$$

Let

$$w = \bar{x} - (\bar{g}-m)(\bar{x},\lambda) + J_\rho^{M_1(\cdot,\bar{e},\lambda)}((\bar{g}-m)(\bar{x},\lambda) + \rho(w_1 - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda))]. \tag{3.7}$$

Then we have $w \in T(\bar{x},\lambda)$.

It follows from (3.4) and (3.7) that

$$\begin{aligned}\|z-w\| &= \|x-(g-m)(x,\lambda) + J_\rho^{M_1(\cdot,e,\lambda)}((g-m)(x,\lambda) \\ &\quad + \rho(w_1 - N_1(a,b,c,d,\lambda))) - [\bar{x} - (\bar{g}-m)(\bar{x},\lambda) \\ &\quad + J_\rho^{M_1(\cdot,\bar{e},\lambda)}((\bar{g}-m)(\bar{x},\lambda) + \rho(w_1 - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)))]\| \\ &\le \|x-\bar{x}-(g(x,\lambda)-\bar{g}(\bar{x},\lambda)\| + \|m(x,\lambda)-m(\bar{x},\lambda)\| \\ &\quad + \|J_\rho^{M_1(\cdot,e,\lambda)}((g-m)(x,\lambda) + \rho(w_1 - N_1(a,b,c,d,\lambda))) \\ &\quad - J_\rho^{M_1(\cdot,e,\lambda)}((\bar{g}-m)(\bar{x},\lambda) + \rho(w_1 - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)))\| \\ &\quad + \|J_\rho^{M_1(\cdot,e,\lambda)}((\bar{g}-m)(\bar{x},\lambda) + \rho(w_1 - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)))\| \\ &\quad - J_\rho^{M_1(\cdot,\bar{e},\lambda)}((\bar{g}-m)(\bar{x},\lambda) + \rho(w_1 - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)))\| \\ &\le 2\|x-\bar{x}-(g(x,\lambda)-\bar{g}(\bar{x},\lambda))\| + 2\|m(x,\lambda)-m(\bar{x},\lambda)\| \\ &\quad + \|x-\bar{x}-\rho(N_1(a,b,c,d,\lambda) - N_1(\bar{a},\bar{b},c,d,\lambda))\| \\ &\quad + \rho\|N_1(\bar{a},\bar{b},c,d,\lambda) - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)\| + \sigma\|e-\bar{e}\|.\end{aligned} \tag{3.8}$$

Since G is α_G-strongly monotone and ℓ_G-Lipschitz continuous in first argument, we have

$$\begin{aligned}&\|x-\bar{x}-(g(x,\lambda)-\bar{g}(\bar{x},\lambda))\|^2 \\ &= \|x-\bar{x}\|^2 - 2\langle g(x,\lambda)-\bar{g}(\bar{x},\lambda), x-\bar{x}\rangle + \|g(x,\lambda)-\bar{g}(\bar{x},\lambda)\|^2 \\ &\le \|x-\bar{x}\|^2 - 2\alpha_G\|x-\bar{x}\|^2 + [\tilde{H}(G(x,\lambda),G(\bar{x},\lambda)]^2 \\ &\le (1-2\alpha_G+\ell_G^2)\|x-\bar{x}\|^2,\end{aligned}$$

and hence

$$\|x-\bar{x}-(g(x,\lambda)-\bar{g}(\bar{x},\lambda))\| \le \sqrt{1-2\alpha_G+\ell_G^2}\ \|x-\bar{x}\|. \tag{3.9}$$

Since m is ℓ_m-Lipschitz continuous in first argument, we have

$$\|m(x,\lambda)-m(\bar{x},\lambda)\| \le \ell_m\|x-\bar{x}\|. \tag{3.10}$$

Since $N_1(\cdot,\cdot,\cdot,\cdot,\cdot)$ is $\alpha_{A,B}$-mixed strongly monotone in first and second arguments with respect to A and B, mixed ℓ_1-ℓ_2-Lipschitz continuous, and A and B are ℓ_A-Lipschitz continuous and ℓ_B-Lipschitz continuous respectively, we have

$$\begin{aligned}&\|x-\bar{x}-\rho(N_1(a,b,c,d,\lambda)-N_1(\bar{a},\bar{b},c,d,\lambda))\|^2 \\ &\le \|x-\bar{x}\|^2 - 2\rho\langle N_1(a,b,c,d,\lambda)-N_1(\bar{a},\bar{b},c,d,\lambda), x-\bar{x}\rangle \\ &\quad + \rho^2\|N_1(a,b,c,d,\lambda)-N_1(\bar{a},\bar{b},c,d,\lambda)\|^2 \\ &\le \|x-\bar{x}\|^2 - 2\rho\alpha_{A,B}\|x-\bar{x}\|^2 + \rho^2(\ell_1\|a-\bar{a}\| + \ell_2\|b-\bar{b}\|)^2\end{aligned}$$

$$\leq (1 - 2\rho\alpha_{A,B} + \rho^2(\ell_1\ell_A + \ell_2\ell_B)^2)\|x - \bar{x}\|^2,$$

and hence we have

$$\|x-\bar{x}-\rho(N_1(a,b,c,d,\lambda)-N_1(\bar{a},\bar{b},c,d,\lambda))\| \leq \sqrt{1 - 2\rho\alpha_{A,B} + \rho^2\Gamma_1^2}\ \|x-\bar{x}\|^2. \tag{3.11}$$

Since $N_1(\cdot,\cdot,\cdot,\cdot,\cdot)$ are mixed ℓ_3-ℓ_4-Lipschitz continuous in third and fourth arguments, and C and D are ℓ_C-Lipschitz continuous and ℓ_D-Lipschitz continuous respectively, we have

$$\begin{aligned}(3.12)\quad &\|N_1(\bar{a},\bar{b},c,d,\lambda) - N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda)\| \\ &\leq \ell_3\|c-\bar{c}\| + \ell_4\|d-\bar{d}\| \leq \ell_3\tilde{H}(C(y,\lambda),C(\bar{y},\lambda)) + \ell_4\tilde{H}(D(y,\lambda),D(\bar{y},\lambda)) \\ &\leq (\ell_3\ell_C + \ell_4\ell_D)\|y-\bar{y}\| = \Gamma_2\|y-\bar{y}\|.\end{aligned}$$

Sine E is ℓ_E-Lipschitz continuous in first argument, we have

$$\|e - \bar{e}\| \leq \tilde{H}(E(y,\lambda),E(\bar{y},\lambda)) \leq \ell_E\|y-\bar{y}\|. \tag{3.13}$$

From (3.8)–(3.13) it follows that

$$\begin{aligned}(3.14)\quad \|z-w\| &\leq [2(\ell_m + \sqrt{1-2\alpha_G+\ell_G^2}\ ) + \sqrt{1-2\rho\alpha_{A,B}+\rho^2\Gamma_1^2}\ ]\ \|x-\bar{x}\| \\ &\quad + [\rho\Gamma_2 + \sigma\ell_E]\|y-\bar{y}\| \\ &= [\Theta_1 + \sqrt{1-2\rho\alpha_{A,B}+\rho^2\Gamma_1^2}]\|x-\bar{x}\| + [\rho\Gamma_2+\sigma\ell_E]\|y-\bar{y}\|,\end{aligned}$$

where $\Theta_1 = 2(\ell_m + \sqrt{1-2\alpha_G+\ell_G^2})$. By (3.5) and (3.6), we have

$$\begin{aligned}(3.15)\quad &\|y-\bar{y}\| \\ &= \|J_\mu^{M_2(\cdot,s,\lambda)}(x+w_2-\mu N_2(f,q,\lambda)) - J_\mu^{M_2(\cdot,\bar{s},\lambda)}(\bar{x}+w_2-\mu N_2(\bar{f},\bar{q},\lambda))\| \\ &\leq \|J_\mu^{M_2(\cdot,s,\lambda)}(x+w_2-\mu N_2(f,q,\lambda)) - J_\mu^{M_2(\cdot,s,\lambda)}(\bar{x}+w_2-\mu N_2(\bar{f},\bar{q},\lambda))\| \\ &\quad + \|J_\mu^{M_2(\cdot,s,\lambda)}(\bar{x}+w_2-\mu N_2(\bar{f},\bar{q},\lambda)) - J_\mu^{M_2(\cdot,\bar{s},\lambda)}(\bar{x}+w_2-\mu N_2(\bar{f},\bar{q},\lambda))\| \\ &\leq \|x-\bar{x}-\mu(N_2(f,q,\lambda)-N_2(\bar{f},\bar{q},\lambda))\| + \eta\|s-\bar{s}\|. \\ &\leq \|x-\bar{x}-\mu(N_2(f,q,\lambda)-N_2(\bar{f},q,\lambda))\| + \mu\|N_2(\bar{f},q,\lambda)-N_2(\bar{f},\bar{q},\lambda))\| \\ &\quad + \eta\|s-\bar{s}\|.\end{aligned}$$

Since $N_2(\cdot,\cdot,\cdot)$ is α_F-strongly monotone in first argument with respect to F, k_1-Lipschitz continuous in first argument, and F is ℓ_F-Lipschitz continuous, we have

$$\begin{aligned}&\|x-\bar{x}-\mu(N_2(f,q,\lambda)-N_2(\bar{f},q,\lambda))\|^2 \\ &= \|x-\bar{x}\|^2 - 2\mu\langle N_2(f,q,\lambda)-N_2(\bar{f},q,\lambda), x-\bar{x}\rangle + \mu^2\|N_2(f,q,\lambda)-N_2(\bar{f},q,\lambda)\|^2 \\ &\leq \|x-\bar{x}\|^2 - 2\mu\alpha_F\|x-\bar{x}\|^2 + \mu^2k_1^2\|f-\bar{f}\|^2 \\ &\leq (1-2\mu\alpha_F+\mu^2k_1^2\ell_F^2)\|x-\bar{x}\|^2.\end{aligned}$$

It follows that

$$\|x-\bar{x}-\mu(N_2(f,q,\lambda)-N_2(\bar{f},q,\lambda))\| \leq \sqrt{1-2\mu\alpha_F+\mu^2k_1^2\ell_F^2}\ \|x-\bar{x}\|. \tag{3.16}$$

Since $N_2(\cdot,\cdot,\cdot)$ is k_2-Lipschitz continuous in second argument and Q is ℓ_Q-Lipschitz continuous, we have

$$\|N_2(\bar{f},q,\lambda)-N_2(\bar{f},\bar{q},\lambda))\| \le k_2\|q-\bar{q}\| \le k_2\ell_Q\|x-\bar{x}\|. \tag{3.17}$$

Since S is ℓ_S-Lipschitz continuous, we have

$$\|s-\bar{s}\| \le \tilde{H}(S(x,\lambda),S(\bar{x},\lambda)) \le \ell_S\|x-\bar{x}\|. \tag{3.18}$$

From (3.15)–(3.18) it follows that

$$\|y-\bar{y}\| \le (\sqrt{1-2\mu\alpha_F+\mu^2k_1^2\ell_F^2}+\mu k_2\ell_Q+\eta\ell_S)\ \|x-\bar{x}\| = \Theta_2\|x-\bar{x}\|. \tag{3.19}$$

where $\Theta_2=\sqrt{1-2\mu\alpha_F+\mu^2k_1^2\ell_F^2}+\mu k_2\ell_Q+\eta\ell_S$. The condition (3.3) implies $\Theta_2<1$.

By (3.14) and (3.19), we have

$$\|z-w\| \le [\Theta_1+\sqrt{1-2\rho\alpha_{A,B}+\rho^2\Gamma_1^2}+\rho\Gamma_1+\sigma\ell_E]\|x-\bar{x}\| = \Theta\|x-\bar{x}\|,$$

where $\Theta=\Theta_1+\sqrt{1-2\rho\alpha_{A,B}+\rho^2\Gamma_1^2}+\rho\Gamma_2+\sigma\ell_E$. The condition (3.3) implies $\Theta<1$.

Hence we have

$$d(z,T(\bar{x},\lambda))=\inf_{u\in T(\bar{x},\lambda)}\|z-u\| \le \|z-w\| \le \Theta\|x-\bar{x}\|.$$

Since $z\in T(x,\lambda)$ is arbitrary, we obtain

$$\sup_{z\in T(x,\lambda)} d(z,T(\bar{x},\lambda)) \le \Theta\|x-\bar{x}\|.$$

By using same argument, we can prove

$$\sup_{w\in T(\bar{x},\lambda)} d(T(x,\lambda),w) \le \Theta\|x-\bar{x}\|.$$

By the definition of the Hausdorff metric $\tilde{H}$ on $C(H)$, we obtain that for all $(x,\bar{x},\lambda)\in H\times H\times\Omega$,

$$\tilde{H}(T(x,\lambda),T(\bar{x},\lambda)) \le \Theta\|x-\bar{x}\|,$$

i.e., $T(x,\lambda)$ is a set-valued contractive mapping which is uniform with respect to $\lambda\in\Omega$. By a fixed point theorem of Nadler [12], for each $\lambda\in\Omega$, $T(x,\lambda)$ has a fixed point $x(\lambda)\in H$, i.e., $x(\lambda)\in T(x(\lambda),\lambda)$. By Lemma 2.1 and Lemma 2.2, for each $\lambda\in\Omega$, there exist $(x,y)=(x(\lambda),y(\lambda))\in H\times H$, $a=a(x,\lambda)\in A(x,\lambda)$, $b=b(x,\lambda)\in B(x,\lambda)$, $c=c(y,\lambda)\in C(y,\lambda)$, $d=d(y,\lambda)\in D(y,\lambda)$, $e=e(y,\lambda)\in E(y,\lambda)$, $g=g(x,\lambda)\in G(x,\lambda)$, $f=f(x,\lambda)\in F(x,\lambda)$, $q=q(x,\lambda)\in Q(x,\lambda)$, and $s=s(x,\lambda)\in S(x,\lambda)$ such that

$$\begin{cases}(g-m)(x,\lambda)=J_\rho^{M_1(\cdot,e,\lambda)}[(g-m)(x,\lambda)+\rho(w_1-N_1(a,b,c,d,\lambda)],\\ y=J_\mu^{M_2(\cdot,s,\lambda)}[x+w_2-\mu N_2(f,q,\lambda)],\end{cases}$$

i.e., (x,y,a,b,c,d,g,e,f,q,s) is a solution of the SPMQVI (2.1). Hence the solution set $Sol(\lambda)$ of the SPMQVI (2.1) is nonempty .

For each $\lambda\in\Omega$, let $(x_n,y_n)=(x_n(\lambda),y_n(\lambda))$ be a sequence such that $y_n(\lambda)\in R(x_n(\lambda),\lambda)$, $x_n(\lambda)\in T(x_n(\lambda),\lambda)$, $x_n(\lambda)\to x_0(\lambda)$ and $y_n(\lambda)\to y_0(\lambda)$, as $n\to\infty$. Hence there exist $a_n=a_n(x_n,\lambda)\in A(x_n,\lambda)$, $b_n=b_n(x_n,\lambda)\in B(x_n,\lambda)$, $c_n=c_n(y_n,\lambda)\in C(y_n,\lambda)$, $d_n=d_n(y_n,\lambda)\in D(y_n,\lambda)$, $e_n=e_n(y_n,\lambda)\in E(y_n,\lambda)$, $g_n=$

$g_n(x_n,\lambda) \in G(x_n,\lambda)$, $f_n = f_n(x_n,\lambda) \in F(x_n,\lambda)$, $q_n = q_n(x_n,\lambda) \in Q(x_n,\lambda)$, and $s_n = s(x_n,\lambda) \in S(x_n,\lambda)$ such that

$$\begin{cases}(g_n - m)(x_n,\lambda) = J_\rho^{M_1(\cdot,e_n,\lambda)}[(g_n-m)(x_n,\lambda) + \rho(w_1 - N_1(a_n,b_n,c_n,d_n,\lambda)],\\ y_n = J_\mu^{M_2(\cdot,s_n,\lambda)}[x_n + w_2 - \mu N_2(f_n,q_n,\lambda)],\end{cases}$$

Since $A, B, C, D, E, F, G, Q, S, m, N_1$ and N_2 all are Lipschitz continuous, and $J_\rho^{M_1(\cdot,e,\lambda)}$ and $J_\rho^{M_2(\cdot,s,\lambda)}$ are nonexpansive and satisfy the conditions (3.1) and (3.2), it is easy to show that $a_n \to a_0 = a_0(x_0,\lambda) \in A(x_0,\lambda)$, $b_n \to b_0 = b_0(x_0,\lambda) \in B(x_0,\lambda)$, $c_n \to c_0 = c_0(y_0,\lambda) \in C(y_0,\lambda)$, $d_n \to d_0 = d_0(y_n,\lambda) \in D(y_0,\lambda)$, $e_n \to e_0 = e_0(y_0,\lambda) \in E(y_0,\lambda)$, $g_n \to g_0(x_0,\lambda) \in G(x_0,\lambda)$, $f_n \to f_0 = f_0(x_0,\lambda) \in F(x_0,\lambda)$, $q_n \to q_0 = q_0(x_0,\lambda) \in Q(x_0,\lambda)$ and $s_n \to s_0 = s(x_0,\lambda) \in S(x_0,\lambda)$ such that

$$\begin{cases}(g_0 - m)(x_0,\lambda) = J_\rho^{M_1(\cdot,e_0,\lambda)}[(g_0-m)(x_0,\lambda) + \rho(w_1 - N_1(a_0,b_0,c_0,d_0,\lambda)],\\ y_0 = J_\mu^{M_2(\cdot,s_0,\lambda)}[x_0 + w_2 - \mu N_2(f_0,q_0,\lambda)],\end{cases}$$

By Lemma 2.1, $(x_0, y_0, a_0, b_0, c_0, d_0, g_0, e_0, s_0, f_0, q_0)$ is a solution of the SPMQVI (2.1) and for each $\lambda \in \Omega$, the solution set $Sol(\lambda) = \{(x,y) \in H \times H : \exists\, y \in R(x,\lambda)$ such that $x \in T(x,\lambda)\}$ of the SPMQVI (2.1) is closed in $H \times H$. □

Theorem 3.2. *Under the hypotheses of Theorem 3.1, further assume*

(i) *$A, B, C, D, E, F, G, Q, S, m$ are all Lipschitz continuous in second arguments with Lipschitz constants L_A, L_B, L_C, L_D, L_E, L_F, L_G, L_Q, L_S, L_m, respectively;*

(ii) *N_1 and N_2 are Lipschitz continuous in fifth argument and third argument with Lipschitz constants L_5 and L_3, respectively;*

(iii) *there exist $\xi, \theta > 0$ such that for any $(e,s,u) \in H \times H \times H$, $\lambda_1, \lambda_2 \in \Omega$,*

$$\|J_\rho^{M_1(\cdot,e,\lambda_1)}(u) - J_\rho^{M_1(\cdot,e,\lambda_2)}(u)\| \le \xi\|\lambda_1 - \lambda_2\|, \tag{3.20}$$

$$\|J_\mu^{M_2(\cdot,s,\lambda_1)}(u) - J_\mu^{M_2(\cdot,s,\lambda_2)}(u)\| \le \theta\|\lambda_1 - \lambda_2\|. \tag{3.21}$$

Then the solution set $Sol(\lambda) = \{(x(\lambda), y(\lambda)) \in H \times H : \exists\, y(\lambda) \in R(x(\lambda),\lambda)$ such that $x(\lambda) \in T(x(\lambda),\lambda)\}$ of the SPMQVI (2.1) is Lipschitz continuous with respect to the parameter $\lambda \in \Omega$.

Proof. For any given $\lambda_1, \lambda_2 \in \Omega$, by Theorem 3.1, $Sol(\lambda_1)$ and $Sol(\lambda_2)$ are both nonempty closed subsets of $H \times H$. By the proof of Theorem 3.1, $T(\cdot,\lambda_1) : H \to C(H)$ and $T(\cdot,\lambda_2) : H \to C(H)$ are both set-valued contractive mappings with same contractive constant $\Theta \in (0,1)$. Let $F_1(\lambda_1)$ and $F_2(\lambda_2)$ denote the fixed point sets of $T(\cdot,\lambda_1)$ and $T(\cdot,\lambda_2)$. By Lemma 2.3, we obtain

$$\tilde{H}(F_1(\lambda_1), F_2(\lambda_2)) \le \frac{1}{1-\Theta}\sup_{x\in H}\tilde{H}(T(x,\lambda_1), T(x,\lambda_2)). \tag{3.22}$$

For each fixed $x \in X$ and any $z \in T(x,\lambda_1)$, there exists $y \in R(x,\lambda_1)$ such that

$$z \in \bigcup_{a\in A(x,\lambda_1), b\in B(x,\lambda_1), g\in G(x,\lambda_1), c\in C(y,\lambda_1), d\in D(y,\lambda_1), e\in E(y,\lambda_1)} [x - (g-m)(x,\lambda_1) + J_\rho^{M_1(\cdot,e,\lambda_1)}((g-m)(x,\lambda_1) + \rho(w_1 - N_1(a,b,c,d,\lambda_1))].$$

Hence there exist $a = a(x,\lambda_1) \in A(x,\lambda_1)$, $b = b(x,\lambda_1) \in B(x,\lambda_1)$, $c = c(y,\lambda_1) \in C(y,\lambda_1)$, $d = d(y,\lambda_1) \in D(y,\lambda_1)$, $e = e(y,\lambda_1) \in E(x,\lambda_1)$, and $g = g(x,\lambda_1) \in G(x,\lambda_1)$ such that

$$z = x-(g-m)(x,\lambda_1)+J_\rho^{M_1(\cdot,e,\lambda_1)}((g-m)(x,\lambda_1)+\rho(w_1-N_1(a,b,c,d,\lambda_1))]. \tag{3.23}$$

From $y \in R(x,\lambda_1)$ it follows that there exist $f = f(x,\lambda_1) \in F(x,\lambda_1)$, $q = q(x,\lambda_1) \in Q(x,\lambda_1)$, and $s = s(x,\lambda_1) \in S(x,\lambda_1)$ such that

$$y = J_\mu^{M_2(\cdot,s,\lambda_1)}[x + w_2 - \mu N_2(f,q,\lambda_1)]. \tag{3.24}$$

Since $F(x,\lambda_2)$, $Q(x,\lambda_2)$, $S(x,\lambda_2) \in C(H)$, there exist $\bar f = \bar f(x,\lambda_2) \in F(x,\lambda_2)$, $\bar q = \bar q(x,\lambda_2) \in Q(x,\lambda_2)$, $\bar s = \bar s(x,\lambda_2) \in S(x,\lambda_2)$ such that

$$\|f-\bar f\| \le \tilde H(F(x,\lambda_1),F(x,\lambda_2)),\ \|q-\bar q\| \le \tilde H(Q(x,\lambda_1),Q(x,\lambda_2)),$$
$$\|s-\bar s\| \le \tilde H(S(x,\lambda_1),S(x,\lambda_2)).$$

Let

$$\bar y = J_\mu^{M_2(\cdot,\bar s,\lambda_2)}[x + w_2 - \mu N_2(\bar f,\bar q,\lambda_2)]. \tag{3.25}$$

Then we have $\bar y \in R(x,\lambda_2)$. Since $A(x,\lambda_2)$, $B(x,\lambda_2)$, $C(\bar y,\lambda_2)$, $D(\bar y,\lambda_2)$, $E(\bar y,\lambda_2)$, $G(x,\lambda_2) \in C(H)$, there exist $\bar a = \bar a(x,\lambda_2) \in A(x,\lambda_2)$, $\bar b = \bar b(x,\lambda_2) \in B(x,\lambda_2)$, $\bar c = \bar c(\bar y,\lambda_2) \in C(\bar y,\lambda_2)$, $\bar d = \bar d(\bar y,\lambda_2) \in D(\bar y,\lambda_2)$, $\bar e = \bar e(\bar y,\lambda_2) \in E(\bar y,\lambda_2)$, $\bar g = \bar g(x,\lambda_2) \in G(x,\lambda_2)$ such that

$$\|a-\bar a\| \le \tilde H(A(x,\lambda_1),A(x,\lambda_2)), \quad \|b-\bar b\| \le \tilde H(B(x,\lambda_1),B(x,\lambda_2)),$$
$$\|c-\bar c\| \le \tilde H(C(y,\lambda_1),C(\bar y,\lambda_2)), \quad \|d-\bar d\| \le \tilde H(D(y,\lambda_1),D(\bar y,\lambda_2)),$$
$$\|e-\bar e\| \le \tilde H(E(y,\lambda_1),\ E(\bar y,\lambda_2)), \quad \|g-\bar g\| \le \tilde H(G(x,\lambda_1),\ G(x,\lambda_2)).$$

Let

$$w = x-(\bar g-m)(x,\lambda_2)+J_\rho^{M_1(\cdot,\bar e,\lambda_2)}((\bar g-m)(x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2))]. \tag{3.26}$$

Then we have $w \in T(x,\lambda_2)$. It follows that

(3.27)

$$\begin{aligned}
&\|z-w\| \\
&= \|x-(g-m)(x,\lambda_1)+J_\rho^{M_1(\cdot,e,\lambda_1)}((g-m)(x,\lambda_1)+\rho(w_1-N_1(a,b,c,d,\lambda_1))) \\
&\quad -[x-(\bar g-m)(x,\lambda_2)+J_\rho^{M_1(\cdot,\bar e,\lambda_2)}((\bar g-m)(x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))]\| \\
&\le \|(g(x,\lambda_1)-\bar g(x,\lambda_2)\| + \|m(x,\lambda_1)-m(x,\lambda_2)\| \\
&\quad + \|J_\rho^{M_1(\cdot,e,\lambda_1)}((g-m)(x,\lambda_1)+\rho(w_1-N_1(a,b,c,d,\lambda_1))) \\
&\quad - J_\rho^{M_1(\cdot,e,\lambda_1)}((\bar g-m)(x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))\| \\
&\quad + \|J_\rho^{M_1(\cdot,e,\lambda_1)}((\bar g-m)(x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))\| \\
&\quad - J_\rho^{M_1(\cdot,\bar e,\lambda_1)}((\bar g-m)(\bar x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))\| \\
&\quad + \|J_\rho^{M_1(\cdot,\bar e,\lambda_1)}((\bar g-m)(x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))\| \\
&\quad - J_\rho^{M_1(\cdot,\bar e,\lambda_2)}((\bar g-m)(\bar x,\lambda_2)+\rho(w_1-N_1(\bar a,\bar b,\bar c,\bar d,\lambda_2)))\| \\
&\le 2\|g(x,\lambda_1)-\bar g(x,\lambda_2)\| + 2\|m(x,\lambda_1)-m(x,\lambda_2)\|
\end{aligned}$$

$$
\begin{aligned}
&+\rho\|N_1(a,b,c,d,\lambda_1)-N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda_2)\|+\sigma\|e-\bar{e}\|+\xi\|\lambda_1-\lambda_2\| \\
\le\ & 2(L_G+L_m)\|\lambda_1-\lambda_2\|+\rho\|N_1(a,b,c,d,\lambda_1)-N_1(\bar{a},\bar{b},c,d,\lambda_1)\| \\
&+\rho\|N_1(\bar{a},\bar{b},c,d,\lambda_1)-N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda_1)\|+\rho\|N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda_1)-N_1(\bar{a},\bar{b},\bar{c},\bar{d},\lambda_2)\| \\
&+\sigma\|e-\bar{e}\|+\xi\|\lambda_1-\lambda_2\| \\
\le\ & 2(L_G+L_m)\|\lambda_1-\lambda_2\|+\rho(\ell_1\|a-\bar{a}\|+\ell_2\|b-\bar{b}\|+\ell_3\|c-\bar{c}\|+\ell_4\|d-\bar{d}\|) \\
&+L_5\|\lambda_1-\lambda_2\|+\sigma\|e-\bar{e}\|+\xi\|\lambda_1-\lambda_2\| \\
\le\ & (2L_G+2L_m+\xi+L_5)\|\lambda_1-\lambda_2\|+\rho(\ell_1\tilde{H}(A(x,\lambda_1),A(x,\lambda_2)) \\
&+\ell_2\tilde{H}(B(x,\lambda_1),B(x,\lambda_2))+\ell_3\tilde{H}(C(y,\lambda_1),C(\bar{y},\lambda_2)) \\
&+\ell_4\tilde{H}(D(y,\lambda_1),D(\bar{y},\lambda_2))+\sigma\tilde{H}(E(y,\lambda_1),E(\bar{y},\lambda_2)) \\
\le\ & (2L_G+2L_m+\xi+L_5+\rho(\ell_1L_A+\ell_2L_B))\|\lambda_1-\lambda_2\| \\
&+\rho\ell_3[\tilde{H}(C(y,\lambda_1),C(\bar{y},\lambda_1))+\tilde{H}(C(\bar{y},\lambda_1),C(\bar{y},\lambda_2))] \\
&+\rho\ell_4[\tilde{H}(D(y,\lambda_1),D(\bar{y},\lambda_1))+\tilde{H}(D(\bar{y},\lambda_1),D(\bar{y},\lambda_2))] \\
&+\sigma[\tilde{H}(E(y,\lambda_1),E(\bar{y},\lambda_1))+\tilde{H}(E(\bar{y},\lambda_1),E(\bar{y},\lambda_2))] \\
\le\ & (2L_G+2L_m+\xi+L_5+\rho(\ell_1L_A+\ell_2L_B+\ell_3L_C+\ell_4L_D)+\sigma L_E)\|\lambda_1-\lambda_2\| \\
&+(\rho\ell_3\ell_C+\rho\ell_4\ell_D+\sigma\ell_E)\|y-\bar{y}\| \\
=\ & \Lambda_1\|\lambda_1-\lambda_2\|+\Lambda_2\|y-\bar{y}\|,
\end{aligned}
$$

where

$$\Lambda_1=2L_G+2L_m+\xi+L_5+\rho(\ell_1L_A+\ell_2L_B+\ell_3L_C+\ell_4L_D)+\sigma L_E$$

and

$$\Lambda_2=\rho\ell_3\ell_C+\rho\ell_4\ell_D+\sigma\ell_E.$$

By (3.24) and (3.25), we have

(3.28)

$$
\begin{aligned}
&\|y-\bar{y}\| \\
&=\|J_\mu^{M_2(\cdot,s,\lambda_1)}[x+w_2-\mu N_2(f,q,\lambda_1)]-J_\mu^{M_2(\cdot,\bar{s},\lambda_2)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)]\| \\
&\le\|J_\mu^{M_2(\cdot,s,\lambda_1)}[x+w_2-\mu N_2(f,q,\lambda_1)]-J_\mu^{M_2(\cdot,s,\lambda_1)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)]\| \\
&\quad+\|J_\mu^{M_2(\cdot,s,\lambda_1)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)]-J_\mu^{M_2(\cdot,\bar{s},\lambda_1)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)]\| \\
&\quad+\|J_\mu^{M_2(\cdot,\bar{s},\lambda_1)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)]-J_\mu^{M_2(\cdot,\bar{s},\lambda_2)}[x+w_2-\mu N_2(\bar{f},\bar{q},\lambda_2)] \\
&\le\mu\|N_2(f,q,\lambda_1)-N_2(\bar{f},\bar{q},\lambda_2)\|+\eta\|s-\bar{s}\|+\theta\|\lambda_1-\lambda_2\| \\
&\le\mu(k_1\|f-\bar{f}\|+k_2\|q-\bar{q}\|+L_3\|\lambda_1-\lambda_2\|)+\eta\|s-\bar{s}\|+\theta\|\lambda_1-\lambda_2\| \\
&\le(\mu L_3+\theta)\|\lambda_1-\lambda_2\|+\mu k_1\tilde{H}(F(x,\lambda_1),F(x,\lambda_2)) \\
&\quad+\mu k_2\tilde{H}(Q(x,\lambda_1),Q(x,\lambda_2))+\eta\tilde{H}(S(x,\lambda_1),S(x,\lambda_2)) \\
&\le[\mu(L_3+k_1L_F+k_2L_Q)+\theta+\eta L_S]\|\lambda_1-\lambda_2\|\le\Lambda_3\|\lambda_1-\lambda_2\|,
\end{aligned}
$$

where $\Lambda_3=\mu(L_3+k_1L_F+k_2L_Q)+\theta+\eta L_S$.
From (3.27) and (3.28) it follows that

$$\|z-w\| \le (\Lambda_1+\Lambda_2\Lambda_3)\|\lambda_1-\lambda_2\|.$$

Hence we obtain

$$\sup_{z\in T(x,\lambda_1)} d(z, T(x,\lambda_2)) \le (\Lambda_1+\Lambda_2\Lambda_3)\|\lambda_1-\lambda_2\|.$$

By using similar argument as above, we can obtain

$$\sup_{w\in T(x,\lambda_2)} d(T(x,\lambda_1), w) \le (\Lambda_1+\Lambda_2\Lambda_3)\|\lambda_1-\lambda_2\|.$$

It follows that

$$\tilde{H}(T(x,\lambda_1)), T(x,\lambda_2)) \le (\Lambda_1+\Lambda_2\Lambda_3)\|\lambda_1-\lambda_2\|.$$

By (3.22), we obtain

$$\tilde{H}(F_1(\lambda_1)), F_2(\lambda_2)) \le \frac{1}{1-\Theta}(\Lambda_1+\Lambda_2\Lambda_3)\|\lambda_1-\lambda_2\|.$$

On the other hand, by the definition of the set-valued mapping $R(x,\lambda)$ and similar argument as in the proof of (3.28), it is easy to show that

$$\tilde{H}(R(x,\lambda_1), R(x,\lambda_2)) \le \Lambda_3\|\lambda_1-\lambda_2\|.$$

This proves that the solution set $Sol(\lambda)$ of the SPMQVI (2.1) is Lipschitz continuous with respect to the parameter $\lambda \in \Omega$. □

Remark 3.1. If each mappings in Theorem 3.2 is assumed to be continuous with respect to the parameter $\lambda \in \Omega$, then by similar argument as above, we can show that the solution set $Sol(\lambda)$ of SPMQVI (2.1) is also continuous with respect to the parameter $\lambda \in \Omega$.

4. Conclusion

In this paper, we introduce a new system of parametric mixed quasi-variational inclusions (SPMQVI) (2.1) in Hilbert spaces. The existence and closeness of solution set for the SPMQVI (2.1) is proved under suitable conditions. The Lipschitz continuity of the solution set mapping with respect to the parameter is also established under suitable conditions.

References

[1] S. Adly, *Perturbed algorithms and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl. **201** (1996), 609–630.

[2] R. P. Agarwal, Y. J. Cho, and N. J. Huang, *Sensitivity analysis for strongly nonlinear quasi-variational inclusions*, Appl. Math. Lett. **13** (2000), 19–24. 2000.

[3] R. P. Agarwal, N. J. Huang, M. Y. Tan, *Sensitivity analysis for a new system of generalized nonlinear mixed quasi-variational inclusions*, Appl. Math. Lett. **17** (2004), 345–352.

[4] S. Dafermos, *Sensitivity analysis in variational inequalities*, Math. Oper. Res. **13** (1988), 421–434.

[5] X. P. Ding and C. L. Lou, *On parametric generalized quasivariational inequalities*, J. Optim. Theory Appl. **100** (1999), 195–205.

[6] X. P. Ding, *Sensitivity analysis for generalized nonlinear implicit quasi-variational inclusions*, Appl. Mathe. Lett. **17** (2004), 225–235.

[7] X. P. Ding, *Sensitivity analysis of solution set for a new class of generalized implicit quasi-variational inclusions*, Fixed Point Theory and Applications, Vol. 6, Nova Sci. Pub. Inc., New York, 2004.

[8] X. P. Ding, *Parametric completely generalized nonlinear implicit quasi-variational inclusions involving h-maximal monotone mappings*, J. Comput. Appl. Math. **182** (2005), 252–269.
[9] T. C. Lim, *On fixed point stability for set-valued contractive mappings with application to generalized differential equations*, J. Mathe. Anal. Appl. **110** (1985), 436–441.
[10] Z. Liu, L. Debnath, S. M. Kang, and J. S. Ume, *Sensitivity analysis for parametric completely generalized nonlinear implicit quasivariational inclusions*, J. Math. Anal. Appl. **277** (2003), 142–154.
[11] R. N. Mukherjee and H. L. Verma, *Sensitivity analysis of generalized variational inequalities*, J. Math. Anal. Appl. **167** (1992), 299–304.
[12] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–485.
[13] M. A. Noor, *Sensitivity analysis for quasi-variational inequalities*, J. Optim. Theory Appl. **95** (1997), 399–407.
[14] M. A. Noor and K. I. Noor, *Sensitivity analysis for quasi-variational inclusions*, J. Math. Anal. Appl. **236** (1999), 290–299.
[15] J. Y. Park and J. U. Jeong, *Parametric generalized mixed variational inequalities*, Appl. Math. Lett. **17** (2004), 43–48.
[16] D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, Sijthoff & Noordhoff Inter. Pub. Romania, 1978.
[17] S. M. Robinson, *Sensitivity analysis of variational inequalities by normal-map techniques*, Variational inequalities and network equilibrium problems (Erice, 1994), Plenum, New York, 1995, pp. 257–269.
[18] Salahuddin, *Parametric generalized set-valued variational inclusions and resolvent equations*, J. Matt. Anal. Appl. **198** (2004), 146–156.
[19] N. D. Yen, *Lipschitz continuity of solutions of variational inequalities with a parametric polyhedral constraint*, Math. Oper. Res. **20** (1995), 695–708.

Manuscript received September 30, 2006
revised February 13, 2007

X. P. Ding
College of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, P.R. China.
E-mail address: xieping_ding@hotmail.com

J. C. Yao
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsinug, Taiwan, 80424, R.O.C.
E-mail address: yaojc@math.nsysu.edu.tw