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ON MUTUALLY NEAREST POINTS OF UNBOUNDED SETS IN BANACH SPACES

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Abstract. Let $\mathcal{A}(X)$ (resp. $\mathcal{C}(X)$) denote the family of all nonempty closed (resp. nonempty closed convex) subsets of a strictly convex and Kadec (resp. uniformly convex) Banach space X and let $\mathcal{A}(X)$ be endowed with the h_ρ-topology. Let $G \in \mathcal{A}(X)$ and consider the minimization problem $\min_{x\in A,z\in G} \|x - z\|$, denoted by $\min(A, G)$. It is proved that the set of all subsets $A \in \mathcal{A}_G(X)$ (resp. $A \in \mathcal{C}_G(X)$) such that the minimization problem $\min(A, G)$ is well-posed is a dense G_δ-subset in $\mathcal{A}_G(X)$ (resp. $\mathcal{C}_G(X)$) provided that G is a relatively weakly compact closed (resp. a bounded closed) subset of X, where $\mathcal{A}_G(X)$ is the closure of the set $\{A \in \mathcal{A}(X) : \lambda_{AG} > 0\}$ with respect to the h_ρ-topology and $\mathcal{C}_G(X) = \mathcal{C}(X)\cap\mathcal{A}_G(X)$. In particular, in the case when X is uniformly convex it is also proved that the set of all subsets $A \in \mathcal{A}_G(X)$ such that the minimization problem $\min(A, G)$ fails to be well-posed is a σ-porous set in $\mathcal{A}_G(X)$. Similar results are given for the family of all nonempty closed boundedly relatively compact subsets of X. The case when G is unbounded is also considered.

1. Introduction

Let X be a real Banach space. Let $\mathcal{A}^b(X)$ (resp. $\mathcal{C}^b(X)$) denote the family of all nonempty closed bounded (resp. nonempty closed bounded convex) subsets of X. Let h denote the Hausdorff distance. It is well-known that the space $(\mathcal{A}^b(X), h)$ is a complete metric space.

For closed disjoint subsets A and G of X, we set

$$\lambda_{AG} := \inf\{\|z - x\| : x \in A, z \in G\}.$$

A pair (x_0, z_0) with $x_0 \in A, z_0 \in G$ is called a solution of the *minimization* problem, denoted by $\min(A, G)$, if $\|x_0 - z_0\| = \lambda_{AG}$. Moreover, any sequence $\{(x_n, z_n)\}$ with $x_n \in A, z_n \in G$, such that $\lim_{n\to\infty} \|x_n - z_n\| = \lambda_{AG}$ is called a *minimizing* sequence for $\min(A, G)$). A minimization problem is said to be *well-posed* if it has a unique solution and every minimizing sequence converges strongly to this solution.

For a given closed subset G of X, let $\bar{\mathcal{C}}^b_G(X)$ stand for the closure (under the Hausdorff distance) of the set $\{A \in \mathcal{C}^b(X) : \lambda_{AG} > 0\}$. In [2], it is proved that the set of all $A \in \bar{\mathcal{C}}^b_G(X)$ such that the minimization problem $\min(A, G)$ is well-posed is a dense G_δ-subset of $\bar{\mathcal{C}}^b_G(X)$, provided that X is a uniformly convex Banach space.

Recently (see [11], [12] and [13]), this result has been extended to the framework of strongly convex and/or strictly convex Banach spaces also for the class of nonempty

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compact, nonempty convex compact or nonempty closed bounded subsets of X, provided that each class under consideration is endowed with Hausdorff distance. For further related results, see [14–19].

In this note we will study similar problem for the class of unbounded sets endowed with the h_ρ-topology (see [1]). Since in the class $\mathcal{A}^b(X)$ the h_ρ-topology is weaker than the topology generated by the Hausdorff distance, the results presented here are independent of those mentioned above.

2. Preliminaries and main results

Recall that a function $d : \mathfrak{X}\times\mathfrak{X} \to [0, +\infty]$ is called a *semimetric* (*pseudometric, gauge*) in $\mathfrak{X}$ if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) \le d(x, z) + d(z, y)$ for every $x, y, z \in \mathfrak{X}$. The d ball of radius ϵ centered at y is the set $U_d(y, \epsilon) = \{x \in \mathfrak{X} : d(x, y) < \epsilon\}$.

A family $\{d_\rho : \rho \in \Gamma\}$ of gauges on $\mathfrak{X}$ is called *separating* if for each pair of points $x, y \in \mathfrak{X}$, $x \neq y$ there is $\rho \in \Gamma$ such that $d_\rho(x, y) > 0$. The topology having for a subbasis the family $\{U_{d_\rho}(x, \epsilon) : \rho \in \Gamma,\ x \in X,\ \epsilon > 0\}$ is called *the gauge topology* in $\mathfrak{X}$ induced by $\{d_\rho : \rho \in \Gamma\}$. Obviously this topology is Hausdorff if the family of gauges is separating.

Let X be a Banach space and let A be a subset of X. Then, as usual, $\overline{A}$ stands for the closure of A, $\operatorname{diam} A$ for the diameter of A, $\overline{\text{co}}A$ for the closed convex hull of A, and $d(x, A)$ for the distance from x to A. By $S(x, r)$ we denote the closed ball in X with center x and radius r. In particular S stands for $S(0, 1)$. By $\mathbb{R}$ we denote the set of all reals and by $\mathbb{N}$ the set of all positive integers. Moreover, we set

$\mathcal{A}(X)$ — the family of all nonempty closed subsets,
$\mathcal{A}^b(X)$ — the family of all nonempty closed bounded subsets,
$\mathcal{D}(X)$ — the family of all nonempty closed boundedly compact subsets,
$\mathcal{D}^b(X)$ — the family of all nonempty compact subsets,
$\mathcal{C}(X)$ — the family of all nonempty closed convex subsets,
$\mathcal{C}^b(X)$ — the family of all nonempty closed bounded convex subsets,
$\mathcal{K}(X)$ — the family of all nonempty closed convex boundedly compact subsets,
$\mathcal{K}^b(X)$ — the family of all nonempty convex compact subsets.

For $A, B \in \mathcal{A}(X)$, we define

$$e(A, B) = \inf\{\epsilon > 0 : A \subset B + \epsilon S\} \tag{2.1}$$

and

$$h(A, B) = \max\{e(A, B),\ e(B, A)\}. \tag{2.2}$$

Note that h is allowed to take value $+\infty$. If $A, B \in \mathcal{A}^b(X)$ then h is the well-known *Hausdorff distance*. Obviously

$$e(A, B) = \begin{cases} \sup_{a\in A} d(a, B) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset. \end{cases} \tag{2.3}$$

Now we introduce the h_ρ-topology on the space $\mathcal{A}(X)$ (*cf.* [1]). For $\rho > 0$ and $A, B \in \mathcal{A}(X)$, we define

$$h_\rho(A, B) = \max\{e(A \cap \rho S, B), e(B \cap \rho S, A)\}. \tag{2.4}$$

Clearly, for any $\rho > 0$ and $A, B \in \mathcal{A}(X)$,

$$h_\rho(A, B) \le h(A, B). \tag{2.5}$$

Clearly h_ρ is a gauge in $\mathcal{A}(X)$ and the family $\{h_\rho : \rho > 0\}$ is separating.

For $A \in \mathcal{A}(X)$, $\rho > 0$ and $r > 0$ we define

$$\mathcal{U}_\rho(A, r) = \{B \in \mathcal{A}(X) : \ h_\rho(A, B) < r\}. \tag{2.6}$$

Clearly, the family $\{\mathcal{U}_\rho(A, r) : \ \rho > 0, \, r > 0\}$ is the neighbourhood basis of the point A and the corresponding (gauge) topology on $\mathcal{A}(X)$ is called *h_ρ-topology* on $\mathcal{A}(X)$. Note that the sequence $\{A_n\}$ converges to A in h_ρ-topology iff $h_\rho(A_n, A) \to 0$ for every $\rho > 0$. It is easy to verify that the set $\mathcal{A}^b(X)$ is dense in the space $\mathcal{A}(X)$ with respect to the h_ρ-topology. The next proposition describes some relationship between the convergences in the h_ρ-topology and in the topology generated by the Hausdorff distance.

Proposition 2.1. *Let $\{A_n\}$ be a sequence of sets from $\mathcal{C}(X)$ and let $A_0 \in \mathcal{C}^b(X)$. Then the following conditions are equivalent:*

(i) *$\{A_n\}$ converges to A_0 in the h_ρ-topology.*

(ii) *There exist $M > 0$ and some integer n_0 such that $\|A_n\| \le M$ for $n \ge n_0$ and $\{A_n\}$ converges to A_0 with respect to the Hausdorff distance.*

Proof. The implication (ii) $\Rightarrow$ (i) follows immediately from (2.5). We need only to prove (i) $\Rightarrow$ (ii). Let $\rho_0 > 0$ and let n_* be such that $A_0 \cap \rho_0 S \neq \emptyset$ and

$$h_{\rho_0}(A_n, A_0) < 1 \quad \text{for each } n \ge n_*. \tag{2.7}$$

Let $b \in A_0 \cap \rho_0 S$. From (2.7) it follows that $b \in A_n + S$ for every $n \ge n_*$. Thus, for every $n \ge n_*$, we can choose $a_n \in A_n$ and $c_n \in S$ such that $b = a_n + c_n$. Obviously the sequence $\{a_n\}$ is bounded.

Further, for every $k \in \mathbb{N}$, let m_k be such that $h_k(A_n, A_0) < 1$ for each $n \ge m_k$. This means that

$$A_n \cap k\, S \subset A_0 + S \quad \text{for each} \ \ n \ge m_k. \tag{2.8}$$

Suppose that the first statement of condition (ii) does not hold. Then, for every $k \in \mathbb{N}$, there exists $n_k \ge \max\{k, m_k\}$ and a point $x_{n_k} \in A_{n_k}$ such that $\|x_{n_k}\| > k$. Without loss of generality we can assume that $k \ge \sup_{n\in\mathbb{N}} \|a_n\|$. Since the set A_{n_k} is convex and a_{n_k}, $x_{n_k} \in A_{n_k}$ satisfy that $\|a_{n_k}\| \le k$, $x_{n_k} > k$, there is $y_{n_k} \in A_{n_k}$ such that $\|y_{n_k}\| = k$. By (2.8) the sequence $\{y_{n_k}\}$ is bounded, a contradiction. Thus the first statement of condition (ii) holds. Obviously, if $\rho > M$ then $h_\rho(A_n, A) = h(A_n, A)$ for each $n \ge n_0$. This implies that $h(A_n, A_0) \to 0$ as $n \to \infty$. The proof is complete. $\square$

Remark 2.1. The assumption about convexity of sets A_n, $n \in \mathbb{N}$, and boundedness of A_0 in Proposition 2.1 cannot be dropped as shown by the examples bellow.

Example 2.1. Let $\{x_n\} \subset X$ be a sequence satisfying $\|x_n\| \to \infty$ and let $x_0 \in X$. Then define $\{A_n\} \subseteq \mathcal{A}(X)$ as follows.

$$A_0 = \{x_0\} \quad \text{and} \quad A_n = \{x_0, x_n\}, \quad n = 1, 2, \ldots.$$

Obviously, A_n converges to A_0 in the h_ρ-topology but $h(A_n, A_0) \to +\infty$.

Example 2.2. Let $X = \mathbb{R} \times \mathbb{R}$. Define, for $n \in \mathbb{N}$,

$$A_0 = \{(x, 0) : x \in \mathbb{R}\} \quad \text{and} \quad A_n = \{(x, x/n) : x \in \mathbb{R}\}.$$

Obviously $\{A_n\}$ converges to A_0 in the h_ρ-topology but $h(A_n, A_0) = +\infty$ for every $n \in \mathbb{N}$.

Definition 2.1. X is said to be

(i) strictly convex if, for any $x_1, x_2 \in S$, the condition $\|x_1 + x_2\| = 2$ implies that $x_1 = x_2$;
(ii) uniformly convex if, for any sequences $\{x_n\}, \{y_n\} \subseteq S$, the condition $\lim_{n\to\infty} \|x_n + y_n\| = 2$ implies that $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
(iii) (sequentially) Kadec if, for any sequence $\{x_n\} \subseteq S$ and $x \in S$, the condition $x_n \to x$ weakly implies that $\lim_{n\to\infty} \|x_n - x\| = 0$.

For a given set $G \in \mathcal{A}(X)$, we denote by $\mathcal{A}_G(X)$ the closure of the set $\{A \in \mathcal{A}(X) : \lambda_{AG} > 0\}$ with respect to the h_ρ-topology. Further, by $\mathcal{C}_G(X)$, $\mathcal{D}_G(X)$ and $\mathcal{K}_G(X)$ we denote the intersections of $\mathcal{A}_G(X)$ with $\mathcal{C}(X)$, $\mathcal{D}(X)$ and $\mathcal{K}(X)$, respectively. Then the main results are stated as follows.

Theorem 2.1. *Let $G \in \mathcal{A}^b(X)$. Suppose that X is a uniformly convex Banach space. Then the set of all $F \in \mathcal{C}_G(X)$ such that the minimization problem* $\min(F, G)$ *is well-posed is a dense G_δ-subset in $\mathcal{C}_G(X)$ with respect to the h_ρ-topology.*

Remark 2.2. The statement of Theorem 2.1 remains true with $\mathcal{A}_G(X)$ in the place of $\mathcal{C}_G(X)$.

Theorem 2.2. *Suppose that X is a strictly convex, Kadec Banach space. Let G be a nonempty closed relatively weakly compact subset of X. Then the set of all $F \in \mathcal{K}_G(X)$ (resp. $F \in \mathcal{D}_G(X)$, $F \in \mathcal{A}_G(X)$) such that the minimization problem* $\min(F, G)$ *is well-posed is a dense G_δ-subset in $\mathcal{K}_G(X)$ (resp. $\mathcal{D}_G(X)$, $\mathcal{A}_G(X)$) with respect to the h_ρ-topology.*

Theorem 2.3. *Suppose that X is a uniformly convex Banach space. Let $G \in \mathcal{A}(X)$. Then the set of all $F \in \mathcal{C}_G(X)$ such that the minimization problem* $\min(F, G)$ *fails to be well-posed is a set of the first Bair category in $\mathcal{C}_G(X)$ with respect to the h_ρ-topology.*

Theorem 2.4. *Suppose that X is a strictly convex, Kadec Banach space. Let G be a nonempty closed, relatively boundedly weakly compact subset of X. Then the set of all $F \in \mathcal{K}_G(X)$ such that the minimization problem* $\min(F, G)$ *fails to be well-posed is a set of the first Bair category in $\mathcal{K}_G(X)$ with respect to the h_ρ-topology.*

3. Proofs of the main results

Let F, $G \in \mathcal{A}(X)$ and $\sigma > 0$. We set $\|A\| = \sup_{a\in A} \|a\|$ and

$$L_F(G, \sigma) := \{g \in G : \, d(g, F) \le \lambda_{FG} + \sigma\}.$$

Obviously the set $L_F(G, \sigma)$ is nonempty, closed and $L_F(G, \sigma_1) \subseteq L_F(G, \sigma_2)$ if $\sigma_1 \le \sigma_2$. The following characterization of well-posedness is direct but very useful (see [2]).

Proposition 3.1. *Let* $G,\ F \in \mathcal{A}(X)$. *Then the problem* $\min(F, G)$ *is well-posed if and only if*

$$\inf_{\sigma>0} \mathrm{diam} L_G(F,\sigma) = 0 \quad \text{and} \quad \inf_{\sigma>0} \mathrm{diam} L_F(G,\sigma) = 0.$$

Proposition 3.2. *Let* $G \in \mathcal{A}^b(X)$. *Let* $m > 0$ *and let* $\rho > \|G\| + m$. *Then, for every* $A,\ B \in \mathcal{A}(X)$ *with* $\lambda_{AG} \le m$ *and* $\lambda_{BG} \le m$, *we have*

$$|\lambda_{AG} - \lambda_{BG}| \le h_\rho(A, B). \tag{3.1}$$

Proof. Let $A,\ B \in \mathcal{A}(X)$ be such that $\lambda_{AG} \le m$ and $\lambda_{BG} \le m$. Let $\epsilon > 0$ be such that

$$\|G\| + m + \epsilon < \rho. \tag{3.2}$$

Choose $b \in B$ and $a \in A$ such that

$$\lambda_{BG} > d(b, G) - \epsilon \quad \text{and} \quad \|a - b\| < d(b, A) + \epsilon.$$

Then

$$\lambda_{AG} - \lambda_{BG} \le d(a, G) - d(b, G) + \epsilon \le \|a - b\| + \epsilon \le d(b, A) + 2\epsilon. \tag{3.3}$$

Obviously

$$\|b\| \le d(b, G) + \|G\| \le \lambda_{BG} + \epsilon + \|G\| \le \|G\| + m + \epsilon.$$

From this and (3.2) it follows immediately that

$$d(b, A) \le \sup_{y \in B \cap \rho S} d(y, A) \le h_\rho(A, B).$$

Using the last inequality in (3.3) we get

$$\lambda_{AG} - \lambda_{BG} \le h_\rho(A, B) + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lambda_{AG} - \lambda_{BG} \le h_\rho(A, B).$$

Changing the order of A and B, we obtain (3.1). □

Given $G \in \mathcal{A}(X)$ and $k \in \mathbb{N}$ define

$$\mathcal{L}_k = \Big\{ F \in \mathcal{C}_G(X) : \inf_{\sigma>0} \mathrm{diam} L_G(F,\sigma) < \epsilon_k \ \text{ and } \ \inf_{\sigma>0} \mathrm{diam} L_A(G,\sigma) < \epsilon_k \Big\}, \tag{3.4}$$

where $\epsilon_k = 1/k$.

Lemma 3.1. *Let* $G \in \mathcal{A}^b(X)$. *Then for every* $k \in \mathbb{N}$ *the set* $\mathcal{L}_k$ *is open in* $\mathcal{C}_G(X)$ *with respect to the* h_ρ*-topology.*

Proof. Fix $k \in \mathbb{N}$ and let $A \in \mathcal{L}_k$. Set

$$\theta = \max\Big\{ \inf_{\sigma>0} \mathrm{diam} L_G(A,\sigma),\ \inf_{\sigma>0} \mathrm{diam} L_A(G,\sigma) \Big\}. \tag{3.5}$$

Fix $m > \lambda_{AG}$ and take $\eta > 0$ such that

$$\lambda_{AG} + \eta < m, \quad \theta + 2\eta < \epsilon_k. \tag{3.6}$$

Let $\tilde{a} \in A$ be such that

$$d(\tilde{a}, G) < \lambda_{AG} + \frac{\eta}{2} \tag{3.7}$$

and let

$$\rho > \|G\| + \|\tilde{a}\| + m.$$

By (3.5), there exists $\sigma_1 > 0$ such that

$$\|G\| + m + \sigma_1 < \rho \tag{3.8}$$

and

$$\max\Big\{\mathrm{diam} L_G(A, \sigma_1),\ \mathrm{diam} L_A(G, \sigma_1)\Big\} < \theta + \eta.$$

Fix $\sigma_2 \in (0, \sigma_1)$ and set

$$\delta = \min\Big\{\frac{1}{3}(\sigma_1 - \sigma_2), \frac{\eta}{2}\Big\}. \tag{3.9}$$

We will show that

$$\mathcal{U}_\rho(A, \delta) \cap \mathcal{C}_G(X) \subseteq \mathcal{L}_k, \tag{3.10}$$

where $\mathcal{U}_\rho(A, \delta)$ is given by (2.6) (with δ in place of r).

Let $B \in \mathcal{U}_\rho(A, \delta) \cap \mathcal{C}_G(X)$. Note that $\tilde{a} \in A \cap \rho S \subseteq B + \delta S$. It follows from conditions (3.7) and (3.6) that

$$\lambda_{BG} \le d(\tilde{a}, G) + \delta < \lambda_{AG} + \frac{\eta}{2} + \delta \le \lambda_{AG} + \eta \le m. \tag{3.11}$$

This means that A, B satisfy the hypotheses of Proposition 3.2 and consequently the condition (3.1) holds.

Let $g \in L_B(G, \sigma_2)$, i.e., $g \in G$ and

$$d(g, B) \le \lambda_{BG} + \sigma_2. \tag{3.12}$$

Let $b \in B$ be such that

$$\|g - b\| < d(g, B) + \delta. \tag{3.13}$$

From the last inequality, (3.12), (3.11). (3.9) and (3.8), we have

$$\|b\| \le \|g\| + d(g, B) + \delta \le \|G\| + \lambda_{BG} + \sigma_2 + \delta \le \|G\| + m + \sigma_1 < \rho.$$

Hence $b \in B \cap \rho S$ and

$$d(b, A) \le \sup_{y \in B \cap \rho S} d(y, A) \le h_\rho(A, B). \tag{3.14}$$

Consequently, by (3.13), (3.14), (3.12), (3.1) and (3.9) we get

$$\begin{aligned} d(g, A) &\le \|g - b\| + d(b, A) \le d(g, B) + \delta + h_\rho(A, B) \\ &\le \lambda_{BG} + \sigma_2 + \delta + h_\rho(A, B) \le \lambda_{AG} + \sigma_2 + \delta + 2h_\rho(A, B) \\ &\le \lambda_{AG} + \sigma_2 + 3\delta \le \lambda_{AG} + \sigma_1. \end{aligned}$$

This shows that $L_B(G, \sigma_2) \subseteq L_A(G, \sigma_1)$ because $g \in L_B(G, \sigma_2)$ is arbitrary. Consequently

$$\mathrm{diam} L_G(B, \sigma_2) \le \mathrm{diam} L_G(A, \sigma_1) < \epsilon_k. \tag{3.15}$$

Now, let $b \in L_G(B, \sigma_2)$ i.e. $b \in B$ and

$$d(b, G) \le \lambda_{BG} + \sigma_2. \tag{3.16}$$

Obviously $\|b\| \le \|G\| + \lambda_{BG} + \sigma_2$ and by (3.11), (3.8) and the choice of σ_2 we have $\|b\| \le \rho$. From this and the fact that $A \cap \rho S \ne \emptyset$ it follows easily that

$$d(b, A) \le h_\rho(A, B) < \delta. \tag{3.17}$$

Let $a \in A$ be such that $\|b - a\| < \delta$. Then, by (3.16), (3.1) and (3.9) we obtain

$$d(a, G) \le d(b, G) + \delta \le \lambda_{BG} + \sigma_2 + \delta \le \lambda_{AG} + \sigma_2 + 2\delta \le \lambda_{AG} + \sigma_1.$$

This means that $a \in L_A(G, \sigma_1)$ and so $b \in L_A(G, \sigma_1) + \delta S$. Since $b \in L_G(B, \sigma_2)$ is arbitrary, $L_B(G, \sigma_2) \subseteq L_A(G, \sigma_1) + \delta S$, which implies

$$\mathrm{diam} L_B(G, \sigma_2) \le \mathrm{diam} L_A(G, \sigma_1) + 2\delta < \epsilon_k. \tag{3.18}$$

From (3.12) and (3.18) it follows that $B \in \mathcal{L}_k$. Since $B \in \mathcal{U}_\rho(A, \delta) \cap \mathcal{C}_G(X)$ is arbitrary, the proof of Lemma 3.1 is complete. □

The following example shows that Lemma 3.1 may fail if $G \in \mathcal{A}(X)$.

Example 3.1. Let $X = \mathbb{R} \times \mathbb{R}$. Define $G = \{(-1, 0)\} \bigcup \bigcup_{n=1}^{\infty} \{(2n, 2)\}$. Let $A_0 = [0, +\infty) \times \{0\}$ and $A_n = \{(t, t/(2n)) : t \in [0, 2n]\}$ for each $n \in \mathbb{N}$. Obviously $A_n \to A_0$ in the h_ρ-topology, $A_0 \in \mathcal{L}_k$ but $A_n \notin \mathcal{L}_k$ for each $k \in \mathbb{N}$. This means that, for arbitrary $k \in \mathbb{N}$, the set $\mathcal{L}_k$ is not open with respect to the h_ρ-topology.

Lemma 3.2. *Let $G \in \mathcal{A}(X)$. Then, for every $k \in \mathbb{N}$, the set $\mathcal{C}_G^b(X) \cap \mathcal{L}_k$ is contained in the set $\mathrm{int}\,\mathcal{L}_k$, where $\mathrm{int}\,\mathcal{L}_k$ denotes the interior of $\mathcal{L}_k$ with respect to the h_ρ-topology.*

Proof. Let $A \in \mathcal{C}^b(X) \cap \mathcal{L}_k$. Then, by Proposition 2.1, there exist $M > 0$, $\bar\rho > 0$ and $\bar\epsilon > 0$ such that

$$\|B\| \le M \quad \text{for each } B \in \mathcal{U}_{\bar\rho}(A, \bar\epsilon) \cap \mathcal{C}_G(X).$$

Indeed, otherwise, then, for every $k \in \mathbb{N}$, there is $B_k \in \mathcal{U}_k(A, 1/k)$ such that $\|B_k\| \ge k$, which is a contradiction by Proposition 2.1. Let $\rho > M$ and let $\delta > 0$ be defined as in the proof of Lemma 3.1. Using the similar arguments as in the proof of Lemma 3.1 one can show that $\mathcal{U}_\rho(A, \delta) \cap \mathcal{C}_G(X) \subseteq \mathcal{L}_k$, which completes the proof. □

Remark 3.1. Note that Lemma 3.2 fails if the class $\mathcal{C}_G^b(X)$ is replaced by $\mathcal{A}_G^b(X)$ or $\mathcal{D}_G^b(X)$ showed by the following examples.

Example 3.2. Let X be a Banach space and let $x_0 \in X$ such that $\|x_0\| = 1$. Let $x_n = (n+1)x_0$ for $n \in \mathbb{N}$. Define

$$G = \left\{\frac{4}{3}x_0\right\} \cup \bigcup_{n=1}^{\infty} \left\{x \in X : \|x - x_n\| = \frac{1}{3}\right\},$$

$$A_0 = \{x_0\} \quad \text{and} \quad A_n = \{x_0, x_n\}, \quad n \in \mathbb{N}.$$

Clearly $A_n \to A_0$ in h_ρ-topology and $A_0 \in \mathcal{L}_k$ but $A_n \notin \mathcal{L}_k$ for each $k \in \mathbb{N}$. This means that $A_0 \notin \mathrm{int}\,\mathcal{L}_k$ for each $k \in \mathbb{N}$.

Let $\mathcal{V}(G)$ denote the set of all $F \in \mathcal{A}_G^b(X)$ such that the minimization problem $\min(F, G)$ is well-posed. Define

$$\mathcal{V}_C(G) = \mathcal{C}^b(X) \cap \mathcal{V}_B(G), \quad \mathcal{V}_D(G) = \mathcal{D}^b(X) \cap \mathcal{V}_B(G), \quad \mathcal{V}_K(G) = \mathcal{K}^b(X) \cap \mathcal{V}_B(G).$$

The following observations are known (see [2], [12], [13]).

Proposition 3.3. *Let $G \in \mathcal{A}(X)$. Then the following assertions hold.*

(i) *If X is uniformly convex, then $\mathcal{V}_C(G)$ is a dense G_δ-subset of $\mathcal{C}_G^b(X)$.*

(ii) *If X is strictly convex Kadec and G is relatively boundedly weakly compact, then $\mathcal{V}_B(G)$ (resp. $\mathcal{V}_D(G)$, $\mathcal{V}_K(G)$) is a dense G_δ-subset of $\mathcal{A}_G^b(X)$ (resp. $\mathcal{D}_G^b(X)$, $\mathcal{K}_G^b(X)$).*

Now we are ready to prove the main theorems. Here we only give the proofs of Theorems 2.1 and 2.3 because the proofs of Theorems 2.2 and 2.4 are almost the same as Theorems 2.1 and 2.3 respectively.

Proof of Theorem 2.1. For $k \in \mathbb{N}$, let $\mathcal{L}_k$ be defined by (3.4). Let

$$\mathcal{L}_0 = \bigcap_{k\geq 1} \mathcal{L}_k. \tag{3.17}$$

Then, for $F \in \mathcal{C}_G(X)$, by Proposition 3.1, the minimization problem $\min(F,G)$ is well-posed if and only if $F \in \mathcal{L}_0$. By virtue of Lemma 3.1 it suffices to show that each $\mathcal{L}_k$ is dense in $\mathcal{C}_G(X)$ with respect to the h_ρ-topology. For this end, let $\mathcal{V}_0^b(G)$ denote the set of all $F \in \mathcal{C}_G^b(X)$ such that the minimization problem $\min(F,G)$ is well-posed. Then, by Proposition 3.3, the set $\mathcal{V}_0^b(G)$ is dense in $\mathcal{C}_G^b(X)$ with respect to the Hausdorff distance and so it is dense in $\mathcal{C}_G^b(X)$ with respect to the h_ρ-topology. Since the set $\mathcal{C}_G^b(X)$ is dense in $\mathcal{C}_G(X)$ with respect to the h_ρ-topology, it follows that $\mathcal{V}_0^b(G)$ is dense in $\mathcal{C}_G(X)$ with respect to h_ρ-topology. Clearly $\mathcal{V}_0^b(G) \subseteq \mathcal{L}_k$. Hence $\mathcal{L}_k$ is dense in $\mathcal{C}_G(X)$ with respect to the h_ρ-topology. The proof is complete. □

Proof of Theorem 2.3. Let $\mathcal{L}_k$ and $\mathcal{L}_0$ be defined by (3.4) and (3.17), respectively. Then, for $F \in \mathcal{C}_G(X)$, the minimization problem $\min(F,G)$ is not well-posed if and only if $F \in \mathcal{C}_G(X) \setminus \mathcal{L}_0$. Since

$$\mathcal{C}_G(X) \setminus \mathcal{L}_0 = \bigcup_{k=1}^{\infty} (\mathcal{C}_G(X) \setminus \mathcal{L}_k),$$

it suffices to verify that each $\mathcal{C}_G(X) \setminus \mathcal{L}_k$ is nowhere dense in $\mathcal{C}_G(X)$. Let $F \in \mathcal{C}_G(X) \setminus \mathcal{L}_k$ and let $\mathcal{U}_\rho(F,\epsilon)$ be any open neighbourhood of F in $\mathcal{C}_G(X)$. Then, as in the proof of Theorem 2.1, one can verify that there exists a bounded set $A \in \mathcal{U}_\rho(F,\epsilon)$ such that the minimization problem $\min(A,G)$ is well-posed; hence $A \in \mathcal{L}_k$. By Lemma 3.2, there exists an open neighbourhood of A, say $\mathcal{U}(A)$, such that $\mathcal{U}(A) \subseteq \mathcal{L}_k$; that is, $\mathcal{U}(A) \cap \mathcal{C}_G(X) \setminus \mathcal{L}_k = \emptyset$. This shows that $\mathcal{C}_G(X) \setminus \mathcal{L}_k$ is nowhere dense in $\mathcal{C}_G(X)$ and the proof is complete. □

4. A porosity result

Let $\{d_\rho : \rho \in \Gamma\}$ be a family of gauges on $\mathfrak{X}$. Assume that $\mathfrak{X}$ is endowed with the topology generated by $\{d_\rho\}_{\rho\in\Gamma}$. Moreover assume that Γ is an ordered set and for every $\rho_1, \rho_2 \in \Gamma$, $\rho_1 \leq \rho_2$ we have

$$d_{\rho_1}(x,y) \leq d_{\rho_2}(x,y) \quad \text{for all } x, y \in \mathfrak{X}.$$

A subset Y of $\mathfrak{X}$ is said to be *porous* in $\mathfrak{X}$ if there exist $s \in (0,1]$, $r_0 > 0$ and $\rho_0 \in \Gamma$ such that for every $x \in \mathfrak{X}$, $r \in (0, r_0]$ and $\rho \in \Gamma$ with $\rho \geq \rho_0$, there is a point

$y \in \mathfrak{X}$ such that $U_{d_\rho}(y, sr) \subseteq U_{d_\rho}(x, r) \cap (\mathfrak{X} \setminus Y)$. A subset Y is said to be σ-porous in $\mathfrak{X}$ if it is a countable union of sets which are porous in $\mathfrak{X}$.

Theorem 4.1. *Suppose that X is a uniformly convex Banach space. Let $G \in \mathcal{A}^b(X)$. Then the set $\mathcal{A}_G(X) \setminus \mathcal{V}(G)$ is σ-porous in $\mathcal{A}_G(X)$.*

To prepare the proof of Theorem 4.1, we will need the following version of Steckin's lens lemma (see [2,13]) and some other lemmas.

Lemma 4.1. *Let X be a uniformly convex Banach space. Let $r_0 > 0$ be arbitrary. Then for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every $0 < \delta \leq \delta(\epsilon)$, $0 < r \leq r_0$, and $x, y \in X$ satisfying $0 < \|x - y\| \leq r/2$ we have*

$$\mathrm{diam} D(x, y, r, \delta) < \epsilon,$$

where

$$D(x, y, r, \delta) = \Big\{ z \in X : \|z - y\| \leq r - \|x - y\|(1 - \delta) \text{ and } \|z - x\| \geq r \Big\}. \tag{4.1}$$

Let $G \in \mathcal{A}^b(X)$ be fixed. Denote by $\mathcal{V}(G)$ the set of all $F \in \mathcal{A}_G(X)$ such that the minimization problem $\min(F, G)$ is well-posed. For $F \in \mathcal{V}(G)$, let (f_F, g_F) denote the unique solution to the problem $\min(F, G)$. For $\alpha \in [0, 1]$, set

$$u_{F,\alpha} = (1 - \alpha) f_F + \alpha g_F$$

and

$$F_\alpha = F \cup \{u_{F,\alpha}\}.$$

Now define

$$\widetilde{\mathcal{A}} = \bigcap_{n \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{F \in \mathcal{V}(G)} \bigcup_{\alpha \in [0,1/2]} \mathcal{U}_{\rho_n}(F_\alpha, \gamma_{F,\alpha,k}),$$

where

$$\rho_n = n, \quad \gamma_{F,\alpha,k} = \frac{1}{k} \min \Big\{ d\big(u_{F,\alpha}, F\big), 1 \Big\}.$$

Lemma 4.2. *Let X be a uniformly convex Banach space. Let $G \in \mathcal{A}^b(X)$. Then $\tilde{\mathcal{A}} \subseteq \mathcal{V}(G)$.*

Proof. Let $F \in \tilde{\mathcal{A}}$. By virtue of Proposition 3.1, it suffices to show that

$$\lim_{\delta \to 0+} \mathrm{diam} L_G(F, \delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0+} \mathrm{diam} L_F(G, \delta) = 0. \tag{4.2}$$

By the definition of $\widetilde{\mathcal{A}}$, for each $n, k \in \mathbb{N}$, there exist $F_{nk} \in \mathcal{V}(G)$ and $\alpha_{nk} \in [0, 1/2]$ such that

$$h_{\rho_n}(F, \widetilde{F}_{nk}) \leq \delta_{nk}, \tag{4.3}$$

where

$$\widetilde{F}_{nk} = F_{nk} \cup \{u_{nk}\}, \quad u_{nk} = (1 - \alpha_{nk}) f_{F_{nk}} + \alpha_{nk} g_{F_{nk}}, \quad \delta_{nk} = \gamma_{F_{nk}, \alpha_{nk}, k}.$$

For notational convenience, set

$$\lambda_{nk} = \lambda_{F_{nk}G}, \quad \widetilde{\lambda}_{nk} = \lambda_{\widetilde{F}_{nk}G}, \quad \bar{r} = \lambda_{FG} + 1, \quad r_* = \|G\| + \lambda_{FG} + 1.$$

Observe that

$$\widetilde{\lambda}_{nk} = \big(1 - \alpha_{nk}\big)\lambda_{nk}, \quad d(u_{nk}, F_{nk}) = \alpha_{nk}\lambda_{nk}, \quad \delta_{nk} \leq \frac{1}{k}\alpha_{nk}\lambda_{nk}. \tag{4.4}$$

Without loss of generality we can assume that $\alpha_{nk} > 0$ for all $n, k \in \mathbb{N}$.

We claim that

$$L_G(\widetilde{F}_{nk}, 4\delta_{nk}) = \{u_{nk}\} \quad \text{for all } k > 4 \text{ and } n > r_*. \tag{4.5}$$

To see this, let $f \in L_G(\widetilde{F}_{nk}, 4\delta_{nk})$, where $k > 4$ and $n > r_*$. Using (4.4) we obtain

$$d(f, G) \leq \widetilde{\lambda}_{nk} + 4\delta_{nk} \leq (1 - \alpha_{nk})\lambda_{nk} + \frac{4}{k}\,\alpha_{nk}\lambda_{nk} < \lambda_{nk}. \tag{4.6}$$

This means that $f \notin F_{nk}$ and so $f = u_{nk}$. Thus (4.5) holds.

On the other hand, by Proposition 3.2, we have

$$|\lambda_{FG} - \tilde{\lambda}_{nk}| \leq h_{\rho_n}(F, \tilde{F}_{nk}) \leq \delta_{nk}. \tag{4.7}$$

Now we can show that

$$\operatorname{diam} L_G(F, \delta_{nk}) \leq 4\delta_{nk} \quad \text{for all } k > 4 \text{ and } n > r_*. \tag{4.8}$$

Indeed, let $k > 4$, $n > r_*$ and let $f \in L_G(F, \delta_{nk})$ be arbitrary. Since $h_{\rho_n}(F, \widetilde{F}_{nk})) \leq \delta_{nk}$, there exists $\bar{f} \in \widetilde{F}_{nk}$ such that $\|f - \bar{f}\| \leq 2\delta_{nk}$. By the definition of $L_G(F, \delta_{nk})$ and (4.7) we have

$$d(\bar{f}, G) \leq \|f - \bar{f}\| + d(f, G) \leq \lambda_{FG} + 3\delta_{nk} < \lambda_{nk} + 4\delta_n k.$$

Thus $\bar{f} \in L_G(\widetilde{F}_{nk}, 4\delta_{nk})$ and from (4.4) it follows that $\bar{f} = u_{nk}$. Consequently, for any $f_1, f_2 \in L_G(F, \delta_{nk})$, we have

$$\|f_1 - f_2\| \leq \|f_1 - u_{nk}\| + \|u_{nk} - f_2\| \leq 4\delta_{nk}.$$

Then (4.8) is proved whence the first relation of (4.2) follows immediately.

To complete the proof, it remains to verify the second relation of (4.2). Let $D(x, y, r, \delta)$ be defined by (4.1). We claim that

$$L_{\widetilde{F}_{nk}}(G, 3\delta_{nk}) \subseteq D\big(f_{F_{nk}},\, u_{nk},\, \lambda_{nk},\, 4/k\big) \quad \text{for all } k > 4 \text{ and } n > r_*. \tag{4.9}$$

In fact, using (4.4), for arbitrary $g \in L_{\widetilde{F}_{nk}}(G, 3\delta_{nk})$, we obtain

$$d(g, \widetilde{F}_{nk}) \leq \widetilde{\lambda}_{nk} + 3\delta_{nk} = (1 - \alpha_{nk})\lambda_{nk} + 3\delta_{nk}.$$

Now, taking $f_{nk} \in \widetilde{F}_{nk}$ such that

$$\|g - f_{nk}\| \leq (1 - \alpha_{nk})\lambda_{nk} + 4\delta_{nk}, \tag{4.10}$$

we have

$$d(f_{nk}, G) \leq \|g - f_{nk}\| \leq (1 - \alpha_{nk})\lambda_{nk} + 4\delta_{nk} \leq \lambda_{nk} - (1 - 4/k)\alpha_{nk}\lambda_{nk} < \lambda_{nk}.$$

This implies that $f_{nk} = u_{nk}$. Hence, by (4.10) and (4.4), we have

$$\begin{aligned}\|g - u_{nk}\| &= \|g - f_{nk}\| \leq (1 - \alpha_{nk})\lambda_{nk} + 4\delta_{nk} \\ &\leq (1 - \alpha_{nk})\lambda_{nk} + \frac{4}{k}\alpha_{nk}\lambda_{nk} = \lambda_{nk} - (1 - 4/k)\|u_{nk} - f_{F_{nk}}\|.\end{aligned}$$

Since $\|g - f_{F_{nk}}\| \geq \lambda_{nk}$, it follows that $g \in D(f_{F_{nk}}, u_{nk}, \lambda_{nk}, 4/k)$ and so (4.9) is proved.

Finally, for arbitrary $g \in L_F(G, \delta_{nk})$ using (4.3) and Proposition 3.2, we have

$$d(g, \widetilde{F}_{nk}) \leq d(g, F) + \delta_{nk} \leq \lambda_{FG} + 2\delta_{nk} \leq \widetilde{\lambda}_{nk} + 3\delta_{nk}.$$

This means that

$$L_F(G, \delta_{nk}) \subset L_{\widetilde{F}_{nk}}\big(G, 3\delta_{nk}\big).$$

Combining the last inclusion and (4.9), we obtain

$$L_F(G, \delta_{nk}) \subseteq D\big(f_{F_{nk}},\, u_{nk},\, \lambda_{nk},\, 4/k\big) \quad \text{for all } k > 4 \text{ and } n > r_*. \tag{4.11}$$

From (4.11) and Lemma 4.1 the second relation of (4.2) follows. The proof is complete. □

Proof of Theorem 4.1. For $n, k, l \in \mathbb{N}$, set

$$\mathcal{A}_{nk} = \mathcal{A}_G(X) \setminus \bigcup_{F \in \mathcal{V}(G)} \bigcup_{\alpha \in [0,1/2]} \mathcal{U}_{\rho_n}(F_\alpha, \gamma_{F,\alpha,k})$$

and

$$\mathcal{A}^l_{nk} = \{F \in \mathcal{A}_{nk} : 1/l < \lambda_{FG} < l\}.$$

By Lemma 4.2, we have

$$\mathcal{A}_G(X) \setminus \mathcal{V}(G) \subseteq \mathcal{A}_G(X) \setminus \widetilde{\mathcal{A}} = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} \mathcal{A}^l_{nk}.$$

To complete the proof it suffices to show that $\mathcal{A}^l_{nk}$ is porous in $\mathcal{A}_G(X)$ for every $n, k, l \in \mathbb{N}$.

Let $n, k, l \in \mathbb{N}$. Define

$$r_0 = \frac{1}{2l}, \quad s = \frac{1}{4k}, \quad \rho_0 = 2(l + \|G\|).$$

Let $F \in \mathcal{A}^l_{nk}$, $0 < r \le r_0$ and $\rho \ge \rho_0$ be arbitrary. Let η be such that

$$0 < \eta < \frac{r}{4} \quad \text{and} \quad \frac{1}{l} - \eta < \lambda_{FG} < l + \eta. \tag{4.12}$$

By Theorem 2.1, there exists $\widehat{F} \in \mathcal{V}(G)$ such that

$$h_\rho(F, \widehat{F}) < \eta. \tag{4.13}$$

Taking $f_\eta \in F$ such that

$$d(f_\eta, G) \le \lambda_{FG} + \eta,$$

we have

$$\|f_\eta\| \le \|G\| + \lambda_{FG} + \eta < \rho.$$

Hence, by (4.13),

$$f_\eta \in F \cap \rho S \subset \widehat{F} + \eta S,$$

which in turns implies that

$$\lambda_{\widehat{F}G} \le d(f_\eta, G) + \eta \le \lambda_{FG} + 2\eta < 2l. \tag{4.14}$$

Thus we can apply Proposition 3.2 to conclude that

$$|\lambda_{\widehat{F}G} - \lambda_{FG}| \le h_\rho(F, \widehat{F}) < \eta. \tag{4.15}$$

From (4.15), (4.12) and (4.13), it follows that

$$h_\rho(F, \widehat{F}) < \frac{r}{4} \quad \text{and} \quad \frac{1}{l} < \lambda_{\widehat{F}G} < l.$$

In particular this implies that $\lambda_{\widehat{F}G} > 2r$. Set

$$\widehat{u}_{1/2} = (f_{\widehat{F}} + g_{\widehat{F}})/2.$$

Observe that $\|\widehat{u}_{1/2}\| < \rho$ because $\|g_{\widehat{F}}\| \leq \|G\| < \rho$ and $\|f_{\widehat{F}}\| \leq \|f_{\widehat{F}} - g_{\widehat{F}}\| + \|g_{\widehat{F}}\| \leq \lambda_{\widehat{F},G} + \|G\| \leq 2l + \|G\| < \rho$.

Putting $\widehat{F}_{1/2} = \widehat{F} \cup \{\widehat{u}_{1/2}\}$ we have

$$\begin{aligned} h_\rho(\widehat{F}_{1/2}, F) &\geq h_\rho(\widehat{F}_{1/2}, \widehat{F}) - h_\rho(\widehat{F}, F) \\ &\geq d(\widehat{u}_{1/2}, \widehat{F}) - r/4 = (1/2)\lambda_{\widehat{F}G} - r/4 \geq 3r/4. \end{aligned}$$

It follows that there exists $\alpha \in [0, 1/2]$ such that $h_\rho(\widehat{F}_\alpha, F) = 3r/4$. Since, for each $A \in \mathcal{U}_\rho(\widehat{F}_\alpha, sr)$,

$$h_\rho(A, F) \leq h_\rho(A, \widehat{F}_\alpha) + h_\rho(\widehat{F}_\alpha, F) \leq sr + 3r/4 \leq r,$$

we have that

$$\mathcal{U}_\rho(\widehat{F}_\alpha, sr) \subseteq \mathcal{U}_\rho(F, r).$$

Thus, in order to complete the proof, we need to show that

$$\mathcal{U}_\rho(\widehat{F}_\alpha, sr) \subseteq \mathcal{A}_G(X) \setminus \mathcal{A}^l_{nk}.$$

Since $\mathcal{A}^l_{nk} \subseteq \mathcal{A}_{nk}$ and $\mathcal{A}_{nk} \cap \mathcal{U}_\rho\big(\widehat{F}_\alpha, \gamma_{\widehat{F}_\alpha,\alpha,k}\big) = \emptyset$, It suffices to show that

$$\mathcal{U}_\rho(\widehat{F}_\alpha, sr) \subseteq \mathcal{U}_\rho\big(\widehat{F}_\alpha, \gamma_{\widehat{F}_\alpha,\alpha,k}\big). \tag{4.16}$$

From relations $h_\rho(\widehat{F}_\alpha, F) = 3r/4$ and $h_\rho(F, \widehat{F}) < r/4$ it follows that

$$h_\rho\big(\widehat{F}_\alpha, \widehat{F}\big) \geq h_\rho\big(\widehat{F}_\alpha, F\big) - h_\rho\big(F, \widehat{F}\big) \geq r/2. \tag{4.17}$$

Moreover, since $\|u_{\widehat{F},\alpha}\| < \rho$ we have

$$d(u_{\widehat{F},\alpha}, \widehat{F}) = h_\rho\big(\widehat{F}_\alpha, \widehat{F}\big) \tag{4.18}$$

By (4.17), (4.18) and equality $s = 1/(4k)$, we have

$$sr \leq 2s h_\rho(\widehat{F}_\alpha, \widehat{F}) \leq d(u_{\widehat{F},\alpha}, \widehat{F})/k.$$

Since $d(u_{\widehat{F},\alpha}, \widehat{F}) < 1$, this implies that $sr \leq \gamma_{\widehat{F}_\alpha,\alpha,k}$ and so (4.16) holds. The proof is complete. □

Similar result can be given for the class $\mathcal{D}_G(X)$.

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