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CHARACTERIZATION OF THE SOLUTION SETS OF PSEUDOLINEAR PROGRAMS AND PSEUDOAFFINE VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, the concept of a pseudoaffine bifunction, defined as a bifunction h for which both h and $-h$ are pseudomonotone, is considered and necessary and sufficient conditions for such maps are obtained. Also the notion of h-pseudolinear function, defined as a function that is both h-pseudoconvex and h-pseudoconcave, is introduced along with some of its characterizations. Using these characterizations the solution sets of h-pseudolinear programs are characterized. Characterizations for the solution sets of variational inequality problems involving pseudoaffine bifunctions are also obtained in the end.

1. Introduction

It is well known that a function defined on a convex set is linear/affine if it is both convex and concave. In order to generalize the concept of convexity/concavity Mangasarian [15] introduced the concept of pseudoconvexity/pseudoconcavity for differentiable functions. Martos [17] defined a class of functions called pseudomonotonic functions, which included those functions, which are both pseudoconvex as well as pseudoconcave. Many authors extended the study of this class of functions, which were later termed as pseudolinear functions. The class of pseudolinear functions includes the class of linear fractional functions, which arise in many practical situations. Efficient simplex type algorithm has been developed for solving a hospital fee optimization problem involving linear fractional objective and linear objective, which is a pseudolinear program having almost a linear structure (see [13] and [18]). Chew and Choo [5] derived first and second order characterizations for differentiable pseudolinear functions, which were later, extended by Komlosi [11]. Komlosi [12] extended the notion of pseudoconvexity/pseudoconcavity to nondifferentiable functions by means of a bifunction h.

Monotone maps play an important role in variational inequality problems, like convex functions in mathematical programming. In 1990 Karamardian and Schaible [10] introduced different concepts of generalized monotone maps. Various notions of generalized monotonicity are related to generalized convexity concepts in a survey article by Schaible [20]. The notion of pseudomonotonicity in terms of a bifunction h was introduced by Komlosi [12] and was related to the notion of pseudoconvexity (for non differentiable functions) given in terms of h. Bianchi and Schaible [2] related the notion of pseudolinearity with a map F, called PPM map, for which both F and

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$-F$ are pseudomonotone. In the same paper, a PPM map F has been characterized in terms of pseudolinearity of f, where F is the gradient of the function f. Similar notions are known in literature and are referred to as bipseudomonotone maps by Ansari, Konnov and Yao [1] and as pseudoaffine maps by Bianchi, Hadjisavvas and Schaible [3].

Characterizations and properties of solution sets are useful for understanding the behavior of solution methods for mathematical programs that have multiple optimal solutions. Mangasarian [16] presented several characterizations of solution set of a finite dimensional convex program and applied it to study monotone linear complimentarity problems. This study was further extended by Burke and Ferris [4] to infinite dimensional convex program. Jeyakumar and Yang [8] provided several simple characterizations for pseudolinear differentiable program. Similar characterizations have been discussed by Jeyakumar and Yang [7] for programs in which the objective function is a convex composite vector valued function. Recently, various characterizations of optimal solution set of cone constrained convex optimization problem in terms of Lagrange multiplier have been derived by Jeyakumar, Lee and Dinh [9], where the results have also been extended to semi-definite and fractional programs. Bianchi and Schaible [2] extended the results of Jeyakumar and Yang [8] to obtain characterizations of solution set of PPM variational inequality problem.

In this paper, we extend the work of Jeyakumar and Yang [8] and Bianchi and Schaible [2] to the nondifferentiable case. The paper is organized as follows. In Section 2, we introduce the notion of pseudoaffinity of a general bifunction h and obtain an equivalent form of pseudoaffine bifunction. In Section 3, we give some characterizations for h-pseudolinear functions, which are used in Section 4 to characterize the solution sets of programs involving such functions. In the last section, we study pseudoaffine variational inequality problems and derive complete characterizations for the solution sets.

2. Pseudoaffine bifunctions

In the sequel, we use the extension of addition to $\bar{R} = R \cup \{+\infty, -\infty\}$, given by

$$(+\infty) + r = +\infty, \qquad \text{for each } \; r \in \bar{R},$$
$$(-\infty) + r = -\infty, \qquad \text{for each } \; r \in R \cup \{-\infty\}\,.$$

According to Sach and Penot [19] a function $s : R^n \to \bar{R}$ is said to be subodd if $s(d) + s(-d) \geq 0$, $\forall\, d \in R^n \backslash \{0\}$. Clearly s is odd if s and $-s$ are both subodd, that is, $s(d) + s(-d) = 0$, $\forall\, d \in R^n \backslash \{0\}$.

Let C be a convex subset of R^n and let $h : C \times R^n \to \bar{R}$ be a bifunction such that for each $x \in C$,

$$h(x; \cdot) \quad \text{is positively homogeneous;} \tag{2.1}$$

$$h(x; 0) = 0\,. \tag{2.2}$$

According to Komlosi [12] a bifunction $h : C \times R^n \to \bar{R}$ is said to be pseudomonotone on C if for any $x, y \in C$, $x \neq y$

$$h(x; y - x) > 0 \quad \Rightarrow \quad h(y; x - y) < 0;$$

or equivalently

$$h(x; y - x) \geq 0 \quad \Rightarrow \quad h(y; x - y) \leq 0.$$

We now have the concept of pseudoaffinity for the bifunction h.

Definition 2.1. A bifunction h is said to be pseudoaffine on C if h and $-h$ are both pseudomonotone on C.

The following is an example of a pseudoaffine bifunction which satisfies conditions (2.1) and (2.2) and is also odd in the second argument.

Example 2.2. The bifunction $h : R^2 \times R^2 \to \bar{R}$ defined as

$$h(x; d) = \begin{cases} e^{x_1}(d_1^2 + d_2^2)/(d_1 + d_2); & \text{if } \ d_1 \neq -d_2\,, \\ 0, & \text{if } \ d_1 = -d_2\,, \end{cases}$$

where $d = (d_1, d_2)$ and $x = (x_1, x_2)$, is pseudoaffine.

For any two distinct points $x, y \in R^n$, we denote by $[x, y]$ the line segment $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$. We recall that a function $g : R^n \to R$ is said to be radially upper semicontinuous (radially lower semicontinuous) on C if for each segment $[x, y]$ in C, the restriction of g to $[x, y]$ is upper semicontinuous (lower semicontinuous) and g is said to be radially continuous on C if g is both radially upper semicontinuous and radially lower semicontinuous on C.

A necessary and sufficient condition for the bifunction h to be pseudoaffine, which extends the corresponding result by Bianchi and Schaible [2] for PPM maps, is as follows.

Theorem 2.3. *Suppose that h is radially continuous on C in the first argument, odd in the second argument and satisfies conditions* (2.1) *and* (2.2). *Then h is pseudoaffine on C if and only if* $\forall\ x, y \in C$

$$h(x; y - x) = 0 \quad \Rightarrow \quad h(y; x - y) = 0. \tag{2.3}$$

Proof. Suppose that h is pseudoaffine on C and $x, y \in C$. In view of condition (2.2) the result is obvious if $x = y$. Let us now suppose $h(x; y - x) = 0$ for $x \neq y$. Pseudomonotonicity of h and $-h$ implies that $h(y; x-y) \leq 0$ and $h(y; x-y) \geq 0$, and hence we have the result. Conversely, let us assume that h is not a pseudomonotone bifunction. Then there exist $x, y \in C$, $x \neq y$ such that $h(x; y - x) > 0$ and $h(y; x - y) \geq 0$. Define $g : [0, 1] \to \bar{R}$ as $g(\alpha) = h(x + \alpha(y - x); y - x)$. We have $g(0) > 0$ and $g(1) = h(y; y-x) = -h(y; x-y) \leq 0$. Since h is radially continuous in the first argument there exists $\alpha^* \in]0, 1]$ such that $g(\alpha^*) = 0$ that is, $h(z; y-x) = 0$, where $z = x+\alpha^*(y-x)$. By positive homogeneity of h we have $h(z; z-x) = 0$ which in turn implies using the suboddness of h that $h(z; x - z) = 0$. Using (2.3) we now obtain $h(x; z - x) = 0$ and positive homogeneity of h leads us to $h(x; y - x) = 0$, a contradiction. Similarly, we can also establish that $-h$ is pseudomonotone on C. □

Corollary 2.4. *Suppose that $C \subseteq R^n$ is convex and h is radially continuous on C in the first argument, odd in the second argument and satisfies conditions* (2.1) *and* (2.2). *Then h is pseudoaffine on C if and only if* $\forall\ x, y \in C$

$$h(x; y - x) = 0 \Rightarrow h(\alpha x + (1 - \alpha)y; x - y) = 0, \quad \forall\ \alpha \in [0, 1]. \tag{2.4}$$

Proof. Let us suppose that h is pseudoaffine on C and $x, y \in C$. By assumption (2.2) the result holds trivially if $x = y$. Let $h(x; y - x) = 0$ for $x \neq y$ and let $z = \alpha x + (1 - \alpha)y$ for $\alpha \in [0, 1]$. Since the result is obvious for $\alpha = 1$, we assume that $\alpha \in [0, 1[$. Now using (2.1) we have

$$0 = h(x; y - x) = h(x; (1 - \alpha)^{-1}(z - x)) = (1 - \alpha)^{-1} h(x; z - x),$$

which implies that $h(x; z - x) = 0$. Since h is pseudoaffine so from the above theorem we have $h(z; x - z) = 0$ and from positive homogeneity of h it follows that $h(z; x - y) = 0$. Conversely, by taking $\alpha = 0$ in (2.4) condition (2.3) is satisfied and hence the result follows from Theorem 2.3. □

We may note that in Theorem 2.3, the oddness assumption is needed only in the sufficiency part. We now give an example to show that this assumption cannot be relaxed.

Example 2.5. Let $h : R \times R \to \bar{R}$ be defined as

$$h(x, d) = \begin{cases} d, & \text{if } d \geq 0\,, \\ -2d, & \text{if } d < 0. \end{cases}$$

Then h is a continuous function of x satisfying (2.1) and (2.2) but h is not odd in the second argument because $h(1; 3) + h(1; -3) \neq 0$. Here, condition (2.3) is trivially satisfied but h is not pseudoaffine on R because for $x = 1$ and $y = 4$, $h(x; y - x) = 3 > 0$ but $h(y; x - y) = 6 \not< 0$.

3. Pseudolinearity in terms of bifunctions

In [19] a relation between the notions of pseudomonotonicity of the bifunction h and pseudoconvexity of a real valued function f was established assuming that h is majorized by the upper Dini directional derivative of f. Following this idea in this section we present the notion of pseudolinearity in relation to pseudoaffinity.

Let f be a real valued function defined on the convex set C. We know that any generalized derivative of f is in the form of a bifunction $h(x; d)$, where x refers to a point in the domain C and d refers to a given direction in R^n. All generalized derivatives are positively homogeneous functions of the direction d and this motivates condition (2.1). We recall that the upper and the lower Dini directional derivatives of f at $x \in C$, in the direction $d \in R^n$, are defined as

$$D^+ f(x; d) = \limsup_{t \to 0+} \frac{f(x + td) - f(x)}{t}\,,$$

$$D_+ f(x; d) = \liminf_{t \to 0+} \frac{f(x + td) - f(x)}{t}\,.$$

The function f is said to be quasiconvex (resp. quasiconcave) on C, if $\forall$ $x, y \in C$, $x \neq y$ and $\forall\ \lambda \in [0, 1]$, $f(x + \lambda(y - x)) \leq \max\{f(x), f(y)\}$ (resp. $f(x + \lambda(y - x)) \geq \min\{f(x), f(y)\}$).

The following lemma which is a consequence of Diewert's mean value theorem [6] will be used in the sequel.

Lemma 3.1. *If f is radially lower semicontinuous on C then for any $x, y \in C$, $x \neq y$ there exists $\theta \in [0, 1[$ such that for $w = x + \theta(y - x)$*

$$D_+ f(w; y - x) \geq f(y) - f(x).$$

The following notions of pseudoconvexity and quasiconvexity for the real valued function f in terms of a bifunction h are due to Komlosi [12].

Definition 3.2. The function f is said to be

(i) h-pseudoconvex on C, if $\forall\ x, y \in C$, $x \neq y$,

$$f(y) < f(x) \quad \Rightarrow \quad h(x; y - x) < 0;$$

(ii) h-quasiconvex on C, if $\forall\ x, y \in C$, $x \neq y$,

$$f(y) \leq f(x) \quad \Rightarrow \quad h(x; y - x) \leq 0.$$

Assuming suboddness assumption Sach and Penot [19] established the following theorem (see Theorem 5.1 and Theorem 3.2 in [19]).

Theorem 3.3. *Suppose that h is subodd in the second argument and satisfies* (2.1)*. Further suppose that*

$$h(x; d) \leq D^+ f(x; d)\,, \quad \forall\ x \in C \ \text{ and } \ d \in R^n. \tag{3.1}$$

If f is h-pseudoconvex on C then $f$is quasiconvex as well as h-quasiconvex on C.

The conditions (2.1), (2.2) and (3.1) are satisfied by certain generalized derivatives such as the contingent derivative (or Hadamard derivative) given by

$$f'(x; d) = \liminf_{\substack{t \to 0+ \\ u \to d}} \frac{f(x + tu) - f(x)}{t}$$

upper epiderivative (or adjacent derivative) given by

$$f'(x; d) = \sup_{U \in N(d)} \limsup_{t \to 0+} \inf_{u \in U} \frac{f(x + tu) - f(x)}{t},$$

where $N(d)$ denotes the family of neighbourhoods of d.

Analogously we have the notions of h-pseudoconcavity and h-quasiconcavity.

Definition 3.4 ([12])**.** The function f is said to be

(i) h-pseudoconcave on C, if $\forall x, y \in C, x \neq y$,

$$f(y) > f(x) \Rightarrow h(x; y - x) > 0;$$

(ii) h-quasiconcave on C, if $\forall x, y \in C, x \neq y$,

$$f(y) \geq f(x) \Rightarrow h(x; y - x) \geq 0.$$

The following theorem, which is a variant of Theorem 3.3, can be established on the same lines.

Theorem 3.5. *Suppose that $-h$ is subodd in the second argument and satisfies* (2.1)*. Further suppose that*

$$D_+ f(x; d) \leq h(x; d)\,, \qquad \forall\ x \in C \ \text{ and } \ d \in R^n. \tag{3.2}$$

If f is h-pseudoconcave on C then f is quasiconcave and h-quasiconcave on C.

It was proved in [19] that if f is a radially upper semicontinuous function on C, h is a bifunction associated to f such that h is subodd in the second argument and satisfies conditions (2.1) and (3.1) then f is h-pseudoconvex on C if and only if h is pseudomonotone on C. We now give an analogous of this theorem using Theorem 3.5.

Theorem 3.6. *Suppose that $-h$ is subodd in the second argument and satisfies* (2.1) *and* (3.2). *Further suppose that f is radially lower semicontinuous on C. Then f is h-pseudoconcave on C if and only if $-h$ is pseudomonotone on C.*

Proof. Assume that f is h-pseudoconcave on C and let $x, y \in C$ be such that $-h(x; y-x) > 0$. From relation (3.2) it follows that $D_+f(x; y-x) < 0$ which implies that $f(x + \lambda(y-x)) < f(x)$, for some $\lambda \in]0, 1[$. Now by Theorem 3.5 f is quasiconcave on C and therefore $f(y) \leq f(x + \lambda(y-x)) < f(x)$. By h-pseudoconcavity of f it follows that $h(y; x-y) > 0$ or $-h(y; x-y) < 0$. Thus, $-h$ is pseudomonotone on C.

Conversely, suppose $-h$ is pseudomonotone on C and let $x, y \in C$ be such that $f(x) < f(y)$. Then by Lemma 3.1 there exists $\theta \in [0, 1[$ such that for $w = x + \theta(y-x)$

$$0 < f(y) - f(x) \leq D_+f(w; y-x) \leq h(w; y-x).$$

If $\theta = 0$, it immediately follows that $h(x; y-x) > 0$. If $\theta > 0$, by positive homogeneity and suboddness of $-h$ we obtain successively $h(w; w-x) > 0$ and $h(w; x-w) < 0$. As $-h$ is pseudomonotone therefore it follows that $h(x; w-x) > 0$ that is, $h(x; y-x) > 0$. □

We are now in a position to introduce the definition of h-pseudolinearity.

Definition 3.7. The function f is said to be h-pseudolinear on C if f is both h-pseudoconvex as well as h-pseudoconcave on C.

If f is a differentiable h-pseudolinear function with $h(x; d) = \langle \bigtriangledown f(x), d \rangle$ then f is pseudolinear in the sense of Chew and Choo [5]. We now give two examples of h-pseudolinear functions. Even though the first one is radially continuous, it may be noted that both the functions are not pseudolinear in the sense of Chew and Choo [5].

Example 3.8. Let $f\colon R \to R$ be defined as

$$f(x) = \begin{cases} x^2 + 2x, & \text{if } x \geq 0, \\ x, & \text{if } x < 0. \end{cases}$$

Then,

$$D^+f(x; d) = D_+f(x; d) = \begin{cases} (2x+2)d, & \text{if } x > 0, \forall\, d \ \text{ or if } \ x = 0, d > 0, \\ d, & \text{if } x < 0, \forall\, d \ \text{ or if } \ x = 0, d < 0. \end{cases}$$

If we take $h(x; d) = D^+f(x; d)$, for each $x, d \in R$ then it can be seen that f is h-pseudolinear on R.

Example 3.9. Let $f : R^2 \to R$ be defined as

$$f(x) = \begin{cases} x_1 + x_2 + 1, & \text{if } x_1 + x_2 > 0, \\ x_1 + x_2, & \text{if } x_1 + x_2 \leq 0, \end{cases}$$

then

$$D^+ f(x; d) = D_+ f(x; d) = \begin{cases} d_1 + d_2, & \text{if } x_1 + x_2 > 0, \forall\, d\,, \\ +\infty, & \text{if } x_1 + x_2 = 0, d_1 + d_2 > 0\,, \\ d_1 + d_2, & \text{if } x_1 + x_2 = 0, d_1 + d_2 \leq 0 \text{ or} \\ & \quad\text{if } x_1 + x_2 < 0, \forall\, d, \end{cases}$$

where $x = (x_1, x_2)$ and $d = (d_1, d_2)$. The function f is h-pseudolinear function on R^2 with $h(x; d) = D^+ f(x; d)$, for each x, $d \in R^2$.

Remark 3.10. We observe that the choice of h affects the nature of the function f. For instance, if we take $h(x; d) = \min\{0, d_1 + d_2\}$ in the above example then f is not h-pseudolinear as f is not h-pseudoconcave because for $x = (0, 0)$ and $y = (1, 1)$, $f(x) < f(y)$ but $h(x; y - x) = 0$.

Using Theorem 3.6 and its analogous we now give a relationship between h-pseudolinear functions and pseudoaffine bifunctions. For the remaining theorems of this section we assume that the bifunction h associated to f satisfies conditions (2.1), (2.2), (3.1) and (3.2).

Theorem 3.11. *Suppose that h is odd in the second argument and f is radially continuous on C. Then f is h-pseudolinear on C if and only if h is pseudoaffine on C.*

Our next result is a consequence of the above theorem, Theorem 2.3 and Corollary 2.4.

Theorem 3.12. *Suppose that h is odd in the second argument and f is radially continuous on C. Further if h is radially continuous on C in the first argument then f is h-pseudolinear on C if and only if one of the following holds:*

(a) $h(x; y - x) = 0 \Rightarrow h(y; x - y) = 0$

(b) $h(x; y - x) = 0 \Rightarrow h(\alpha x + (1 - \alpha)y; x - y) = 0$, $\forall\, \alpha \in [0, 1]$.

The following result, which we shall use in the sequel, has been established by Lalitha and Mehta [14] but with a different approach.

Theorem 3.13. *Let h be odd in the second argument. If f is h-pseudolinear on C then for each $x, y \in C$,*

$$h(x; y - x) = 0 \Leftrightarrow f(x) = f(y). \tag{3.3}$$

Proof. Let us suppose $x, y \in C$. As the result holds trivially if $x = y$, we assume that $x \neq y$. If $h(x; y - x) = 0$, than by h-pseudoconvexity and h-pseudoconcavity of f it follows that $f(x) = f(y)$. Conversely, let us suppose that $f(x) = f(y)$. As f is h-pseudolinear on C it follows by Theorems 3.3 and 3.5 that f is both h-quasiconvex as well as h-quasiconcave on C and hence we have $h(x; y - x) = 0$. □

Our next example shows that the conclusion of the above theorem may not hold in the absence of oddness assumption.

Example 3.14. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is irrational,} \\ -1, & \text{if } x \text{ is rational.} \end{cases}$$

Define h as follows

$$h(x; d) = \begin{cases} -|d|, & \text{if } x \text{ is irrational,} \\ |d|, & \text{if } x \text{ is rational.} \end{cases}$$

Then f is a h-pseudolinear function and h satisfies (2.1), (2.2), (3.1) and (3.2). Here, h fails to be odd and we observe that $f(1) = f(-1)$ but $h(1; -2) \neq 0$.

4. Characterizations of solution sets of pseudolinear programs

Consider the following problem:

$$\text{(P)} \qquad \min f(x) \quad \text{subject to } x \in C,$$

where f is an h-pseudolinear function and h is a bifunction associated to f such that h is odd in the second argument and satisfies conditions (2.1), (2.2), (3.1) and (3.2). We assume that the solution set X^* of (P) is nonempty.

In this section, we characterize the solution sets of problem (P) taking into account the properties of h-pseudolinear functions.

Theorem 4.1. *The solution set X^* of* (P) *is a convex set.*

Proof. Let $x_1, x_2 \in X^*$. Then, $f(x_1) = f(x_2)$ and by Theorem 3.13 we have $h(x_1; x_2 - x_1) = 0$ and $h(x_2; x_1 - x_2) = 0$. Also, for $\alpha \in [0, 1]$

$$\begin{aligned} h(x_2; \alpha x_1 + (1-\alpha)x_2 - x_2) &= h(x_2; \alpha(x_1 - x_2)) \\ &= \alpha h(x_2; x_1 - x_2) = 0. \end{aligned}$$

From Theorem 3.13 it follows that $f(\alpha x_1 + (1-\alpha)x_2) = f(x_2)$ and thus we have $\alpha x_1 + (1-\alpha)x_2 \in X^*, \forall \alpha \in [0, 1]$. □

Theorem 4.2. *If $x^* \in X^*$, then the solution set X^* can be characterized as*

$$\begin{aligned} X^* &= \{x \in C | h(x; x^* - x) = 0\} \\ &= \{x \in C | h(x^*; x - x^*) = 0\} \\ &= \{x \in C | h(\alpha x + (1-\alpha)x^*; x^* - x) = 0, \ \forall \ \alpha \in [0, 1]\}. \end{aligned}$$

Proof. Using Theorem 3.13 it is obvious that

$$X^* = \{x \in C | h(x; x^* - x) = 0\} = \{x \in C | h(x^*; x - x^*) = 0\}. \tag{4.1}$$

We now prove that $X^* = \{x \in C | h(\alpha x + (1-\alpha)x^*; x^* - x) = 0, \forall \ \alpha \in [0, 1]\}$. Let us suppose that $x \in X^*$. Now by the previous theorem $\alpha x + (1-\alpha)x^* \in X^*$, $\forall$ $\alpha \in [0, 1]$ and therefore from (4.1) it follows that

$$0 = h(\alpha x + (1-\alpha)x^*; x^* - \alpha x - (1-\alpha)x^*)$$

$$= \quad \alpha h(\alpha x+(1-\alpha)x^*;x^*-x).$$

If $\alpha>0$ we have $h(\alpha x+(1-\alpha)x^*;x^*-x)=0$. Also from (4.1) and oddness of h we have $h(x^*,x^*-x)=0$. Therefore, from the above two statements we have

$$X^*\subseteq\{x\in C|h(\alpha x+(1-\alpha)x^*;x^*-x)=0,\ \forall\ \alpha\in[0,1]\}.$$

Conversely let $x\in\{x\in C|h(\alpha x+(1-\alpha)x^*;x^*-x)=0,\ \forall\ \alpha\in[0,1]\}$. Then by taking $\alpha=1$ in particular, we have $h(x;x^*-x)=0$ and again from (4.1) it follows that $x\in X^*$. Thus, we have $\{x\in C|h(\alpha x+(1-\alpha)x^*;x^*-x)=0,\ \forall\ \alpha\in[0,1]\}\subseteq X^*$. □

Theorem 4.3. *If $x^*\in X^*$, then the solution set $X^*=X'=X''=X'''$, where*

$$\begin{aligned} X' &= \{x\in C|h(x;x^*-x)\geq 0\}, \\ X'' &= \{x\in C|h(x^*;x-x^*)\leq 0\}, \\ X''' &= \{x\in C|h(\alpha x+(1-\alpha)x^*,x^*-x)\geq 0,\forall\alpha\in[0,1]\}. \end{aligned}$$

Proof. The inclusion $X^*\subseteq X'$ follows immediately from Theorem 4.2. For the converse inclusion assume that $x\in C$ satisfies $h(x;x^*-x)\geq 0$. As f is h-pseudoconvex on C it follows that $f(x)\leq f(x^*)$, but x^* is a solution of (P) and hence $f(x^*)\leq f(x)$. Therefore $f(x)=f(x^*)$, that is, $x\in X^*$. We can similarly prove using the h-pseudoconcavity of f that $X^*=X''$.

To show that $X^*=X'''$ we first note that $X'''\subseteq X'=X^*$. Now let $x\in X^*$, then from Theorem 4.2 we have $h(\alpha x+(1-\alpha)x^*,x^*-x)=0\ \forall\ \alpha\in[0,1]$. This implies that $x\in X'''$ and hence $X^*\subseteq X'''$. □

Theorem 4.4. *If $x^*\in X^*$ then*

$$\begin{aligned} X^* &= \{x\in C|h(x^*;x-x^*)=h(x;x^*-x)\} \\ &= \{x\in C|h(x^*;x-x^*)\leq h(x;x^*-x)\}. \end{aligned}$$

Proof. It is clear from Theorem 4.2 that the inclusion

$$X^*\subseteq\{x\in C|h(x^*;x-x^*)=h(x;x^*-x)\} \tag{4.2}$$

holds. Now let $x\in C$ be such that $h(x^*;x-x^*)\leq h(x;x^*-x)$. Suppose that $x\notin X^*$. Then $f(x)>f(x^*)$. Using h-pseudoconcavity of f we have $h(x^*;x-x^*)>0$ and thus $h(x;x^*-x)>0$. By applying h-pseudoconvexity of f we get $f(x)\leq f(x^*)$, which is a contradiction. Thus, $x\in X^*$ and we have

$$\{x\in C|h(x^*;x-x^*)\leq h(x;x^*-x)\}\subseteq X^*. \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$\begin{aligned} X^* &\subseteq \{x\in C|h(x^*;x-x^*)=h(x;x^*-x)\} \\ &\subseteq \{x\in C|h(x^*;x-x^*)\leq h(x;x^*-x)\} \\ &\subseteq X^*. \end{aligned}$$

5. Characterizations of solution sets of pseudoaffine variational inequality problems

Consider the variational inequality problem:

$$\text{(VIP)} \qquad \text{Find } x^* \in C \text{ such that} \quad h(x^*; x - x^*) \geq 0, \quad \forall x \in C,$$

where C is a convex subset of R^n and $h : C \times R^n \to \bar{R}$ is a pseudoaffine bifunction satisfying (2.1) and (2.2). Let K^* denote the solution set of this problem and assume that $K^* \neq \phi$.

The following proposition is a direct consequence of pseudoaffinity of h.

Proposition 5.1. *A point $x^* \in C$ is a solution of* (VIP) *if and only if $h(x; x^* - x) \leq 0$, $\forall\ x \in C$.*

Theorem 5.2. *Suppose that the bifunction h is radially continuous in the first argument and odd in the second argument. If $x^* \in K^*$ then*

$$\begin{aligned} K^* &\subseteq \{x \in C | h(x; x^* - x) = 0\} \\ &= \{x \in C | h(x^*; x - x^*) = 0\} \\ &= \{x \in C | h(\alpha x + (1-\alpha)x^*; x^* - x) = 0\ \forall\ \alpha \in [0, 1]\}. \end{aligned}$$

Proof. Let $x \in K^*$, then $h(x; y - x) \geq 0$, $\forall\ y \in C$. In particular, for $y = x^*$ we have $h(x; x^* - x) \geq 0$. As x^* is a solution of (VIP) from Proposition 5.1 it follows that $h(x; x^* - x) \leq 0$. Thus, we have $h(x; x^* - x) = 0$ which implies that

$$K^* \subseteq \{x \in C | h(x^*; x - x^*) = 0\}.$$

The rest of the result follows from Theorem 2.3 and Corollary 2.4. □

Remark 5.3. Above theorem gives a partial characterization of the solution set K^*, that is, the implication in the above theorem, is in general, one way. If $x^* \in K^*$ and $h(x^*; x - x^*) = 0$ for some $x \in C$, it does not always imply that $x \in K^*$ as can be seen from the following example.

Example 5.4. Let $C = [-1, 1] \times [-1, 1]$ and let $h : C \times R^2 \to \bar{R}$ be defined as

$$h(x; d) = \begin{cases} (1 + x_1^2 + x_2^2) d_1^3 / d_2^2, & d_2 \neq 0\,, \\ 0, & d_2 = 0, \end{cases}$$

where $d = (d_1, d_2)$ and $x = (x_1, x_2)$. We note that h is a continuous function of x and an odd function of d satisfying (2.1) and (2.2). It can be easily verified using (2.3) that h is a pseudoaffine bifunction on C. Clearly, $x^* = (-1, 1)$ is a solution of (VIP) and $x = (1, 1)$ satisfies $h(x^*; x - x^*) = 0$ but $x \notin K^*$ because for $y = (0, 1/2)$, $h(x, y - x) = -12 < 0$.

However, Bianchi and Schaible [2] proved that if $F : C \subseteq R^n \to R^n$, is a continuous G-map then the solution set K^* coincides with the set $\{x \in C | h(x^*; x - x^*) = 0\}$, where $h(x; x^* - x) = \langle F(x), x - x^* \rangle$. i.e Jeyakumar and Yang [8] proved that if F is a gradient of some pseudolinear function then $K^* = X^* = \{x \in C | \langle F(x), x - x^* \rangle = 0\}$.

We now prove that the solution set K^* of (VIP) coincides with the set $\{x \in C | h(x^*; x - x^*) = 0\}$ when h is a bifunction associated to some h-pseudolinear function. The following proposition, which relates the solution sets of the problems (P) and (VIP) was proved in [14], however we again present the proof here for completeness.

Proposition 5.5. *Suppose that f is a h-pseudolinear function defined on a convex set C, where the bifunction h, associated to f, is odd in the second argument and satisfies* (2.1), (2.2), (3.1) *and* (3.2). *Then $x^* \in C$ is a solution of* (P) *if and only if x^* is a solution of* (VIP).

Proof. Let x be an optimal solution for (P). Then, for any $y \in C$

$$f(x) \leq f(x + \alpha(y - x)), \quad \forall\ \alpha \in]0, 1].$$

This implies that

$$D_+ f(x; y - x) \geq 0, \quad \forall\ y \in C$$

and using (3.2) it follows that $h(x; y - x) \geq 0$, $\forall\ y \in C$. Thus, x is a solution of (VIP).

Conversely, suppose x is a solution of (VIP) not optimal for (P). Then there exists $y \in C$ such that $f(y) < f(x)$. Then from h-pseudoconvexity of f it follows that $h(x; y - x) < 0$, which contradicts the fact that x is a solution of (VIP). □

Theorem 5.6. *Suppose that f is h-pseudolinear on C, the bifunction h is odd in the second argument and satisfies conditions* (2.1), (2.2), (3.1) *and* (3.2). *Let $x^* \in K^*$ then*

$$\begin{aligned} K^* &= \{x \in C | h(x; x^* - x) = 0\} \\ &= \{x \in C | h(x^*; x - x^*) = 0\} \\ &= \{x \in C | h(\alpha x + (1 - \alpha)x^*; x^* - x) = 0\ \forall\ \alpha \in [0, 1]\}. \end{aligned}$$

Proof. The result follows from Proposition 5.5 and Theorem 4.2. □

Other characterizations for solution set K^* of (VIP) can be obtained from Theorem 4.3 and Theorem 4.4.

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