Journal of Nonlinear and Convex Analysis
Volume 8, Number 1, 2007, 99–103
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

AN APPLICATION OF THE RESOLVENT METHOD TO RIGIDITY THEORY FOR HOLOMORPHIC MAPPINGS

MARINA LEVENSHTEIN, SIMEON REICH, AND DAVID SHOIKHET

Abstract. Let f be the generator of a one-parameter continuous semigroup of holomorphic self-mappings of the open unit disk Δ in the complex plane. We use the resolvent method to show that if for some boundary point τ of Δ, the angular limit $\angle\lim_{z\to\tau}\frac{f(z)}{|z-\tau|^3}=0$, then f vanishes identically in Δ.

We denote by $\mathrm{Hol}(\Delta, D)$ the set of all holomorphic functions on the open unit disk $\Delta = \{z : |z| < 1\}$ which map Δ into a set $D \subset \mathbb{C}$, and by $\mathrm{Hol}(\Delta)$ the set of all holomorphic self-mappings of Δ.

A family $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ is called **a one-parameter continuous semigroup on** Δ (a semigroup, in short) if

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$
and
(ii) $\lim_{t\to 0^+} F_t(z) = z$ for all $z \in \Delta$.

It follows from a result of E. Berkson and H. Porta [3] that each semigroup is differentiable with respect to $t \in \mathbb{R}^+ = [0,\infty)$. So, for each one-parameter continuous semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$, the limit

$$\lim_{t\to 0^+}\frac{z-F_t(z)}{t} = f(z), \quad z \in \Delta,$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. This mapping f is called the **(infinitesimal) generator** of $S = \{F_t\}_{t\geq 0}$.

Recall that $\tau \in \overline{\Delta}$ is a fixed point of $F \in \mathrm{Hol}(\Delta)$ if either $F(\tau) = \tau$, where $\tau \in \Delta$, or $\lim_{r\to 1^-} F(r\tau) = \tau$, where $\tau \in \partial\Delta = \{z : |z| = 1\}$. If F is not an automorphism of Δ with an interior fixed point, then by the Schwarz–Pick lemma and the Julia–Wolff–Carathéodory theorem, there is a unique fixed point $\tau \in \overline{\Delta}$ such that for each $z \in \Delta$, $\lim_{n\to\infty} F_n(z) = \tau$, where the n-th iterate F_n of F is defined by $F_1 = F$, $F_n = F \circ F_{n-1}$, $n = 2, 3, \ldots$. Moreover, if $\tau \in \Delta$, then $|F'(\tau)| < 1$, and if $\tau \in \partial\Delta$, then the so-called angular derivative at the point τ, $F'(\tau) \in (0, 1]$. This point is called the Denjoy–Wolff point (or sink point) of F. We say that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has an angular limit L at a point $\tau \in \partial\Delta$ and write $L := \angle\lim_{z\to\tau} f(z)$ if $f(z) \to L$ as $z \to \tau$ in each nontangential approach region $\Gamma_{\tau,k} = \left\{z \in \Delta : \frac{|z-\tau|}{1-|z|} < k\right\}$, where $k > 1$. In a similar way, we define the angular derivative at τ as $\angle\lim_{z\to\tau} f'(z)$.

2000 *Mathematics Subject Classification.* Primary 30C45, 30D05, 47H20.

Key words and phrases. Angular derivative, boundary fixed point, continuous one-parameter semigroup, Denjoy-Wolff point, holomorphic generator, resolvent.

The problem of finding conditions for a holomorphic function f on Δ to coincide identically with a given function g when they behave similarly in a neighborhood of a point of $\overline{\Delta}$ was studied by many mathematicians. For example, if for some point $\tau \in \Delta$, a holomorphic self-mapping f of Δ satisfies the equalities $f(\tau) = \tau$ and $f'(\tau) = 1$, then f is the identity mapping I on Δ by the Schwarz lemma. In the boundary case ($\tau \in \partial\Delta$) the problem was studied for $g = I$ in the papers of F. Bracci, R. Tauraso and F. Vlacci [4], D. M. Burns and S. G. Krantz [5], and R. Tauraso [15]. In particular, it has been shown in [4] and [5] that if a holomorphic self-mapping of Δ coincides with the identity up to the third order of expansion at a boundary point, then it is, in fact, the identity mapping.

For a constant mapping g the following facts are known. If $f \in \mathrm{Hol}(\Delta, \overline{\Delta})$, then the conditions $\lim_{r\to 1^-} f(r\tau) = \tau$ and $\lim_{r\to 1^-} f'(r\tau) = 0$ for some $\tau \in \partial\Delta$ imply that $f \equiv \tau$ by the Julia–Wolff–Carathéodory theorem. S. Migliorini and F. Vlacci [10] proved that if the image of a holomorphic function f on Δ is contained in a wedge of angle $\pi\alpha$, $0 < \alpha \le 2$, with vertex at the origin, and f satisfies the equality $\angle \lim_{z\to\tau} \frac{f(z)}{(z-\tau)^\alpha} = 0$, then $f \equiv 0$.

More recently, M. Elin, M. Levenshtein, D. Shoikhet and R. Tauraso have established in [7] a new rigidity principle for holomorphic generators of one-parameter continuous semigroups. We formulate this principle as the Theorem below and present a different proof of it, based on the resolvent method which plays an important rôle in the study of nonlinear semigroups (see, for example, [2], [11] and [13]).

Let f be a function defined on Δ. Consider the equation

$$w = z + sf(z), \quad w \in \Delta, \quad s > 0. \tag{1}$$

If $J_s := (I + sf)^{-1}$ $(s > 0)$ is a well-defined (single-valued) self-mapping of Δ, it is called the **(nonlinear) resolvent** of f and for each $w \in \Delta$ and $s > 0$, the solution of equation (1) is then $z = J_s(w)$. It has been established in [12] that J_s $(s > 0)$ is a well-defined holomorphic self-mapping of Δ if and only if f is a holomorphic generator. Moreover, the set of null points of a generator coincides with the set of fixed points of its resolvent (see [13], Lemma 8.2). So for the study of the null point sets of generators (for instance, to determine their structure, to solve the existence problem and to find methods for approximating null points), it is often convenient to investigate the fixed point sets of their resolvents. We take this approach in our proof. For other recent applications of the resolvent method see, for example, [8] and [9] and the references therein.

First we quote [4, Theorem 2.4(4)], a result which we are going to use in the proof of the Theorem.

Proposition. *Let F be a holomorphic self-mapping of Δ. If there exists $\tau \in \partial\Delta$ such that $\lim_{r\to 1^-} F(r\tau) = \tau$, $\lim_{r\to 1^-} F'(r\tau) = 1$ and $\lim_{r\to 1^-} F''(r\tau) = \lim_{r\to 1^-} F'''(r\tau) = 0$, then $F \equiv I$.*

Theorem. *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ be the generator of a one-parameter continuous semigroup. Suppose that for some $\tau \in \partial\Delta$,*

$$\angle \lim_{z \to \tau} \frac{f(z)}{|z-\tau|^3} = 0. \tag{2}$$

Then $f \equiv 0$ in Δ.

Proof. Since f is a holomorphic generator of a one-parameter semigroup, equation (1) has a unique solution $z = J_s(w)$. Moreover, the mapping J_s (the resolvent of f) is a well-defined holomorphic self-mapping of Δ for all $s > 0$.

We claim that τ is the Denjoy–Wolff point of J_s for each $s > 0$. Indeed, by our assumption, $\angle \lim\limits_{z \to \tau} f(z) = \angle \lim\limits_{z \to \tau} f'(z) = 0$. Consequently, if f does not vanish identically in Δ, then τ is the Denjoy–Wolff point of the corresponding semigroup (see Lemma 3 in [6]) and f admits the representation

$$f(z) = (z-\tau)(1-z\overline{\tau})g(z), \quad z \in \Delta, \tag{3}$$

where $g \in \mathrm{Hol}(\Delta, \mathbb{C})$ and $\mathrm{Re}\, g(z) \ge 0$ for all $z \in \Delta$ (see [3]). This implies, in particular, that f has no interior null point. So, by equality (1), J_s $(s > 0)$ has no interior fixed point. Suppose that $\varsigma \in \partial\Delta$ is the common boundary Denjoy–Wolff point of all the resolvents J_s $(s > 0)$ [13, Theorem 8.3(ii)], that is, $\angle \lim\limits_{z \to \varsigma} J_s(z) = \varsigma$ and $0 < \angle \lim\limits_{z \to \varsigma} J_s'(z) \le 1$. By the Julia–Wolff–Carathéodory theorem (see, for example, [14]),

$$\frac{|\varsigma - J_s(z)|^2}{1-|J_s(z)|^2} \le \frac{|\varsigma - z|^2}{1-|z|^2}, \quad z \in \Delta.$$

Hence

$$\frac{1}{s}\left[|\varsigma - J_s(z)|^2(1-|z|^2) - (1-|J_s(z)|^2)|\varsigma - z|^2\right] \le 0, \quad s > 0.$$

A calculation shows that the limit as $s \to 0^+$ of the left-hand side of this inequality exists and equals $-2\,\mathrm{Re}[f(z)(\overline{z}-\overline{\varsigma})(1-\overline{z}\varsigma)]$. Consequently, $\mathrm{Re}\,\frac{f(z)}{(z-\varsigma)(1-z\overline{\varsigma})} \ge 0$. Denote $h(z) := \frac{f(z)}{(z-\varsigma)(1-z\overline{\varsigma})}$. Then

$$f(z) = (z-\varsigma)(1-z\overline{\varsigma})h(z), \quad z \in \Delta, \tag{4}$$

where $h \in \mathrm{Hol}(\Delta, \mathbb{C})$ and $\mathrm{Re}\, h(z) \ge 0$ for all $z \in \Delta$. By Theorem 2.1 in [1], the generator f has a unique representation of this form. So, comparing (3) with (4), we conclude that $\varsigma = \tau$, as claimed.

Next, we show that for all $s > 0$,

$$\lim_{r \to 1^-} J_s'(r\tau) = 1 \quad \text{and} \quad \lim_{r \to 1^-} J_s''(r\tau) = \lim_{r \to 1^-} J_s'''(r\tau) = 0. \tag{5}$$

Note that $J_s(r\tau)$ tends to τ nontangentially as $r \to 1^-$. Indeed, by Julia's lemma,

$$\frac{|1-J_s(r\tau)\overline{\tau}|^2}{1-|J_s(r\tau)|^2} \le \alpha \frac{(1-r)^2}{1-r^2},$$

where $\alpha := \angle \lim\limits_{z \to \tau} \frac{1-|J_s(z)|}{1-|z|}$. Hence,

$$\left(\frac{|J_s(r\tau)-\tau|}{1-|J_s(r\tau)|}\right)^2 \le \alpha \frac{1-r}{1+r} \cdot \frac{1-|J_s(r\tau)|^2}{(1-|J_s(r\tau)|)^2}$$

$$= \alpha \frac{1+|J_s(r\tau)|}{1+r} \cdot \frac{1-r}{1-|J_s(r\tau)|} < \frac{2\alpha}{1+r} \cdot \frac{1-r}{1-|J_s(r\tau)|} \to 1 \quad \text{as} \quad r \to 1^-.$$

For $w \in \Delta$, we find from (1) that

$$J_s'(w) = \frac{1}{1+sf'(J_s(w))}, \quad J_s''(w) = -s\frac{f''(J_s(w))(J_s'(w))^2}{1+sf'(J_s(w))}$$

and

$$J_s'''(w) = -s\frac{f'''(J_s(w))(J_s'(w))^3 + 3f''(J_s(w))J_s'(w)J_s''(w)}{1+sf'(J_s(w))}.$$

By our assumption, $\angle \lim\limits_{z\to\tau} f^{(j)}(z) = 0$, $j = 1, 2, 3$. Consequently, for each $s > 0$, equalities (5) do hold.

Now it follows from the Proposition that $J_s \equiv I$ and, by equality (1), $f \equiv 0$. □

Remark 1 (M. Elin)**.** *The proof of the Theorem can also be concluded by observing that since*

$$w = J_s(w) + sf(J_s(w)),$$

we have

$$\frac{w - J_s(w)}{(w-\tau)^3} = s\frac{f(J_s(w))}{(J_s(w)-\tau)^3} \cdot \left(\frac{J_s(w)-\tau}{w-\tau}\right)^3$$

for all $w \in \Delta$*.*

Remark 2. *Note that condition (2) in the Theorem cannot be replaced by the requirement that* $\angle \lim\limits_{z\to\tau} \frac{f(z)}{|z-\tau|^{3-\varepsilon}} = 0$ *for some* $\varepsilon > 0$*.*

Indeed, the function $f(z) = -(1-z)^3$, for example, is a generator and for each $\varepsilon > 0$, $\angle \lim\limits_{z\to 1} \frac{f(z)}{|z-1|^{3-\varepsilon}} = 0$. However, f does not vanish in Δ.

Acknowledgment

The second author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund - B. and G. Greenberg Research Fund (Ottawa).

References

[1] D. Aharonov, M. Elin, S. Reich and D. Shoikhet, *Parametric representations of semi-complete vector fields on the unit balls in* C^n *and in Hilbert space*, Atti. Accad. Naz. Lincei 10 (1999), 229-253.

[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden, 1976.

[3] E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators*, Michigan Math. J. 25 (1978), 101-115.

[4] F. Bracci, R. Tauraso and F. Vlacci, *Identity principles for commuting holomorphic self-maps of the unit disk*, J. Math. Anal. Appl. 270 (2002), 451-473.

[5] D. M. Burns and S. G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. 7 (1994), 661-676.

[6] M. Elin and D. Shoikhet, *Dynamic extension of the Julia–Wolff–Carathéodory theorem*, Dynamic Systems and Applications 10 (2001), 421-438.

[7] M. Elin, M. Levenshtein, D. Shoikhet and R. Tauraso, *Rigidity of holomorphic generators and one-parameter semigroups*, Preprint, arXiv.math.CV/0512482v1, 2005.

[8] L. A. Harris, S. Reich and D. Shoikhet, *Dissipative holomorphic functions, Bloch radii, and the Schwarz lemma*, J. Anal. Math. 82 (2000), 221-232.
[9] E. Kopecká and S. Reich, *Hyperbolic monotonicity in the Hilbert ball*, Fixed Point Theory Appl. 2006, Article ID 78104, 1-15.
[10] S. Migliorini and F. Vlacci, *A new regidity result for holomorphic maps*, Indag. Math. 13 (2002), 537-549.
[11] S. Reich, *Product formulas, nonlinear semigroups, and accretive operators*, J. Funct. Anal. 36 (1980), 147-168.
[12] S. Reich and D. Shoikhet, *Generation theory for semigroups of holomorphic mappings in Banach spaces*, Abstr. Appl. Anal. 1 (1996), 1-44.
[13] S. Reich and D. Shoikhet, Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces, Imperial College Press, London, 2005.
[14] D. Shoikhet, Semigroups in Geometrical Function Theory, Kluwer, Dordrecht, 2001.
[15] R. Tauraso, *Commuting holomorphic maps of the unit disk*, Ergodic Theory Dynam. Systems 24 (2004), 945-953.

Manuscript received October 24, 2006
revised December 6, 2006

M. Levenshtein
Department of Mathematics, The Technion-Israel Institute of Technology
32000 Haifa, Israel
E-mail address: marlev@list.ru

S. Reich
Department of Mathematics, The Technion-Israel Institute of Technology
32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il

D. Shoikhet
Department of Mathematics, ORT Braude College
P.O. Box 78, 21982 Karmiel, Israel
E-mail address: davs27@netvision.net.il