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GENERALIZED NONEXPANSIVE RETRACTIONS AND A PROXIMAL-TYPE ALGORITHM IN BANACH SPACES

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Abstract. We study the relation between generalized nonexpansive retractions in Banach spaces and generalized projections in dual Banach spaces. Using our results, we deal with a proximal-type algorithm in Banach spaces.

1. Introduction

Let H be a (real) Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then we study the problem of finding a solution z to the equation

$$0 \in Az. \tag{1.1}$$

We identify the mapping A and its graph $G(A) = \{(x, y) : y \in Ax\}$. This problem is connected with convex minimization problems, variational inequality problems and minimax problems.

A well-known method for approximating a solution to (1.1) is the *proximal point algorithm* first introduced by Martinet [20]. This is an iterative procedure, which generates a sequence $\{x_n\}$ iteratively by $x_1 = x \in H$ and

$$x_n \in x_{n+1} + r_n A x_{n+1} \quad (n = 1, 2, \dots)$$

or equivalently $x_{n+1} = J_{r_n} x_n \quad (n = 1, 2, \dots)$, where $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rA)^{-1}$ is the resolvent of A for all $r > 0$.

In 1976, Rockafellar [29] proved that if the solution set $A^{-1}0$ is nonempty and $\liminf_n r_n > 0$, then $\{x_n\}$ is weakly convergent to an element of $A^{-1}0$; see also Brézis and Lions [2] and Lions [18]. It was shown by Güler [8] that the sequence $\{x_n\}$ generated by this algorithm does not converge strongly in general. In 2000, motivated by Mann's type iteration [19, 24] and Halpern's type iteration [9, 30] for nonexpansive mappings, Kamimura and Takahashi [14] modified the proximal point algorithm and obtained weak and strong convergence theorems for maximal monotone operators in Hilbert spaces. Solodov and Svaiter [31] also obtained a modification of the proximal point algorithm with *metric projections*. Solodov and Svaiter's strong convergence theorem is stated as follows:

Theorem 1.1 (Solodov & Svaiter [31]). *Let H be a Hilbert space and let $A \subset H \times H$ be a maximal monotone operator such that $A^{-1}0$ is nonempty. Let $J_r = (I + rA)^{-1}$*

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for all $r > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and

$$\begin{cases} y_n = J_{r_n} x_n; \\ H_n = \{z \in H : \langle z - y_n, x_n - y_n \rangle \leq 0\}; \\ W_n = \{z \in H : \langle z - x_n, x - x_n \rangle \leq 0\}; \\ x_{n+1} = P_{H_n \cap W_n}(x) \qquad\qquad (n = 1, 2, \dots), \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_n r_n > 0$ and $P_{H_n \cap W_n}$ denotes the metric projection from H onto $H_n \cap W_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P(x)$, where P denotes the metric projection from H onto $A^{-1}0$.

Later, Ohsawa and Takahashi [23] obtained a generalization of Theorem 1.1 to maximal monotone operators in Banach spaces with *metric projections*. Kamimura and Takahashi [15] obtained another generalization of Theorem 1.1 with *generalized projections*, which is a generalization of metric projections in Hilbert spaces to Banach spaces. See also [3, 4, 5, 13, 16, 17, 25, 32, 33] for proximal-type algorithms for maximal monotone operators in Banach spaces.

The purpose of the present paper is to obtain a generalization of Theorem 1.1 to maximal monotone operators defined in a dual Banach space with *sunny generalized nonexpansive retractions* recently introduced by Ibaraki and Takahashi [10, 11, 12] (Theorem 4.1). Before proving it, we study the relation between sunny generalized nonexpansive retractions in a Banach space and generalized projections in its dual space (Theorem 3.3). Finally, we deal with the problem of finding a minimizer of a proper lower semicontinuous convex function in a dual Banach space (Corollary 5.1).

2. Preliminaries

Throughout the present paper, all linear spaces are real. Let E be a Banach space and let E^* denote the dual space of E. The value of $x^* \in E^*$ at a point $x \in E$ is denoted by $\langle x, x^* \rangle$. The sets of all real numbers and all positive integers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. For a sequence $\{x_n\}$ of E, the strong convergence and the weak convergence of x_n to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $S(E)$ and $B(E)$ denote the unit sphere and the closed unit ball centered at the origin of E, respectively. A Banach space E is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$ imply $\|(x + y)/2\| \leq 1 - \delta$. It is known that every uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. It is said to be *uniformly smooth* if the limit (2.1) converges uniformly for $x, y \in S(E)$. The space E is said to have a *uniformly Gâteaux differentiable norm* if for all $y \in S(E)$, the limit (2.1) converges uniformly for $x \in S(E)$. It is known that E is uniformly convex if and only if E^* is uniformly smooth.

Let J be the *duality mapping* from E into E^* defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. We know that E is smooth if and only if J is single-valued. We also know that if E is smooth, strictly convex and reflexive, then J is single-valued, one-to-one and onto. In this case, the duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$. See [6, 7, 34, 35] for geometry of Banach spaces.

Let E be a smooth Banach space. Following Alber [1] and Kamimura and Takahashi [15], we denote by $\phi : E \times E \to [0, \infty)$ the mapping defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $(x, y) \in E \times E$. Let $\phi_* : E^* \times E^* \to [0, \infty)$ be the mapping defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^* \rangle + \|y^*\|^2$$

for all $(x^*, y^*) \in E^* \times E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.3}$$

for all $x, y \in E$. It is also easy to see that if E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \iff x = y. \tag{2.4}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for all $x \in E$, there exists a unique $z \in C$ (denoted by $\Pi_C x$) such that $\phi(z, x) = \min_{y \in C} \phi(y, x)$. The mapping Π_C is called the *generalized projection* from E onto C. We know the following lemmas:

Lemma 2.1 ([1, 15])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then the following hold:*

(a) *$z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$;*
(b) *$\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(z, x)$.*

Lemma 2.2 ([15])**.** *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. Then* $\lim_n \phi(x_n, y_n) = 0$ *implies that* $\lim_n \|x_n - y_n\| = 0$.

Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then T is said to be *generalized nonexpansive* ([10, 11, 12]) if $F(T)$ is nonempty and $\phi(Tx, u) \leq \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty closed subset of E and let R be a mapping from E onto C. Then R is said to be a *retraction* if $R^2 = R$. It is known that if R is a retraction from E onto C, then $F(R) = C$. The mapping R is also said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \geq 0$. A nonempty closed subset C of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) ([10, 11, 12]) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. We need the following lemmas:

Lemma 2.3 ([12]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then the following are equivalent:*

(a) *R is sunny and generalized nonexpansive;*
(b) *$\langle x - Rx, Jy - JRx\rangle \leq 0$ for all $(x, y) \in E \times C$.*

Lemma 2.4 ([12]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([12]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;*
(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a set-valued mapping with range $R(A) = \{x^* : x^* \in Ax\}$ and domain $D(A) = \{x \in E : Ax \neq \emptyset\}$. Then the mapping A is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* if A is monotone and there is no monotone operator from E into E^* whose graph properly contains the graph of A. It is known that if $A \subset E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex. We know the following theorem:

Theorem 2.6 ([28]). *Let E be a smooth, strictly convex and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.*

By Theorem 2.6, if E is smooth, strictly convex and reflexive and $A \subset E^* \times E$ $(= E^* \times E^{**})$ is a maximal monotone operator, then $R(J^{-1} + rA) = E$ for all $r > 0$. Thus, if $r > 0$ and $x \in E$, then there exists $z \in E$ such that

$$J^{-1}(Jx) \in J^{-1}(Jz) + rA(Jz)$$

or equivalently $x \in z + rAJz$. It follows from the strict convexity of E and E^* that such a point z is unique. Thus we can define the *resolvent* of A by $P_r x = z$, that is, $P_r = (I + rAJ)^{-1}$. The *Yosida approximation* of A is also defined by $A_r = (I - P_r)/r$. We know that $(JP_r x, A_r x) \in A$ for all $x \in E$; see Ibaraki and Takahashi [12] for more details.

3. Results on generalized nonexpansive retracts

Using the techniques developed by Matsushita and Takahashi [21], we prove the following lemma:

Lemma 3.1. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed generalized nonexpansive retract of E. Then JC is closed and convex.*

Proof. Let R be a generalized nonexpansive retraction from E onto C. Since R is a retraction from E onto C, we have $F(R) = C$. We first show that JC is convex. Let x^* and y^* be elements of JC, let $\alpha \in (0,1)$ and put $\beta = 1-\alpha$. Then we have $x, y \in C$ such that $x^* = Jx$ and $y^* = Jy$. Then we have

$$\begin{aligned}
&\phi(RJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) \\
&= \|RJ^{-1}(\alpha Jx + \beta Jy)\|^2 - 2\langle RJ^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&\quad + \|J^{-1}(\alpha Jx + \beta Jy)\|^2 + \alpha\|x\|^2 + \beta\|y\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= \alpha\phi(RJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(RJ^{-1}(\alpha Jx + \beta Jy), y) \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2)
\end{aligned} \tag{3.1}$$

Since $x, y \in C = F(R)$ and R is generalized nonexpansive, we have

$$\begin{aligned}
&\alpha\phi(RJ^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(RJ^{-1}(\alpha Jx + \beta Jy), y) \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&\le \alpha\phi(J^{-1}(\alpha Jx + \beta Jy), x) + \beta\phi(J^{-1}(\alpha Jx + \beta Jy), y) \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= \alpha\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jx\rangle + \|x\|^2\} \\
&\quad + \beta\{\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), Jy\rangle + \|y\|^2\} \\
&\quad + \|\alpha Jx + \beta Jy\|^2 - (\alpha\|x\|^2 + \beta\|y\|^2) \\
&= 2\|\alpha Jx + \beta Jy\|^2 - 2\langle J^{-1}(\alpha Jx + \beta Jy), \alpha Jx + \beta Jy\rangle \\
&= 2\|\alpha Jx + \beta Jy\|^2 - 2\|\alpha Jx + \beta Jy\|^2 = 0.
\end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we have $\phi(RJ^{-1}(\alpha Jx + \beta Jy), J^{-1}(\alpha Jx + \beta Jy)) = 0$. By (2.4), we have

$$RJ^{-1}(\alpha Jx + \beta Jy) = J^{-1}(\alpha Jx + \beta Jy).$$

Hence we obtain $J^{-1}(\alpha Jx + \beta Jy) \in C$, that is, $\alpha x^* + \beta y^* = \alpha Jx + \beta Jy \in JC$. This shows that JC is convex.

We next show that JC is closed. Let $\{x_n^*\}$ be a sequence of JC converging strongly to $x^* \in E^*$. Then we have $x \in E$ and $x_n \in C$ such that $x^* = Jx$ and $x_n^* = Jx_n$ for all $n \in \mathbb{N}$. By $x_n \in C$, we have

$$\begin{aligned}
\phi(Rx, x_n) &\le \phi(x, x_n) \\
&= \|x\|^2 - 2\langle x, x_n^*\rangle + \|x_n^*\|^2 \\
&\to \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 = \phi(x, x) = 0
\end{aligned}$$

as $n \to \infty$. Hence we have $\lim_n \phi(Rx, x_n) = 0$. On the other hand,

$$\begin{aligned}
\phi(Rx, x_n) &= \|Rx\|^2 - 2\langle Rx, Jx_n\rangle + \|x_n\|^2 \\
&= \|Rx\|^2 - 2\langle Rx, x_n^*\rangle + \|x_n^*\|^2
\end{aligned}$$

$$\to \|Rx\|^2 - 2\langle Rx, x^*\rangle + \|x^*\|^2 = \phi(Rx, x)$$

as $n \to \infty$. Thus we have $\phi(Rx, x) = 0$. Then it follows from (2.4) that $Rx = x$. This gives us that $x^* = Jx = JRx \in JC$. Thus JC is closed. This completes the proof. □

Lemma 3.2. *Let E be a smooth, strictly convex and reflexive Banach space, let C_* be a nonempty closed convex subset of E^* and let Π_{C_*} be the generalized projection from E^* onto C_*. Then the mapping R defined by $R = J^{-1}\Pi_{C_*}J$ is a sunny generalized nonexpansive retraction from E onto $J^{-1}C_*$.*

Proof. We first show that $J^{-1}C_*$ is closed. Let $\{x_n\}$ be a sequence of $J^{-1}C_*$ converging strongly to $x \in E$. Then we have $Jx_n \in C_*$. Since E is smooth and reflexive, the duality mapping J is norm-to-weak continuous from E into E^*; see Takahashi [34, 35]. This implies that $Jx_n \rightharpoonup Jx$. Since C_* is closed and convex, it is also weakly closed. Hence we have $Jx \in C_*$, that is, $x \in J^{-1}C_*$. Thus $J^{-1}C_*$ is closed.

It is obvious that R is a mapping from E into $J^{-1}C_*$. We next show that R is a retraction from E onto $J^{-1}C_*$. If $x \in J^{-1}C_*$, then we have $Jx \in C_*$ and hence $\Pi_{C_*}Jx = Jx$. This implies that

$$Rx = J^{-1}\Pi_{C_*}Jx = J^{-1}Jx = x.$$

Thus R is onto and $Rx = x$ for all $x \in J^{-1}C_*$. It also holds that

$$R^2y = R(Ry) = Ry$$

for all $y \in E$. This shows that R is a retraction.

We finally show that R is sunny and generalized nonexpansive. Since R is a retraction from E onto $J^{-1}C_*$, we have $F(R) = J^{-1}C_*$. Thus $F(R)$ is nonempty. On the other hand, by Lemma 2.1, we have

$$\phi_*(y^*, \Pi_{C_*}x^*) + \phi_*(\Pi_{C_*}x^*, x^*) \le \phi_*(y^*, x^*)$$

for all $(x^*, y^*) \in E^* \times C_*$, which is equivalent to

$$\phi_*(Jy, \Pi_{C_*}Jx) + \phi_*(\Pi_{C_*}Jx, Jx) \le \phi_*(Jy, Jx) \tag{3.3}$$

for all $(x, y) \in E \times J^{-1}C_*$. By (2.3) and (3.3), we have

$$\phi(J^{-1}\Pi_{C_*}Jx, J^{-1}Jy) + \phi(J^{-1}Jx, J^{-1}\Pi_{C_*}Jx) \le \phi(J^{-1}Jx, J^{-1}Jy)$$

for all $(x, y) \in E \times J^{-1}C_*$. Thus we obtain

$$\phi(Rx, y) + \phi(x, Rx) \le \phi(x, y)$$

for all $(x, y) \in E \times J^{-1}C_*$. If $(x, y) \in E \times J^{-1}C_*$, then it follows from the last inequality that

$$\begin{aligned}
0 &\le \phi(x, y) - \{\phi(Rx, y) + \phi(x, Rx)\} \\
&= \{\|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2\} - \{\|Rx\|^2 - 2\langle Rx, Jy\rangle + \|y\|^2 \\
&\quad + \|x\|^2 - 2\langle x, JRx\rangle + \|Rx\|^2\} \\
&= 2\{\langle Rx, Jy\rangle + \langle x, JRx\rangle - \langle x, Jy\rangle - \langle Rx, JRx\rangle\} \\
&= 2\langle x - Rx, JRx - Jy\rangle.
\end{aligned}$$

Thus we have $\langle x - Rx, Jy - JRx\rangle \le 0$ for all $(x, y) \in E \times J^{-1}C_*$. By Lemma 2.3, R is sunny and generalized nonexpansive. Therefore R is a sunny generalized nonexpansive retraction from E onto $J^{-1}C_*$. This completes the proof. $\square$

Using Lemmas 3.1 and 3.2, we have the following theorem:

Theorem 3.3. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

In this case, the unique sunny generalized nonexpansive retraction from E onto C is given by $J^{-1}\Pi_{JC}J$, where Π_{JC} is the generalized projection from E^* onto JC.

Proof. It is obvious that (a) implies (b). By Lemma 3.1, (b) implies (c). Lemma 3.2 ensures that if JC is closed and convex, then $R = J^{-1}\Pi_{JC}J$ is a sunny generalized nonexpansive retraction from E onto $C = J^{-1}JC$. Thus (c) implies (a). This completes the proof. $\square$

We can also show the following proposition:

Proposition 3.4. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(a) *$z = Rx$;*
(b) *$\phi(x, z) = \min_{y\in C} \phi(x, y)$.*

Proof. By Theorem 3.3, we have the following equality: $R = J^{-1}\Pi_{JC}J$. So, using (2.3), we have

$$\begin{aligned} z = Rx &\Longleftrightarrow Jz = \Pi_{JC}Jx \\ &\Longleftrightarrow \phi_*(Jz, Jx) = \min_{y^*\in JC} \phi_*(y^*, Jx) \\ &\Longleftrightarrow \phi_*(Jz, Jx) = \min_{y\in C} \phi_*(Jy, Jx) \\ &\Longleftrightarrow \phi(x, z) = \min_{y\in C} \phi(x, y). \end{aligned}$$

This completes the proof. $\square$

4. Strong convergence theorem

Now, we are ready to prove our main result in this paper, which generalizes Solodov and Svaiter's theorem (Theorem 1.1) in Hilbert spaces to that in Banach spaces.

Theorem 4.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $A \subset E^* \times E$ be a maximal monotone operator such that*

$A^{-1}0$ is nonempty. Let $P_r = (I + rAJ)^{-1}$ for all $r > 0$ and let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and

$$\begin{cases} y_n = P_{r_n} x_n; \\ H_n = \{z \in E : \langle x_n - y_n, Jz - Jy_n \rangle \leq 0\}; \\ W_n = \{z \in E : \langle x - x_n, Jz - Jx_n \rangle \leq 0\}; \\ x_{n+1} = R_{H_n \cap W_n}(x) \qquad\qquad\qquad (n = 1, 2, \dots), \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_n r_n > 0$ and $R_{H_n \cap W_n}$ denotes the sunny generalized nonexpansive retraction from E onto $H_n \cap W_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $R(x)$, where R denotes the sunny generalized nonexpansive retraction from E onto $J^{-1}A^{-1}0$.

Proof. It follows from the maximal monotonicity of A that $A^{-1}0$ is closed and convex and hence Theorem 3.3 ensures that $J^{-1}A^{-1}0$ is a sunny generalized nonexpansive retract of E. Since J is norm-to-weak* continuous, H_n and W_n are closed for all $n \in \mathbb{N}$. It should be noted that the surjectivity of J implies that

$$JH_n = \{z^* \in E^* : \langle x_n - y_n, z^* - Jy_n \rangle \leq 0\}$$

and

$$JW_n = \{z^* \in E^* : \langle x - x_n, z^* - Jx_n \rangle \leq 0\}$$

for all $n \in \mathbb{N}$. It also follows from the injectivity of J that

$$J(H_n \cap W_n) = JH_n \cap JW_n$$

for all $n \in \mathbb{N}$. Thus JH_n, JW_n and $J(H_n \cap W_n)$ are closed and convex for all $n \in \mathbb{N}$. So, if we can show that $H_n \cap W_n$ is nonempty, then Theorem 3.3 ensures that H_n, W_n and $H_n \cap W_n$ are sunny generalized nonexpansive retracts of E for all $n \in \mathbb{N}$.

We first show that $J^{-1}A^{-1}0 \subset H_n \cap W_n$ for all $n \in \mathbb{N}$ by induction. It is obvious that $W_1 = E$. Let $u \in J^{-1}A^{-1}0$ be given. Then it follows from

$$A_{r_1} x_1 = \frac{x_1 - y_1}{r_1} \in AJy_1$$

and $0 \in AJu$ that

$$\langle x_1 - y_1, Jy_1 - Ju \rangle = r_1 \langle A_{r_1} x_1 - 0, Jy_1 - Ju \rangle \geq 0.$$

So, we have $u \in H_1$ and hence $u \in H_1 \cap W_1$. Hence $J^{-1}A^{-1}0 \subset H_1 \cap W_1$. This implies that $H_1 \cap W_1$ is nonempty. By Theorem 3.3, $H_1 \cap W_1$ is a sunny generalized nonexpansive retract of E. Thus we can define $x_2 = R_{H_1 \cap W_1}(x)$ and $y_2 = P_{r_2} x_2$. Suppose that for some $m \in \mathbb{N}$, $J^{-1}A^{-1}0 \subset H_k \cap W_k$ for all $k = 1, 2, \dots, m$. Then x_k and y_k are well-defined for all $k = 1, 2, \dots, m+1$. If $u \in J^{-1}A^{-1}0$, then we can show that $u \in H_{m+1}$ as in the proof of $u \in H_1$. By $u \in J^{-1}A^{-1}0 \subset H_m \cap W_m$ and $x_{m+1} = R_{H_m \cap W_m}(x)$, it follows from Lemma 2.5 that

$$\langle x - x_{m+1}, Ju - Jx_{m+1} \rangle \leq 0,$$

which implies that $u \in W_{m+1}$. Hence $u \in H_{m+1} \cap W_{m+1}$. So, we have $J^{-1}A^{-1}0 \subset H_{m+1} \cap W_{m+1}$. Thus we obtain

$$J^{-1}A^{-1}0 \subset H_n \cap W_n$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined.

We next prove that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.1}$$

Fix $n \in \mathbb{N}$. Note that $x_n \in W_n$ and

$$\langle x - x_n, Jz - Jx_n \rangle \le 0$$

for all $z \in W_n$. So, by Lemma 2.5, we have $x_n = R_{W_n} x$. Thus, using Lemma 2.5, it follows from $x_{n+1} \in W_n$ that

$$\begin{aligned} \phi(x_n, x_{n+1}) &= \phi(R_{W_n} x, x_{n+1}) \\ &\le \phi(x, x_{n+1}) - \phi(x, R_{W_n} x) \\ &= \phi(x, x_{n+1}) - \phi(x, x_n), \end{aligned} \tag{4.2}$$

which implies that $\phi(x, x_n) \le \phi(x, x_{n+1})$. Using Proposition 3.4, we also have

$$\phi(x, x_{n+1}) = \phi(x, R_{H_n \cap W_n} x) \le \phi(x, Rx) \tag{4.3}$$

because $Rx \in J^{-1}A^{-1}0 \subset H_n \cap W_n$. Hence the limit $\lim_n \phi(x, x_n)$ exists. It also follows from $(\|x\| - \|x_n\|)^2 \le \phi(x, x_n)$ that $\{x_n\}$ is bounded. By the existence of $\lim_n \phi(x, x_n)$ and (4.2), we have

$$\phi(x_n, x_{n+1}) \le \phi(x, x_{n+1}) - \phi(x, x_n) \to 0$$

as $n \to \infty$. This implies that $\lim_n \phi(x_n, x_{n+1}) = 0$. Since E is uniformly convex, Lemma 2.2 ensures that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

On the other hand, since $y_n = R_{H_n} x_n$ and $x_{n+1} \in H_n$, it follows from Lemma 2.5 that

$$\phi(y_n, x_{n+1}) = \phi(R_{H_n} x_n, x_{n+1}) \le \phi(x_n, x_{n+1}) \to 0$$

as $n \to \infty$. Using Lemma 2.2, we have

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

Therefore we obtain $\lim_n \|x_n - y_n\| = 0$.

We next show that $Jx_n \rightharpoonup JRx$. Let $\{Jx_{n_i}\}$ be any subsequence of $\{Jx_n\}$ converging weakly to an element z^* of E^*. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping J is uniformly norm-to-weak* continuous on each bounded subset of E; see Takahashi [34, 35] for more details. Thus it follows from (4.1) that

$$\lim_{n\to\infty} \langle p, Jx_n - Jy_n \rangle = 0$$

for all $p \in E$. This implies that $Jy_{n_i} \rightharpoonup z^*$. Since $\liminf_n r_n > 0$, we also know that

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{\|x_n - y_n\|}{r_n} = 0.$$

If $(w^*, w) \in A$, then it follows from the monotonicity of A that

$$\langle w - A_{r_n} x_n, w^* - Jy_n \rangle \ge 0$$

for all $n \in \mathbb{N}$. Letting $n_i \to \infty$ in the last inequality, we obtain

$$\langle w, w^* - z^* \rangle \ge 0.$$

By the maximality of A, we have $z^* \in A^{-1}0$. Putting $z = J^{-1}z^*$, by Proposition 3.4, we see that

$$\phi(x, Rx) \le \phi(x, z). \tag{4.4}$$

Since $Jx_{n_i} \rightharpoonup Jz$ and the norm square of E^* is weakly lower semicontinuous, we have

$$\begin{aligned} \phi(x, z) &= \|x\|^2 - 2\langle x, Jz\rangle + \|Jz\|^2 \\ &\le \liminf_{i\to\infty}\{\|x\|^2 - 2\langle x, Jx_{n_i}\rangle + \|Jx_{n_i}\|^2\} \\ &= \liminf_{i\to\infty} \phi(x, x_{n_i}) \\ &\le \limsup_{i\to\infty} \phi(x, x_{n_i}) \\ &\le \phi(x, Rx), \end{aligned} \tag{4.5}$$

where the last inequality follows from (4.3). By (4.4) and (4.5), we have $\phi(x, Rx) = \phi(x, z)$ and hence $Rx = z$. Thus we obtain $z^* = Jz = JRx$. Consequently, the whole sequence $\{Jx_n\}$ converges weakly to JRx.

We finally show that $x_n \to Rx$. By (2.2), we have

$$\phi(Rx, x_n) = \phi(Rx, x) + \phi(x, x_n) + 2\langle Rx - x, Jx - Jx_n\rangle \tag{4.6}$$

for all $n \in \mathbb{N}$. By (4.3) and (4.6), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \phi(Rx, x_n) &\le \limsup_{n\to\infty}\{\phi(Rx, x) + \phi(x, Rx) + 2\langle Rx - x, Jx - Jx_n\rangle\} \\ &= \phi(Rx, x) + \phi(x, Rx) + 2\langle Rx - x, Jx - JRx\rangle \\ &= \phi(Rx, Rx) = 0. \end{aligned}$$

Thus $\limsup_n \phi(Rx, x_n) = 0$. This gives us that $\lim_n \phi(Rx, x_n) = 0$. By Lemma 2.2, we obtain $\lim_n \|Rx - x_n\| = 0$. Therefore the sequence $\{x_n\}$ converges strongly to Rx. This completes the proof. $\square$

5. Application to a convex minimization problem

In this section, we deal with a convex minimization problem in dual Banach spaces. Let E be a Banach space and let $f : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. By Rockafellar's theorem [26, 27], the subdifferential $\partial f \subset E^* \times E$ defined by

$$\partial f(x^*) = \{x \in E : f(x^*) + \langle x, y^* - x^*\rangle \le f(y^*) \quad (\forall y^* \in E^*)\}$$

for all $x^* \in E^*$ is maximal monotone. It is well-known that

$$0 \in \partial f(u^*) \iff f(u^*) = \min_{y^* \in E^*} f(y^*).$$

Let P_r be the resolvent of ∂f, that is, $P_r = (I + r\partial f J)^{-1}$ $(r > 0)$. Then we can show that

$$\begin{aligned} z = P_r x &\iff x \in z + r\partial f(Jz) \\ &\iff 0 \in \frac{1}{r}(J^{-1}(Jz) - J^{-1}(Jx)) + \partial f(Jz) \end{aligned}$$

$$
\begin{aligned}
&\iff 0 \in \partial\Big(\frac{1}{2r}\|\cdot\|^2 - \frac{1}{r}\langle x, \cdot\rangle + f\Big)(Jz)\\
&\iff Jz = \arg\min_{y^*\in E^*}\Big\{f(y^*) + \frac{1}{2r}\|y^*\|^2 - \frac{1}{r}\langle x, y^*\rangle\Big\}\\
&\iff z = J^{-1}\Big(\arg\min_{y^*\in E^*}\Big\{f(y^*) + \frac{1}{2r}\|y^*\|^2 - \frac{1}{r}\langle x, y^*\rangle\Big\}\Big).
\end{aligned}
$$

Thus, using Theorem 4.1, we have the following corollary:

Corollary 5.1. *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let $f : E^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$
\begin{cases}
y_n = J^{-1}\Big(\arg\min_{y^*\in E^*}\Big\{f(y^*) + \frac{1}{2r_n}\|y^*\|^2 - \frac{1}{r_n}\langle x_n, y^*\rangle\Big\}\Big);\\
H_n = \{z \in E : \langle x_n - y_n, Jz - Jy_n\rangle \le 0\};\\
W_n = \{z \in E : \langle x - x_n, Jz - Jx_n\rangle \le 0\};\\
x_{n+1} = R_{H_n\cap W_n}(x) \qquad\qquad (n = 1, 2, \dots),
\end{cases}
$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_n r_n > 0$ and $R_{H_n\cap W_n}$ denotes the sunny generalized nonexpansive retraction from E onto $H_n \cap W_n$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $R(x)$, where R denotes the sunny generalized nonexpansive retraction from E onto $J^{-1}(\partial f)^{-1}(0)$.

References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type (A. G. Kartsatos Ed.), Lecture Notes in Pure and Appl. Math., **178**, Dekker, New York, 1996, pp.15–50.

[2] H. Brézis and P. L. Lions, *Produits infinis de résolvantes*, Israel J. Math. **29** (1978), 329–345.

[3] R. S. Burachik and S. Scheimberg, *A proximal point method for the variational inequality problem in Banach spaces*, SIAM J. Control. Optim. **39** (2001), 1633–1649.

[4] D. Butnariu and A. N. Iusem, *On a proximal point method for convex optimization in Banach spaces*, Numer. Funct. Anal. Optim. **18** (1997), 723–744.

[5] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers Group, Dordrecht (2000).

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers Group, Dordrecht (1990).

[7] J. Diestel, *Geometry of Banach Spaces -Selected Topics*, Lecture Notes in Mathematics, 485. Springer-Verlag, Berlin-New York (1975).

[8] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[10] T. Ibaraki and W. Takahashi, *Convergence theorems for new projections in Banach spaces*, RIMS Kokyuroku **1484** (2006), 150–160.

[11] T. Ibaraki and W. Takahashi, *Mosco convergence of sequences of retracts of four nonlinear projections in Banach spaces*, Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2007, pp.139–147.

[12] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory, to appear.

[13] S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.
[14] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.
[15] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.
[16] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. **2004** (2004), 239–249.
[17] F. Kohsaka and W. Takahashi, *Proximal point algorithms with Bregman functions in Banach spaces*, J. Nonlinear Convex Anal. **6** (2005), 505–523.
[18] P. L. Lions, *Une méthode itérative de résolution d'une inéquation variationnelle*, Israel J. Math. **31** (1978), 204–208.
[19] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[20] B. Martinet, *Régularisation d'inéuations variationnelles par approximations successives*, Rev. Francaise Informat. Recherche Opérationnelle **4** (1970), 154–158.
[21] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.
[22] J. J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France **93** (1965), 273–299.
[23] S. Ohsawa and W. Takahashi, *Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces*, Arch. Math. **81** (2003), 439–445.
[24] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.
[25] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type (A. G. Kartsatos Ed.), Lecture Notes in Pure and Appl. Math., **178**, Dekker, New York, 1996, pp.313–318.
[26] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510.
[27] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.
[28] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
[29] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.
[30] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), 3641–3645.
[31] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.
[32] W. Takahashi, *Fixed point theorems and proximal point algorithms*, Proceedings of the Second International Conference on Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2003, pp.471–481.
[33] W. Takahashi, *Convergence theorems and nonlinear projections in Banach spaces*, Proceedings of the International Symposium on Banach and Function Spaces (M. Kato and L. Maligranda Eds.), Yokohama Publ., Yokohama, 2004, pp.145–174.
[34] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama (2000) (Japanese).
[35] W. Takahashi, *Nonlinear Functional Analysis -Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama (2000).

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