Journal of Nonlinear and Convex Analysis
Volume 8, Number 2, 2007, 179–196

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

VARIATIONAL PRINCIPLES FOR VECTOR EQUILIBRIUM PROBLEMS RELATED TO CONJUGATE DUALITY

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Abstract. This paper deals with the characterization of solutions for vector equilibrium problems by means of conjugate duality. By using the Fenchel duality we establish variational principles, that is, optimization problems with set-valued objective functions, the solution sets of which contain the ones of the vector equilibrium problems. The set-valued objective mappings depend on the data, but not on the solution sets of the vector equilibrium problems. As a particular instance we obtain gap functions for the weak vector variational inequality problem.

1. Introduction

In analogy to the scalar case, vector equilibrium problems can be considered as being generalizations of vector variational inequalities, vector optimization and equilibrium problems (cf. [3]). In the past some results established for these special cases have been extended to vector equilibrium problems. By generalizing the similar concept from the scalar case (see [7]), gap functions for vector variational inequalities have been proposed for the first time in [9]. Moreover, by generalizing the similar concept from the scalar case (cf. [6] and [8]) so-called variational principles for vector equilibrium problems have been also given (see [4] and [5]). These are optimization problems with set-valued objective functions, the solution sets of which contain the ones of the vector equilibrium problems, being actually extensions of the concept of gap (merit) functions for vector variational inequalities.

Recently, in the scalar case, the construction of gap functions for variational inequalities and equilibrium problems has been associated to Lagrange duality ([10]) but also, more generally, to conjugate duality (see [1], [2], and [12]). On the other hand, a conjugate duality theory in vector optimization has been developed by Tanino and Sawaragi (see [14], [17] and [19]), by introducing some new concepts of conjugate maps and set-valued subgradients, based on Pareto efficiency or on some weak orderings.

In this paper we focus on the construction of set-valued mappings on the basis of the so-called Fenchel duality which allow us to propose new variational principles for vector equilibrium problems. The set-valued mappings depend on the data, but not on the solution sets of the vector equilibrium problems.

In Section 2 we introduce some notions and results regarding conjugate duality in vector optimization based on weak orderings. In Section 3, by using a special

2000 *Mathematics Subject Classification.* Primary 58E35, 49J53, 49N15.

Key words and phrases. Vector equilibrium problem, dual vector equilibrium problem, variational principle, weak vector variational inequality, Minty weak vector variational inequality, gap function, conjugate duality.

perturbation function, we introduce a Fenchel-type dual problem for a vector optimization problem with set constraints. In Section 4 we formulate some variational principles for vector equilibrium problems. The set-valued mappings we introduce here are formulated by means of the Fenchel-type dual and depends on the data, but not on the solution sets of the vector equilibrium problems. In the last section we obtain as a particular case gap functions for the weak vector variational inequalities.

2. Mathematical preliminaries

Let Y be a real topological vector space partially ordered by a pointed closed convex cone C with a nonempty interior int C in Y. For any $\xi, \mu \in Y$, we use the following ordering relations:

$$\xi \leq \mu \Leftrightarrow \mu - \xi \in C;$$
$$\xi < \mu \Leftrightarrow \mu - \xi \in \text{int } C;$$
$$\xi \not< \mu \Leftrightarrow \mu - \xi \notin \text{int } C.$$

The relations $\geq$, $>$ and $\not>$ are defined similarly. Next we introduce the notions of weak maximum and weak supremum of a set Z given in the space $\overline{Y}$, obtained as an extension of Y by adding two imaginary points $+\infty$ and $-\infty$, respectively. These two elements must fulfill $-\infty < y < +\infty$ for $y \in Y$. Further we make the following conventions

$$(\pm\infty) + (\pm\infty) = \pm\infty,$$
$$(\pm\infty) + y = y + (\pm\infty) = \pm\infty \text{ for all } y \in Y,$$
$$\lambda(\pm\infty) = \pm\infty \text{ for } \lambda > 0 \text{ and } \lambda(\pm\infty) = \mp\infty \text{ for } \lambda < 0.$$

The sum $+\infty + (-\infty)$ is not considered, since we can avoid it.

For a given set $Z \subseteq \overline{Y}$, we define the *set* $A(Z)$ *of all points above* Z and the *set* $B(Z)$ *of all points below* Z by

$$A(Z) = \Big\{ y \in \overline{Y} |\ y > y' \text{ for some } y' \in Z \Big\}$$

and

$$B(Z) = \Big\{ y \in \overline{Y} |\ y < y' \text{ for some } y' \in Z \Big\},$$

respectively. Clearly, $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 2.1. A point $\widehat{y} \in \overline{Y}$ is said to be a weak maximal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \in Z$ and $\widehat{y} \notin B(Z)$, that is, if $\widehat{y} \in Z$ and there is no $y' \in Z$ such that $\widehat{y} < y'$.

The set of all weak maximal points of Z is called the *weak maximum of* Z and is denoted by $\operatorname{WMax} Z$.

Definition 2.2. A point $\widehat{y} \in \overline{Y}$ is said to be a weak supremal point of $Z \subseteq \overline{Y}$ if $\widehat{y} \notin B(Z)$ and $B(\{\widehat{y}\}) \subseteq B(Z)$, that is, if there is no $y \in Z$ such that $\widehat{y} < y$ and if the relation $y' < \widehat{y}$ implies the existence of some $y \in Z$ such that $y' < y$.

The set of all weak supremal points of Z is called the *weak supremum* of Z and is denoted by $\operatorname{WSup} Z$. Remark that $\operatorname{WMax} Z = Z \cap \operatorname{WSup} Z$. Moreover it holds $-\operatorname{WMax}(-Z) = \operatorname{WMin} Z$ and $-\operatorname{WSup}(-Z) = \operatorname{WInf} Z$, where the *weak minimum* and the *weak infimum* are defined analogously to the weak maximum and weak supremum, respectively. More about these concepts can be found in the works [18] and [19].

In the following we define the *conjugate mapping* and *subgradient* of a set-valued mapping by using the notions of weak supremum and weak maximum of a set. Let X be another real topological vector space and let $\mathcal{L}(X, Y)$ be the space of all linear continuous operators from X to Y. For $x \in X$ and $l \in \mathcal{L}(X, Y)$, $\langle l, x\rangle$ denotes the value of l at x.

Definition 2.3 (see [19])**.** Let $G : X \rightrightarrows \overline{Y}$ be a set-valued mapping.

(i) The set-valued mapping $G^* : \mathcal{L}(X, Y) \rightrightarrows \overline{Y}$ defined by

$$G^*(T) = \text{WSup} \bigcup_{x\in X} \Big[\langle T, x\rangle - G(x)\Big], \text{ for } T \in \mathcal{L}(X, Y)$$

is called the conjugate mapping of G.

(ii) The set-valued mapping $G^{**} : X \rightrightarrows \overline{Y}$ defined by

$$G^{**}(x) = \text{WSup} \bigcup_{T\in\mathcal{L}(X,Y)} \Big[\langle T, x\rangle - G^*(T)\Big], \text{ for } x \in X$$

is called the biconjugate mapping of G.

(iii) $T \in \mathcal{L}(X, Y)$ is said to be a subgradient of the set-valued mapping G at $(x_0; y_0)$ if $y_0 \in G(x_0)$ and

$$\langle T, x_0\rangle - y_0 \in \text{WMax} \bigcup_{x\in X} \Big[\langle T, x\rangle - G(x)\Big].$$

The set of all subgradients of G at $(x_0; y_0)$ is called *the subdifferential of G at* $(x_0; y_0)$ and is denoted by $\partial G(x_0; y_0)$. If $\partial G(x_0; y_0) \neq \varnothing$ for every $y_0 \in G(x_0)$, then G is said to be *subdifferentiable at x_0*.

Next we recall some notions and results from the conjugate duality theory in vector optimization introduced and investigated in [19]. Let X and Y be real topological vector spaces. Assume that $\overline{Y}$ is the extended space of Y and h is a function from X to $Y \cup \{+\infty\}$. We consider the vector optimization problem

$$(P) \qquad \text{WInf}\{h(x)|\, x \in X\}.$$

Let U be another real topological vector space, the so-called perturbation space. Let $\Phi : X \times U \to Y \cup \{+\infty\}$ be the perturbation function, namely fulfilling

$$\Phi(x, 0) = h(x),\ \forall x \in X.$$

The perturbed problem of (P) is

$$(P_u) \qquad \text{WInf}\Big\{\Phi(x, u)|\ x \in X\Big\},$$

where $u \in U$ is the so-called perturbation variable.

Definition 2.4. The set-valued mapping $W : U \rightrightarrows Y$ defined by

$$W(u) = \text{WInf}(P_u) = \text{WInf}\Big\{\Phi(x, u)|\ x \in X\Big\}$$

is called the value mapping of (P).

It is clear that $\mathrm{WInf}(P) = W(0)$. The conjugate mapping of Φ is

$$\Phi^*(T,\Lambda) = \mathrm{WSup}\Big\{\langle T,x\rangle + \langle \Lambda,u\rangle - \Phi(x,u) \,|\, x \in X,\ u \in U\Big\}$$

for $T \in \mathcal{L}(X, Y)$ and $\Lambda \in \mathcal{L}(\mathrm{U},\mathrm{Y})$. Then

$$\begin{aligned} -\Phi^*(0,\Lambda) &= -\,\mathrm{WSup}\Big\{\langle \Lambda,u\rangle - \Phi(x,u) \,|\, x \in X,\ u \in U\Big\} \\ &= \mathrm{WInf}\Big\{\Phi(x,u) - \langle \Lambda,u\rangle \,|\, x \in X,\ u \in U\Big\}. \end{aligned}$$

A dual problem to (P) can be defined as follows

$$(D) \qquad \mathrm{WSup} \bigcup_{\Lambda\in\mathcal{L}(U,Y)} \Big[-\Phi^*(0,\Lambda)\Big].$$

Since $\Lambda \mapsto -\Phi^*(0,\Lambda)$ is a set-valued mapping, the dual problem is not an usual vector optimization problem.

Proposition 2.1 ([19, Proposition 5.1] (Weak duality))**.** *For any $x \in X$ and $\Lambda \in \mathcal{L}(U,Y)$ it holds*

$$\Phi(x,0) \notin B\Big(-\Phi^*(0,\Lambda)\Big).$$

Definition 2.5 ([19, Definition 5.2])**.** The primal problem (P) is said to be stable if the value mapping W is subdifferentiable at 0.

Theorem 2.1 ([19, Theorem 5.1], [15, Theorem 3.1])**.** *If the problem (P) is stable, then*

$$\mathrm{WInf}(P) = \mathrm{WSup}(D) = \mathrm{WMax}(D).$$

Let us notice that some results on conjugate duality for set-valued vector optimization problems has been given by Song in [15]. Moreover, some stability criteria can be found in [15], [16] and [19].

3. Fenchel duality for vector optimization

In this section we specialize the theory described above and introduce, by using a special perturbation function, a Fenchel-type dual problem to the vector optimization problem with set constraints. Let the spaces X and Y be the same as in Section 2. Assume that h is a function from X to $Y \cup \{+\infty\}$ and $G \subseteq X$. We consider the vector optimization problem

$$(P_c) \qquad \mathrm{WInf}\{h(x) |\, x \in G\}.$$

We choose as perturbation space $U := X$ and consider the perturbation function $\Phi : X \times X \to Y \cup \{+\infty\}$ defined by

$$\Phi(x,u) = \begin{cases} h(x+u), & \text{if } x \in G; \\ +\infty, & \text{otherwise.} \end{cases}$$

For the proof of the next proposition we need the following obvious relations.

Remark 3.1. Let $g : X \to Y$ be a function and $Z \subseteq X$. The following assertions are true:

(i) For any $y \in Y$ it holds

$$\{g(x) + y|\ x \in Z\} = \{g(x)|\ x \in Z\} + y;$$

(ii) For any set $A \subseteq Y$ it holds

$$\bigcup_{x \in Z} [A + g(x)] = A + \bigcup_{x \in Z} \{g(x)\}.$$

Proposition 3.1. *Let $T \in \mathcal{L}(X, Y)$. Then*

$$\Phi^*(0, T) = \mathrm{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle|\ x \in G\}\Big\}.$$

Proof. Let $T \in \mathcal{L}(X, Y)$ be fixed. By definition

$$\begin{aligned}\Phi^*(0, T) &= \mathrm{WSup}\{\langle T, u\rangle - \Phi(x, u)|\ x \in X, u \in X\} \\ &= \mathrm{WSup}\{\langle T, u\rangle - h(x + u)|\ x \in G, u \in X\}.\end{aligned}$$

We set $\bar{u} := x + u$ and, by applying Remark 3.1 and Proposition 2.6 in [19], we obtain

$$\begin{aligned}\Phi^*(0, T) &= \mathrm{WSup}\Big\{\{\langle T, \bar{u}\rangle - h(\bar{u})|\ \bar{u} \in X\} + \{-\langle T, x\rangle|\ x \in G\}\Big\} \\ &= \mathrm{WSup}\Big\{\ \mathrm{WSup}\{\langle T, \bar{u}\rangle - h(\bar{u})|\ \bar{u} \in X\} + \{-\langle T, x\rangle|\ x \in G\}\Big\} \\ &= \mathrm{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle|\ x \in G\}\Big\}.\end{aligned}$$

□

Consequently, we can introduce the following dual problem to (P_c)

$$(D_c) \qquad \mathrm{WSup} \bigcup_{T \in \mathcal{L}(X, Y)} \mathrm{WInf}\Big\{ - h^*(T) + \{\langle T, x\rangle|\ x \in G\}\Big\}.$$

Proposition 3.2 (Weak duality)**.** *For any $x \in G$ and any $T \in \mathcal{L}(X, Y)$ it holds*

$$h(x) \notin B\Big(- \Phi^*(0, T)\Big).$$

Proposition 3.3. *If the primal problem (P_c) is stable, then*

$$\mathrm{WInf}(P_c) = \mathrm{WSup}(D_c) = \mathrm{WMax}(D_c).$$

Remark 3.2. According to Proposition 2.6 in [19], one can use for $\Phi^*(0, T)$ the following equivalent formulations

$$\begin{aligned}\Phi^*(0, T) &= \mathrm{WSup}\Big\{\{\langle T, u\rangle - h(u)|\ u \in X\} + \{-\langle T, x\rangle|\ x \in G\}\Big\} \\ &= \mathrm{WSup}\Big\{h^*(T) + \{-\langle T, x\rangle|\ x \in G\}\Big\} \\ &= \mathrm{WSup}\Big\{h^*(T) + \mathrm{WSup}\{-\langle T, x\rangle|\ x \in G\}\Big\}.\end{aligned}$$

The following result deals with the stability of the problem (P_c), assuming that its objective function has the form $h(x) = \langle C, x\rangle,\ C \in \mathcal{L}(X, Y)$.

Proposition 3.4. *Let $C \in \mathcal{L}(X, Y)$ and the objective function $h : X \to Y$ be defined by $h(x) = \langle C, x\rangle, x \in X$. Then the problem (P_c) is stable.*

Proof. Let $W : X \rightrightarrows Y$ be the value mapping of (P_c)

$$\begin{aligned} W(y) &= \mathrm{WInf}\{\Phi(x,y) |\ x \in X\} \\ &= \mathrm{WInf}\{\langle C, x+y\rangle |\ x \in G\} = \langle C, y\rangle + \mathrm{WInf}\{\langle C, x\rangle |\ x \in G\}. \end{aligned}$$

Let $z \in W(0)$ be fixed. By definition, $\partial W(0; z) \neq 0$ means that there exists $T \in \mathcal{L}(X, Y)$ such that (see Definition 2.3(iii))

$$-z \in \mathrm{WMax} \bigcup_{y \in X} [\langle T, y\rangle - W(y)]. \tag{3.1}$$

Let us show that (3.1) holds. By applying Remark 3.1 we have

$$\begin{aligned} W^*(T) &= \mathrm{WSup} \bigcup_{y \in X} [\langle T, y\rangle - W(y)] \\ &= \mathrm{WSup} \bigcup_{y \in X} [\langle T, y\rangle - \langle C, y\rangle - \mathrm{WInf}\{\langle C, x\rangle |\ x \in G\}] \\ &= \mathrm{WSup} \Big\{ - \mathrm{WInf}\{\langle C, x\rangle |\ x \in G\} + \{\langle T - C, y\rangle |\ y \in X\}\Big\}. \end{aligned}$$

Taking $T = C$, in view of Corollary 2.3 in [19], one has

$$\begin{aligned} W^*(C) &= \mathrm{WSup}\,\mathrm{WSup}\{-\langle C, x\rangle |\ x \in G\} \\ &= \mathrm{WSup}\{-\langle C, x\rangle |\ x \in G\} = - \mathrm{WInf}\{\langle C, x\rangle |\ x \in G\} = -W(0). \end{aligned}$$

This means that $-z \in W^*(C) = \mathrm{WSup} \bigcup_{y \in X} [\langle C, y\rangle - W(y)]$. On the other hand, as $-z = \langle C, 0\rangle - z \in \bigcup_{y \in X} [\langle C, y\rangle - W(y)]$, it follows that

$$-z \in \mathrm{WMax} \bigcup_{y \in X} [\langle C, y\rangle - W(y)].$$

In other words, W is subdifferentiable at 0. □

4. Variational principles for vector equilibrium problems

In the following we assume that K is a nonempty convex set in X and $f : K \times K \to Y$ is a bifunction such that $f(x,x) = 0,\ \forall x \in K$. We consider the vector equilibrium problem which consists in finding $x \in K$ such that

$$f(x,y) \not< 0,\ \forall y \in K. \tag{VEP}$$

By K^p we denote the solution set of (VEP).

We say that a *variational principle* (see [4], [5]) holds for (VEP) if there exists a set-valued map $G : K \rightrightarrows Y$, depending on the data of (VEP) but not on its solution set such that the solution set of (VEP) is contained in the solution set of the set-valued optimization problem

$$\mathrm{WMin} \bigcup_{x \in K} G(x). \tag{P_G}$$

(P_G) is nothing else than the problem of finding $x_0 \in K$ such that

$$G(x_0) \cap \mathrm{WMin} \bigcup_{x \in K} G(x) \neq \varnothing.$$

Auchmuty ([6]) was the first who proposed variational principles as generalizations of the concept of gap functions for variational inequalities. Later, Blum and Oettli ([8]) introduced variational principles also for equlibrium problems. Variational principles for vector equilibrium problems have been investigated by Ansari, Konnov and Yao in [4] and [5] as generalizations of the above mentioned variational principles.

In this section we give variational principles for (VEP) obtained on the basis of the Fenchel-type duality introduced in the previous section. Let us notice that the Fenchel duality has been used by the authors in [1] and [2], when constructing gap functions for scalar equilibrium problems.

One can easily notice that $\bar{x} \in K$ is a solution to (VEP) if and only if 0 is a weak minimal point of the set $\{f(\bar{x}, y)|\ y \in K\}$. For a fixed $x \in K$ consider the following vector optimization problem

$$(P^{VEP}; x) \qquad \mathrm{WInf}\Big\{f(x, y)|\ y \in K\Big\}.$$

Further let be $\widetilde{f} : X \times X \to \overline{Y}$

$$\widetilde{f}(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in K \times K; \\ +\infty, & \text{otherwise.} \end{cases}$$

The Fenchel-type dual of $(P^{VEP}; x)$ introduced in Section 3 turns out to be

$$\begin{aligned}(D^{VEP}; x) \quad & \mathrm{WSup} \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{\widetilde{f}(x, y) - \langle T, y\rangle|\ y \in X\} + \{\langle T, y\rangle|\ y \in K\}\Big\} \\ &= \mathrm{WSup} \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{f(x, y) - \langle T, y\rangle|\ y \in K\} + \{\langle T, y\rangle|\ y \in K\}\Big\}.\end{aligned}$$

In view of Proposition 2.6 in [19], the dual becomes

$$(D^{VEP}; x) \qquad \mathrm{WSup} \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{ - f^*_K(T; x) + \{\langle T, y\rangle|\ y \in K\}\Big\},$$

where $f^*_K(\cdot; x) : \mathcal{L}(X, Y) \rightrightarrows \overline{Y}$ is defined by $f^*_K(T; x) = \mathrm{WSup}\{\langle T, y\rangle - f(x, y)|\ y \in K\}$. For any $x \in K$ we introduce the following mapping

$$\gamma_p(x) := \bigcup_{T \in \mathcal{L}(X,Y)} \Big[- \Phi^*_p(0, T; x)\Big],$$

where $\Phi^*_p(0, T; x) = \mathrm{WSup}\Big\{f^*_K(T; x) + \{-\langle T, y\rangle|\ y \in K\}\Big\}$. This can be rewritten as

$$\begin{aligned}\gamma_p(x) &= \bigcup_{T \in \mathcal{L}(X,Y)} \Big[- \mathrm{WSup}\Big\{f^*_K(T; x) + \{-\langle T, y\rangle|\ y \in K\}\Big\}\Big] \\ &= \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{ - f^*_K(T; x) + \{\langle T, y\rangle|\ y \in K\}\Big\} \\ &= \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{f(x, y) - \langle T, y\rangle|\ y \in K\} + \{\langle T, y\rangle|\ y \in K\}\Big\}.\end{aligned}$$

We consider the following optimization problem

$$(P_\gamma) \qquad \mathrm{WSup} \bigcup_{x\in K} \gamma_p(x),$$

which delivers, as proved in the following, a variational principle for (VEP).

Lemma 4.1. *For any $x \in K$, if $z \in \gamma_p(x)$, then $z \not> 0$.*

Proof. Let $x \in K$ be fixed and

$$z \in \gamma_p(x) = \bigcup_{T\in\mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{f(x,y) - \langle T, y\rangle |\ y \in K\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

Then exists $\overline{T} \in \mathcal{L}(X, Y)$ such that

$$z \in \mathrm{WInf}\Big\{\{f(x,y) - \langle \overline{T}, y\rangle |\ y \in K\} + \{\langle \overline{T}, y\rangle |\ y \in K\}\Big\}.$$

We assume that $z > 0$. This can be equivalently written as

$$z > f(x,x) - \langle \overline{T}, x\rangle + \langle \overline{T}, x\rangle,$$

which leads to a contradiction. □

Theorem 4.1. *Let the problem $(P^{VEP}; x)$ be stable for each $x \in K^p$. Then*
(i) *$\bar{x} \in K$ is a solution to (VEP) if and only if $0 \in \gamma_p(\bar{x})$;*
(ii) *$K^p \subseteq K^p_\gamma$, where K^p_γ denotes the solution set of (P_γ).*

Proof. (i) If $\bar{x} \in K$ is a solution to (VEP), then by Proposition 3.3 one has

$$0 \in \mathrm{WInf}(P^{VEP}; \bar{x}) = \mathrm{WMax}(D^{VEP}; \bar{x}).$$

Whence

$$0 \in \mathrm{WMax} \bigcup_{T\in\mathcal{L}(X,Y)} \mathrm{WInf}\Big\{ - f^*_K(T, \bar{x}) + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

Consequently, $0 \in \gamma_p(\bar{x})$. Let us now assume that

$$\begin{aligned} 0 \in \gamma_p(\bar{x}) &= \bigcup_{T\in\mathcal{L}(X,Y)} \mathrm{WInf}\Big\{ - f^*_K(T, \bar{x}) + \langle T, y\rangle |\ y \in K\}\Big\} \\ &= \bigcup_{T\in\mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{f(\bar{x},y) - \langle T, y\rangle |\ y \in K\} \\ &\qquad + \{\langle T, y\rangle |\ y \in K\}\Big\}. \end{aligned}$$

Thus exists $\overline{T} \in \mathcal{L}(X, Y)$ such that

$$0 \in \mathrm{WInf}\Big\{\{f(\bar{x},y) - \langle \overline{T}, y\rangle |\ y \in K\} + \{\langle \overline{T}, y\rangle |\ y \in K\}\Big\}.$$

Assume that $0 \notin \mathrm{WInf}\{f(\bar{x}, y)|\ y \in K\}$. Then one must have

$$0 \notin \mathrm{WMin}\{f(\bar{x}, y)|\ y \in K\}$$

and so there exists $y' \in K$ such that $f(\bar{x}, y') < 0$ or, equivalently, $f(\bar{x}, y') - \langle \overline{T}, y'\rangle + \langle \overline{T}, y'\rangle < 0$, which leads to a contradiction.

(ii) Let $\bar{x} \in K^p$. In view of (i) we have $0 \in \gamma_p(\bar{x})$. On the other hand, by Lemma 4.1, if $z \in \gamma_p(x)$ for $x \in K$, then $z \not> 0$. In conclusion, from $z \in \bigcup_{x\in K} \gamma_p(x)$ follows $z \not> 0$. This means that

$$0 \in \text{WMax} \bigcup_{x\in K} \gamma_p(x) \subseteq \text{WSup} \bigcup_{x\in K} \gamma_p(x),$$

which means that $\bar{x} \in K^p_\gamma$. $\square$

Remark 4.1. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. Then the linear continuous operator $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ can be identified with a n-dimensional vector. In this case for a given set $Z \subseteq \mathbb{R}$ we have (cf. [18])

$\widehat{y} \in \text{WSup}\, Z$ if and only if $\widehat{y} > y$, $\forall y \in Z$ and if $y' < \widehat{y}$, then $\exists y \in Z$ such that $y' < y$.

In other words $\text{WSup}\, Z$ is nothing else than the usual concept of the supremum of the set Z in $\mathbb{R}$.

Assume that $\varphi : X \times X \to \mathbb{R} \cup \{+\infty\}$ is a bifunction satisfying $\varphi(x, x) = 0$, $\forall x \in K$. Consider the equilibrium problem which consists in finding $x \in K$ such that

$$(EP) \qquad \varphi(x, y) \geq 0, \, \forall y \in K,$$

which is a special case of (VEP). Taking in the formulation of $(D^{VEP}; x)$ the function φ instead of $\widetilde{f}$, the dual becomes

$$\begin{aligned}(D^{EP}; x) \qquad & \sup_{T\in\mathbb{R}^n} \inf \Big\{ \{\varphi(x, y) - \langle T, y\rangle |\ y \in \mathbb{R}^n\} + \{\langle T, y\rangle |\ y \in K\} \Big\} \\ & = \sup_{T\in\mathbb{R}^n} \Big\{ \inf_{y\in\mathbb{R}^n} \{\varphi(x, y) - \langle T, y\rangle\} + \inf_{y\in K} \langle T, y\rangle \Big\} \\ & = \sup_{T\in\mathbb{R}^n} \Big\{ -\varphi^*_y(x, T) + \inf_{y\in K} \langle T, y\rangle \Big\},\end{aligned}$$

where $\varphi^*_y(x, T) := \sup_{y\in\mathbb{R}^n} \{\langle T, y\rangle - \varphi(x, y)\}$ is the conjugate function of $\varphi(x, \cdot) : X \to \mathbb{R} \cup \{+\infty\}$ with respect to the variable y for a fixed x. The function γ_p turns out in this case to be

$$\gamma^{EP}(x) := -v(D^{EP}; x) = \inf_{T\in\mathbb{R}^n} \Big\{ \varphi^*_y(x, T) + \sup_{y\in K} \langle -T, y\rangle \Big\},$$

where $v(D^{EP}; x)$ is the optimal objective value of the problem $(D^{EP}; x)$. This is nothing else than the gap function introduced in [2].

Example 4.1. Let $u : X \to Y \cup \{+\infty\}$ be a given function and the bifunction $\widetilde{f} : \text{dom}\, u \times X \to Y \cup \{+\infty\}$ defined by $\widetilde{f}(x, y) = u(y) - u(x)$, where $\text{dom}\, u := \{x \in X |\ u(x) \in Y\}$. We assume that $K \times K \subseteq \text{dom}\, \widetilde{f}$. Then (VEP) becomes the vector optimization problem of finding $x \in K$ such that

$$(\widetilde{P}_u) \qquad \widetilde{f}(x, y) = u(y) - u(x) \not< 0, \ \forall y \in K.$$

For any $x \in K$, γ_p turns out to be

$$\widetilde{\gamma}_p(x) = -u(x) + \bigcup_{T\in\mathcal{L}(X,Y)} \text{WInf} \Big\{ \{u(y) - \langle T, y\rangle |\ y \in X\} + \{\langle T, y\rangle |\ y \in K\} \Big\}.$$

Assuming the stability of $(\widetilde{P}_u)$, by Proposition 3.3, it holds

$$\mathrm{WInf}(\widetilde{P}_u) = \mathrm{WSup}(\widetilde{D}_u) = \mathrm{WMax}(\widetilde{D}_u), \tag{4.1}$$

where $(\widetilde{D}_u)$ is the Fenchel-type dual problem to $(\widetilde{P}_u)$.

Let $\bar{x} \in K$ be a solution to $(\widetilde{P}_u)$. From (4.1) follows

$$u(\bar{x}) \in \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{u(y) - \langle T, y\rangle |\ y \in X\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

In other words $0 \in \widetilde{\gamma}_p(\bar{x})$. The inverse implication follows analogously (see the proof of Theorem 4.1). On the other hand, by Proposition 3.3 and Proposition 2.6 in [19], one has $\mathrm{WSup} \bigcup\limits_{x \in K} \widetilde{\gamma}_p(x) = \{0\}$. If $\bar{x} \in K$ solves $(\widetilde{P}_u)$, then as shown before, $0 \in \widetilde{\gamma}_p(\bar{x})$. This means that $\bar{x} \in K^p_\gamma$ and so $K^p \subseteq K^p_\gamma$. In other words, for this particular form of the vector equilibrium problem the assertions of Theorem 4.1 are automatically fulfilled. Thus the latter proves to be a natural generalization of some results which hold for some special cases of (VEP).

Example 4.2 (see [16]). Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and the vector-valued function $\varphi_1 : \mathbb{R} \to \mathbb{R}^2 \cup \{+\infty\}$ be given by

$$\varphi_1(x) = \begin{cases} (x, 0), & \text{if } x \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

For the bifunction $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \cup \{+\infty\}$

$$f_1(x,y) = \begin{cases} \varphi_1(y) - \varphi_1(x), & \text{if } (x,y)^T \in [0,1] \times [0,1], \\ +\infty, & \text{otherwise,} \end{cases}$$

we consider the vector equilibrium problem of finding $x \in K = [0,1]$ such that

$$f_1(x,y) = \varphi_1(y) - \varphi_1(x) \not< 0,\ \forall y \in K. \tag{VEP_1}$$

According to γ_p, we have $\forall x \in \mathbb{R}$

$$\gamma_{p_1}(x) = \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \mathrm{WInf}\Big\{\{\varphi_1(y) - \varphi_1(x) - \langle T, y\rangle |\ y \in K\} + \{\langle T, y\rangle |\ y \in K\}\Big\}.$$

This can be written as (see Remark 3.2)

$$\begin{aligned} \gamma_{p_1}(x) = -\varphi_1(x) - \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \mathrm{WSup}\Big\{&\{\langle T, y\rangle - \varphi_1(y) |\ y \in K\} \\ &+ \{-\langle T, y\rangle |\ y \in K\}\Big\} \\ = -\varphi_1(x) - \bigcup_{T \in \mathcal{L}(\mathbb{R},\mathbb{R}^2)} \mathrm{WSup}\Big\{&\mathrm{WSup}\{\langle T, y\rangle - \varphi_1(y) |\ y \in K\} \\ &+ \mathrm{WSup}\{-\langle T, y\rangle |\ y \in K\}\Big\}. \end{aligned}$$

Notice that the linear continuous operator $T \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ can be represented as $T = (\alpha, \beta)^T \in \mathbb{R}^2$. Using the notations

$$\begin{aligned}\psi_1(T) &:= \mathrm{WSup}\{\langle T, y\rangle - \varphi_1(y)|\ y \in K\} = \mathrm{WSup}\{(\alpha-1, \beta)y|\ y \in [0,1]\},\\ \psi_2(T) &:= \mathrm{WSup}\{-\langle T, y\rangle|\ y \in K\} = \mathrm{WSup}\{(-\alpha, -\beta)y|\ y \in [0,1]\},\end{aligned}$$

one can find for any $T = (\alpha, \beta)^T \in \mathbb{R}^2$ how the sets $\psi_1(T), \psi_2(T)$ and $\mathrm{WSup}\{\psi_1(T) +\psi_2(T)\}$ are looking.

(i) If $\alpha \geq 1$ and $\beta \geq 0$, then

$$\begin{aligned}\psi_1(T) &= \{(x,y)^T \in \mathbb{R}^2|\ (x = \alpha - 1,\ y \leq \beta) \vee (y = \beta,\ x \leq \alpha - 1)\},\\ \psi_2(T) &= \{(x,y)^T \in \mathbb{R}^2|\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}.\end{aligned}$$

Whence $\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \psi_1(T)$.

(ii) If $\alpha > 1$ and $\beta < 0$, then

$$\begin{aligned}\psi_1(T) = \Big\{&(x,y)^T \in \mathbb{R}^2|\ (x = \alpha - 1,\ y \leq \beta) \vee (y = 0,\ x \leq 0)\\ &\vee \Big(y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1\Big)\Big\},\\ \psi_2(T) = \Big\{&(x,y)^T \in \mathbb{R}^2|\ (x = 0,\ y \leq 0) \vee (y = -\beta,\ x \leq -\alpha)\\ &\vee \Big(y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0\Big)\Big\}.\end{aligned}$$

Consequently we have

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \Big\{&(x,y)^T \in \mathbb{R}^2|\ (x = \alpha - 1,\ y \leq \beta)\\ &\vee (y = -\beta,\ x \leq -\alpha) \vee \Big(y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0\Big)\\ &\vee \Big(y = \frac{\beta}{\alpha - 1}x,\ 0 \leq x \leq \alpha - 1\Big)\Big\}.\end{aligned}$$

If $\alpha = 1$ and $\beta < 0$, then one can easily see that

$$\begin{aligned}\mathrm{WSup}\{\psi_1(T) + \psi_2(T)\} = \Big\{&(x,y)^T \in \mathbb{R}^2|\ (x = 0,\ y \leq 0)\\ &\vee (y = -\beta,\ x \leq -\alpha) \vee \Big(y = \frac{\beta}{\alpha}x,\ -\alpha \leq x \leq 0\Big)\Big\}.\end{aligned}$$

(iii) If $0 < \alpha < 1$ and $\beta \geq 0$, then

$$\begin{aligned}\psi_1(T) = \Big\{&(x,y)^T \in \mathbb{R}^2|\ (x = 0,\ y \leq 0) \vee (y = \beta,\ x \leq \alpha - 1)\\ &\vee \Big(y = \frac{\beta}{\alpha - 1}x,\ \alpha - 1 \leq x \leq 0\Big)\Big\},\\ \psi_2(T) = \{&(x,y)^T \in \mathbb{R}^2|\ (x = 0,\ y \leq 0) \vee (y = 0,\ x \leq 0)\}.\end{aligned}$$

As a consequence one has $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_1(T)$. If additional, $\alpha = 0$ and $\beta \geq 0$, then it holds

$$\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \Big\{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq 0) \\ \vee (y=\beta,\ x\leq \alpha-1) \vee \Big(y=\frac{\beta}{\alpha-1}x,\ \alpha-1\leq x\leq 0\Big)\Big\}.$$

(iv) If $0<\alpha<1$ and $\beta<0$, then

$$\psi_1(T) = \{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq 0) \vee (y=0,\ x\leq 0)\},$$
$$\psi_2(T) = \Big\{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq 0) \vee (y=-\beta,\ x\leq -\alpha) \\ \vee \Big(y=\frac{\beta}{\alpha}x,\ -\alpha\leq x\leq 0\Big)\Big\}.$$

Hence $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_2(T)$. Moreover, if $\alpha=0$ and $\beta<0$, then it holds

$$\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq \beta) \\ \vee (y=-\beta,\ x\leq 0)\}.$$

(v) If $\alpha<0$ and $\beta\geq 0$, then

$$\psi_1(T) = \Big\{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq 0) \vee (y=\beta,\ x\leq \alpha-1) \\ \vee \Big(y=\frac{\beta}{\alpha-1}x,\ \alpha-1\leq x\leq 0\Big)\Big\},$$
$$\psi_2(T) = \Big\{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y\leq -\beta) \vee (y=0,\ x\leq 0) \\ \vee \Big(y=\frac{\beta}{\alpha}x,\ 0\leq x\leq -\alpha\Big)\Big\}.$$

and we get further

$$\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \Big\{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y\leq -\beta) \\ \vee (y=\beta,\ x\leq \alpha-1) \vee \Big(y=\frac{\beta}{\alpha-1}x,\ \alpha-1\leq x\leq 0\Big) \\ \vee \Big(y=\frac{\beta}{\alpha}x,\ 0\leq x\leq -\alpha\Big)\Big\}.$$

(vi) Finally, if $\alpha<0$ and $\beta<0$, then

$$\psi_1(T) = \{(x,y)^T \in \mathbb{R}^2 |\ (x=0,\ y\leq 0) \vee (y=0,\ x\leq 0),$$
$$\psi_2(T) = \{(x,y)^T \in \mathbb{R}^2 |\ (x=-\alpha,\ y\leq -\beta) \vee (y=-\beta,\ x\leq -\alpha)\}.$$

and in this case we have $\mathrm{WSup}\{\psi_1(T)+\psi_2(T)\} = \psi_2(T)$.

Summarizing all above cases, we obtain the complete description of the mapping γ_{p_1}.

In the following we deal with another class of vector equilibrium problems, the so-called dual vector equilibrium problem. The dual vector equilibrium problem is the problem of finding $x \in K$ such that

$$(DVEP) \qquad f(y,x) \not> 0, \ \forall y \in K.$$

Variational principles for $(DVEP)$ can be given in a similar way to the ones for (VEP). Indeed, let us denote by K^d the solution set of $(DVEP)$. One can notice that $\widehat{x} \in K$ is a solution to $(DVEP)$ if and only if 0 is a weak maximal point of the set $\{f(y,\widehat{x}) |\ y \in K\}$. For any $x \in K$ we consider the vector optimization problem

$$(P^{DVEP};x) \qquad \mathrm{WSup}\{f(y,x) |\ y \in K\} = -\,\mathrm{WInf}\{-f(y,x) |\ y \in K\}.$$

Instead of considering $(P^{DVEP};x)$, we work with the vector optimization problem

$$(\widetilde{P}^{DVEP};x) \qquad \mathrm{WInf}\{-f(y,x) |\ y \in K\}.$$

By using the function $\widehat{f} : X \times X \to \overline{Y}$,

$$\widehat{f}(x,y) = \begin{cases} -f(y,x), & \text{if } (x,y) \in K \times K; \\ +\infty, & \text{otherwise,} \end{cases}$$

the Fenchel-type dual to $(\widetilde{P}^{DVEP};x)$ can be written as

$$\begin{aligned}(\widetilde{D}^{DVEP};x) \quad & \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{\widehat{f}(x,y) - \langle \Lambda, y\rangle |\ y \in X\} + \{\langle \Lambda, y\rangle |\ y \in K\}\Big\} \\ &= \mathrm{WSup} \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WInf}\Big\{\{-f(y,x) - \langle \Lambda, y\rangle |\ y \in K\} + \{\langle \Lambda, y\rangle |\ y \in K\}\Big\}.\end{aligned}$$

For every $x \in K$ we define the following mapping

$$\gamma_d(x) := \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \Phi_d^*(0,\Lambda;x),$$

where $\Phi_d^*(0,\Lambda;x) = \mathrm{WSup}\Big\{\{f(y,x) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}$.

To the problem $(DVEP)$ we associate the set-valued vector optimization problem

$$(D_\gamma) \qquad \mathrm{WInf} \bigcup_{x \in K} \gamma_d(x),$$

which represents the starting point in formulating a variational principle for $(DVEP)$.

Lemma 4.2. *For any $x \in K$, if $z \in \gamma_d(x)$, then $z \not< 0$.*

Proof. Let $x \in K$ be fixed and

$$z \in \gamma_d(x) = \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WSup}\Big\{\{f(y,x) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

Consequently, there exists $\widetilde{\Lambda} \in \mathcal{L}(X,Y)$ such that

$$z \in \mathrm{WSup}\Big\{\{f(y,x) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}.$$

Let $z < 0$. In other words

$$z < f(x,x) + \langle \widetilde{\Lambda}, x\rangle - \langle \widetilde{\Lambda}, x\rangle.$$

This contradicts the fact that z is a weak supremal element of the set $\Big\{\{f(y,x) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}$. □

Theorem 4.2. *Let the problem $(\widetilde{P}^{DVEP}; x)$ be stable for each $x \in K^d$. Then*

(i) *$\widetilde{x} \in K$ is a solution to $(DVEP)$ if and only if $0 \in \gamma_d(\widetilde{x})$;*

(ii) *$K^d \subseteq K^d_\gamma$, where K^d_γ denotes the solution set of (D_γ).*

Proof. (i) Let $\widetilde{x} \in K$ be a solution to $(DVEP)$. Then, by Proposition 3.3, it follows that

$$0 \in \mathrm{WSup}(P^{DVEP}; \widetilde{x}) = -\,\mathrm{WInf}(\widetilde{P}^{DVEP}; \widetilde{x}) = -\,\mathrm{WMax}(\widetilde{D}^{DVEP}; \widetilde{x}).$$

Therefore

$$0 \in \mathrm{WMin} \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WSup}\Big\{\{f(y,\widetilde{x}) + \langle \Lambda, y\rangle |\ y \in K\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

In other words we have $0 \in \gamma_d(\widetilde{x})$. Let now $0 \in \gamma_d(\widetilde{x})$. Then there exists $\widetilde{\Lambda} \in \mathcal{L}(X, Y)$ such that

$$0 \in \mathrm{WSup}\Big\{\{f(y,\widetilde{x}) + \langle \widetilde{\Lambda}, y\rangle |\ y \in K\} + \{-\langle \widetilde{\Lambda}, y\rangle |\ y \in K\}\Big\}.$$

If $0 \notin \mathrm{WSup}(P^{DVEP}; \widetilde{x})$, then $0 \notin \mathrm{WMax}(P^{DVEP}; \widetilde{x})$. Whence there exists $\widetilde{y} \in K$ such that $f(\widetilde{y}, \widetilde{x}) > 0$ or, equivalently, $f(\widetilde{y}, \widetilde{x}) + \langle \widetilde{\Lambda}, \widetilde{y}\rangle - \langle \widetilde{\Lambda}, \widetilde{y}\rangle > 0$. But this leads to a contradiction.

(ii) Let $\widetilde{x} \in K^d$. Taking into account (i), one has $0 \in \gamma_d(\widetilde{x})$. By Lemma 4.2 we obtain that

$$0 \in \mathrm{WMin} \bigcup_{x \in K} \gamma_d(x) \subseteq \mathrm{WInf} \bigcup_{x \in K} \gamma_d(x),$$

which means $\widetilde{x} \in K^d_\gamma$. □

Under some (generalized) convexity and monotonicity assumptions the relations between the solution sets of (VEP) and $(DVEP)$ have been investigated in [5] and [13]. In this way we can relate the mapping γ_d to the vector equilibrium problem (VEP). Before doing this, let us recall some definitions and preliminary results.

Definition 4.1 ([5, Definition 2.1])**.** A function $f : K \times K \to Y$ is called

(i) monotone if, for all $x, y \in K$, we have

$$f(x,y) + f(y,x) \leq 0;$$

(ii) pseudomonotone if, for all $x, y \in K$, we have

$$f(x,y) \not< 0 \text{ implies } f(y,x) \not> 0$$

or, equivalently,

$$f(x,y) > 0 \text{ implies } f(y,x) < 0.$$

Definition 4.2 ([5, cf. Definition 2.2])**.** A function $h : K \to Y$ is called

(i) quasiconvex if, for all $\alpha \in Y$, the set $L(\alpha) = \{x \in K|\ h(x) \leq \alpha\}$ is convex;
(ii) explicitly quasiconvex if h is quasiconvex and, for all $x, y \in K$ such that $h(x) < h(y)$, we have
$$h(z_t) < h(y), \text{ for all } z_t = tx + (1-t)y \text{ and } t \in (0,1);$$
(iii) hemicontinuous if, for any $x, y \in K$ and $t \in [0,1]$, the mapping $t \mapsto h(tx + (1-t)y)$ is continuous at 0^+.

Proposition 4.1 ([5, Proposition 2.1])**.** *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $f : K \times K \to Y$ be a bifunction such that $f(x,x) = 0,\ \forall x \in K$.*
(i) *If f is pseudomonotone, then $K^p \subseteq K^d$;*
(ii) *If $f(x,\cdot)$ is explicitly quasiconvex and $f(\cdot,y)$ is hemicontinuous for all $x, y \in K$, then $K^d \subseteq K^p$.*

By using Theorem 4.2 and Proposition 4.1 one can easily verify the following assertion.

Proposition 4.2. *Let all the assumptions of Proposition 4.1 and Theorem 4.2 be fulfilled. Then*
(i) *$\widetilde{x} \in K$ is a solution to (VEP) if and only if $0 \in \gamma_d(\widetilde{x})$;*
(ii) *$K^p \subseteq K^d_\gamma$.*

5. Gap functions for weak vector variational inequalities

This section deals with the construction of gap functions for the weak vector variational inequality problem. To this end we use the results given in the previous sections for the vector equilibrium problems. As before, let X and Y be real topological spaces, K is a convex subset of X and $F : X \to \mathcal{L}(X,Y)$ a given mapping. The *weak vector variational inequality problem* consists in finding $x \in K$ such that

$$(WVVI) \qquad \langle F(x), y - x\rangle \not< 0,\ \forall y \in K.$$

Definition 5.1 ([9, Definition 5(ii)])**.** A set-valued mapping $\psi : X \rightrightarrows Y$ is said to be a gap function for the problem $(WVVI)$ if it satisfies the following conditions
(i) $0 \in \psi(x)$ if and only if $x \in K$ solves $(WVVI)$;
(ii) $0 \not> \psi(y),\ \forall y \in K$.

As one has that $\bar{x} \in K$ is a solution to $(WVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(\bar{x}), y - \bar{x}\rangle|\ y \in K\}$, let us consider the vector optimization problem

$$(P^{WVVI}; x) \qquad \mathrm{WInf}\{\langle F(x), y - x\rangle|\ y \in K\}.$$

We take in the definition of γ_p for any $x \in K, f(x,y) := \langle F(x), y - x\rangle$ and then define the gap function as being $-\gamma_p$. Thus one gets the following set-valued mapping

$$\begin{aligned}\psi_p(x) &:= \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WSup}\Big\{\{\langle T, y\rangle - \langle F(x), y - x\rangle|\ y \in X\} + \{-\langle T, y\rangle|\ y \in K\}\Big\}\\ &= \bigcup_{T \in \mathcal{L}(X,Y)} \mathrm{WSup}\Big\{\{\langle T - F(x), y\rangle|\ y \in X\} + \{-\langle T, y\rangle|\ y \in K\}\Big\} + \langle F(x), x\rangle.\end{aligned}$$

Theorem 5.1. *The mapping ψ_p is a gap function for the problem $(WVVI)$.*

Proof. (i) Since $y \to \langle F(x), y - x\rangle$ is a linear mapping, Proposition 3.4 ensures the stability of the problem $(P^{WVVI}; x)$ for any $x \in K$. Thus the first condition in the definition of a gap function follows from Theorem 4.1(i).

(ii) By Lemma 4.1, for any $y \in K$ and any $z \in -\psi_p(y)$ one has $z \not> 0$. It follows that $0 \not> \psi_p(y),\ \forall y \in K$. □

The relations between $(WVVI)$ and the so-called *Minty weak vector variational inequality* have been investigated by several authors (see [11], [13], [20] and [21]). This is the problem which consists in finding $x \in K$ such that

$$(MWVVI) \qquad \langle F(y), x - y\rangle \not> 0,\ \forall y \in K.$$

As done in Section 4, $(MWVVI)$ can be related to the following vector optimization problem

$$(P^{MWVVI}; x) \qquad \mathrm{WInf}\{\langle F(y), y - x\rangle |\ y \in K\},$$

in the sense that $x \in K$ is a solution to $(MWVVI)$ if and only if 0 is a weak minimal point of the set $\{\langle F(y), y - x\rangle |\ y \in K\}$. Taking in the formula of γ_d, $\widehat{f}(x, y) := \langle F(x), y - x\rangle$, this becomes the following mapping

$$\psi_d(x) = \bigcup_{\Lambda \in \mathcal{L}(X,Y)} \mathrm{WSup}\Big\{\{\langle F(y), x - y\rangle + \langle \Lambda, y\rangle |\ y \in X\} + \{-\langle \Lambda, y\rangle |\ y \in K\}\Big\}.$$

From Theorem 4.2(i) and Lemma 4.2 one has the following assertion.

Theorem 5.2. *Let the problem $(P^{MWVVI}; x)$ be stable for any solution $x \in K$ to $(MWVVI)$. Then ψ_d is a gap function for the problem $(MWVVI)$.*

Next we give some conditions which guarantee that the mapping ψ_d is also a gap function for $(WVVI)$. To this end we need first the following definitions.

Definition 5.2. [21] Let $F : K \to \mathcal{L}(X, Y)$ be a given function.

(i) F is weakly C-pseudomonotone on K if for each $x, y \in K$, we have

$$\langle F(x), y - x\rangle \not< 0 \text{ implies } \langle F(y), x - y\rangle \not> 0;$$

(ii) F is v-hemicontinuous if for each $x, y \in K$ and $t \in [0, 1]$, the mapping $t \mapsto \langle F(x + t(y - x)), y - x\rangle$ is continuous at 0^+.

Proposition 5.1. [21, Lemma 2.1]
Let X, Y be Banach spaces and let K be a nonempty convex subset of X. Assume that $F : K \to \mathcal{L}(X, Y)$ is weakly C-pseudomonotone on K and v-hemicontinuous. Then $x \in K$ is a solution to $(WVVI)$ if and only if it is also a solution to $(MWVVI)$.

Combining Proposition 5.1 and Theorem 5.2 one gets the following result.

Proposition 5.2. *Let the assumptions of Proposition 5.1 and Theorem 5.2 be fulfilled. Then ψ_d is a gap function for $(WVVI)$.*

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Manuscript received May 15, 2006
revised September 19, 2006

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