Journal of Nonlinear and Convex Analysis
Volume 8, Number 1, 2007, 81–85

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

APPROXIMATION OF CONVEX FUNCTIONS IN ASPLUND GENERATED SPACES

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ABSTRACT. This paper shows that every continuous convex function defined on an Asplund generated space can be represented as the point-wise limit of a non-decreasing sequence of continuous convex functions which are locally affine at all points of dense open sets.

The aim of this note is to prove the following approximation statement:

Theorem 1. *Suppose that f is a continuous convex function on an Asplund generated space X. Then there exists a sequence $f_1, f_2, \dots$ of convex functions on X such that*

(i) $f_n(x) \uparrow f(x)$ *as $n \to \infty$ for every $x \in X$; and*

(ii) *for every $n \in \mathbb{N}$ there is an open dense set in X on which f_n is locally affine.*

This is an extension of a result from [1] which dealt with Asplund spaces. We should note that the functions f_n above are Fréchet differentiable on open dense sets even if the space X is not Asplund.

Our notation and terminology is standard and follows the books [5, 6] and [8]. We just recall that a Banach space X is called *Asplund generated* provided that it contains the image of an Asplund space, under a linear continuous mapping, as a dense subset.

We shall profit from the following simple fact.

Lemma 2. *Let φ be a convex function defined on an open interval containing $[0,1]$. Assume that it is differentiable at 0 and 1 and that $\varphi'(0) = \varphi'(1)$. Then $\varphi(1) = \varphi(0) + \varphi'(0)$.*

Proof. The subdifferential inequality for convex functions yields

$$\varphi'(0) = \varphi'(0)(1-0) \le \varphi(1) - \varphi(0) \le \varphi'(1)(1-0) = \varphi'(0).$$

□

We recall that a function $p : Y \to [0, +\infty)$ is called sublinear if $p(y_1 + y_2) \le p(y_1) + p(y_2)$ and $p(ty) = tp(y)$ for all $y, y_1, y_2 \in Y$ and all $t \ge 0$. Our theorem will follow from its special case, when all the functions involved are sublinear:

2000 *Mathematics Subject Classification.* Primary 58C20.

Key words and phrases. Convex function, locally affine function, Fréchet differentiability, approximation, Asplund generated space.

The research has been supported by a grant of the Natural Science Foundation of China under contract NSFC-10071063, and by grants A 1019301 and GA ČR 201/04/0090.

Proposition 3. *Let Y be an Asplund generated space, and $p : Y \to [0, +\infty)$ be a sublinear continuous function. Then there exist sublinear functions $p_n : Y \to [0, +\infty)$, $n \in \mathbb{N}$, such that*

(i) *$p_n \uparrow p$ as $n \to \infty$ for every $y \in Y$; and*

(ii) *for every $n \in \mathbb{N}$ there is an open dense set in X on which p_n is locally affine.*

Proof. Find an Asplund space X and a linear bounded mapping $T : X \to Y$ with dense range; note that T^* is then one-to-one. Put

$$D = \{y^* \in Y^*;\ \langle y^*, y\rangle \leq 1 \text{ whenever } y \in Y \text{ and } p(y) \leq 1\} \quad \text{and} \quad C = T^*(D).$$

Note that Hahn-Banach theorem yields that $p(\cdot) = \sup\langle D, \cdot\rangle$. Also D and C are bounded weak* closed, convex sets. For $n \in \mathbb{N}$ we find a maximal set $M_n \subset C$ such that $\|\xi_1 - \xi_2\| > \frac{1}{n}$ whenever $\xi_1, \xi_2 \in M_n$ and are different; such a set exists owing to Zorn's lemma. Proceeding subsequently for $n = 1, 2, \ldots$, we can arrange that $M_1 \subset M_2 \subset \cdots$. Fix any $n \in \mathbb{N}$. Let C_n denote the weak* closed convex hull of M_n and put $D_n = T^{*-1}(C_n) \cap D$. Further define functions $q_n : X \to [0, +\infty)$, $p_n : Y \to [0, +\infty)$ by

$$q_n(\cdot) = \sup\langle M_n, \cdot\rangle \ (= \sup\langle C_n, \cdot\rangle), \quad p_n(\cdot) = \sup\langle D_n, \cdot\rangle.$$

Clearly, these functions are sublinear and moreover $p_n \circ T = q_n$. Also, $p_1 \leq p_2 \leq \cdots \leq p$. Fix any $0 \neq y \in Y$. We shall show that $p_n(y) \to p(y)$ as $n \to \infty$. Since p is continuous, $p_n \leq p$, and p_n's are subadditive, it is enough to consider the case when $y = Tx$ for some $x \in X$. Let $\epsilon > 0$ be arbitrary. Take $n \in \mathbb{N}$ so that $n > \frac{2\|x\|}{\epsilon}$. Find $\eta \in D$ so that $p(y) - \langle \eta, y\rangle < \epsilon/2$. Find $\xi \in M_n$ so that $\|T^*\eta - \xi\| \leq \frac{1}{n}$. Find $\eta_1 \in D_n$ so that $T^*\eta_1 = \xi$. Then

$$0 \leq p(y) - p_n(y) = (p(y) - \langle \eta, y\rangle) + (\langle T^*\eta, x\rangle - \langle \xi, x\rangle) + \big(\langle \eta_1, y\rangle - p_n(y)\big) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon.$$

This proves (i).

Let us prove (ii). Fix any $n \in \mathbb{N}$. Let G_n denote the set of all points in X where q_n is Fréchet differentiable; this is a dense set as X is Asplund. Fix any $x \in G_n$. We observe that

$$\begin{aligned}\sup_{h \in B_X} \big[p_n(Tx + tTh) &+ p_n(Tx - tTh) - 2p_n(Tx)\big] \\ &= \sup_{h \in B_X} \big[q_n(x + th) + q_n(x - th) - 2q_n(x)\big] \\ &= o(t)\end{aligned}$$

for $t \downarrow 0$. And, as $\overline{TX} = Y$, we get that p_n is Gâteaux differentiable (even differentiable in a stronger sense) at Tx and $T^*(p_n'(Tx)) = q_n'(x)$.

Fix now any point $x_0 \in G_n$ and denote

$$y_0 = Tx_0, \quad q_n'(x_0) = x_0^*, \quad p_n'(y_0) = y_0^*.$$

Šmulyan's lemma, see [5, Theorem I.1.4 (i)] yields $\alpha > 0$ such that the slice

$$S(C_n, x_0, \alpha) := \big\{x^* \in C_n;\ \langle x^*, x_0\rangle > \langle x_0^*, x_0\rangle - \alpha\big\}$$

has norm diameter less than $\frac{1}{n}$. Since the subdifferential mapping $\partial p_n : Y \to 2^{Y^*}$ is norm to weak* upper semicontinuous, there exists an open ball $U \subset Y$ such that $U \ni Tx_0$ and

$$\partial p_n(U) \subset S(D_n, y_0, \alpha) := \{y^* \in D_n;\ \langle y^*, y_0\rangle > \langle y_0^*, y_0\rangle - \alpha\}.$$

Fix for a while any $y \in U \cap T(G_n)$ and find $x \in G_n$ so that $y = Tx$. Then $p_n'(y) = p_n'(y_0)$. Indeed, since p_n is differentiable at y and $p_n'(y) \in S(D_n, y_0, \alpha)$, we have that $q_n'(x) = T^*(p_n'(y)) \in T^*(S(D_n, y_0, \alpha)) = S(C_n, x_0, \alpha)$. But the latter slice has diameter less than $\frac{1}{n}$, and Šmulyan's lemma [5, Theorem I.1.4] easily implies that the (Fréchet) derivatives $q_n'(x_0)$ and $q_n'(x)$ must belong to M_n. Therefore $q_n'(x) = q_n'(x_0)$, and so $p_n'(y) = T^{*-1}(q_n'(x)) = T^{*-1}(q_n'(x_0)) = p_n'(y_0)$. Now, as p_n is convex, Lemma applied for $\varphi(t) := p_n(y_0 + t(y - y_0))$, $t \in \mathbb{R}$, yields that

$$p_n(y) = p_n(y_0) + p_n'(y_0)(y - y_0) \tag{1}$$

This holds for every $y \in U \cap T(G_n)$. And as the latter set is dense in U (Yes, the density of G_n implies the density of $T(G_n)$ in Y.), and p_n is continuous, (1) is valid for every $y \in U$. This means that p_n is affine on U. Finally, as $T(G_n)$ was dense in Y, the proof of (ii) is finished. □

Proof of Theorem 1. By adding a suitable constant to f, we may and do assume that $f(0) = -1$. Let $p : X \times \mathbb{R} \to [0, +\infty)$ be the Minkowski functional of the epigraph $\operatorname{epi} f := \{(x,t) \in X \times \mathbb{R};\ f(x) \le t\}$, that is,

$$p(x,t) = \inf\{\lambda > 0;\ f(\tfrac{x}{\lambda}) \le \tfrac{t}{\lambda}\}, \quad (x,t) \in X \times \mathbb{R}.$$

From the continuity of f find $\delta > 0$ so small that $\sup f(\delta B_X) < -\frac{1}{2}$. We observe that for every $x \in \delta B_X$ and every $t \in (-1, +\infty)$ we have $(x,t) \in \operatorname{epi} f$ and so $p(x,t) \le 1$. Hence, p is continuous. Let $p_n : X \times \mathbb{R} \to [0, +\infty)$, $n \in \mathbb{N}$, be the functions found in Proposition for our p. We may and do assume that $p_n(0,-1) > p(0,-1) - 1$ $(= 0)$ for all $n \in \mathbb{N}$. Fix any $n \in \mathbb{N}$ for a while. Consider any fixed $x \in X$ and put

$$f_n(x) = \inf\{t \in \mathbb{R};\ p_n(x,t) \le 1\};$$

then $p_n(x, f(x)) \le p(x, f(x)) = 1$ and so $f_n(x) \le f(x)$. Also, $p_n(x, f_{n+1}(x)) \le p_{n+1}(x, f_{n+1}(x)) = 1$ and so $f_n(x) \le f_{n+1}(x)$. Note that $f_n(x) > -\infty$. Indeed, otherwise we would have

$$(0 <)\ \ p_n(0,-1) = \lim_{t \to -\infty} p_n(\tfrac{x}{-t}, -1) = \lim_{t \to -\infty} \tfrac{1}{-t} p_n(x,t) \le \lim_{t \to -\infty} \tfrac{1}{-t} = 0,$$

a contradiction. Thus $f_n(x) \in \mathbb{R}$. Doing so for every $x \in X$, we get a function $f_n : X \to \mathbb{R}$, with $f_n \le f$. Let us prove that $f_n(x) \to f(x)$ as $n \to \infty$. Assume, by contrary that there exists $\Delta > 0$ such that $f_n(x) \le f(x) - \Delta$ for all $n \in \mathbb{N}$. Then $p(x, f(x) - \Delta) = \lim_{n \to \infty} p_n(x, f(x) - \Delta) \le 1$ and so $(x, f(x) - \Delta) \in \operatorname{epi} f$, i.e., $f(x) - \Delta \ge f(x)$, a contradiction. The function f_n is also convex. Indeed, for $x_i \in X$, $\alpha_i > 0$, $i = 1, 2$, with $\alpha_1 + \alpha_2 = 1$, we have

$$p_n(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 f_n(x_1) + \alpha_2 f_n(x_2)) \le \alpha_1 p_n(x_1, f_n(x_1)) + \alpha_2 p_n(x_2, f_n(x_2)) = 1,$$

and so $f_n(\alpha_1 x_1 + \alpha_2 x_2) \le \alpha_1 f_n(x_1) + \alpha_2 f_n(x_2)$.

Now from Proposition we have that p_n is locally affine on an open dense subset $G_n \subset X \times \mathbb{R}$. Let $H_n \subset X$ be the set of all $x \in X$ such that $(x, f_n(x)) \in G_n$. It is easy to check that H_n is an open set. We shall show that it is dense in X. So fix any $0 \neq x_0 \in X$ and any $\epsilon > 0$. Find $(x,t) \in G_n$ so that $\|x - x_0\| < \frac{\epsilon}{2}$ and $0 < p(x,t) - 1 < \frac{\epsilon}{2\|x_0\|}$; this can be done as G_n is dense in X. Put $(x_1, t_1) = \frac{1}{p_n(x,t)}(x,t)$. The positive homogeneity of p_n guarantees that $p_n(x_1,t_1) = 1$. Hence $t_1 = f_n(x_1)$ and so $x_1 \in H_n$. Moreover

$$\begin{aligned} \|x_1 - x_0\| &\leq \Big\| x_1 - \frac{x_0}{p_n(x,t)} \Big\| + \Big\| \frac{x_0}{p_n(x,t)} - x_0 \Big\| \\ &\leq \|x - x_0\| + \|x_0\| \big(p_n(x,t) - 1\big) < \tfrac{\epsilon}{2} + \|x_0\| \tfrac{\epsilon}{2\|x_0\|} = \epsilon. \end{aligned}$$

It remains to show that f_n is locally affine on H_n. Fix any $x_0 \in H_n$. Thus $(x_0, f_n(x_0)) \in G_n$ and $p_n(x_0, f_n(x_0)) = 1$. Then for all $x \in X$ close enough to x_0 we have

$$\begin{aligned} 1 &= p_n(x, f_n(x)) = p_n(x_0, f_n(x_0)) + p_n'(x_0, f_n(x_0))\big((x, f_n(x)) - (x_0, f_n(x_0))\big) \\ &= 1 + p_n'(x_0, f_n(x_0))(x - x_0, 0) + p_n'(x_0, f_n(x_0))(0, f_n(x) - f_n(x_0)), \end{aligned}$$

and so, putting

$$A(\cdot) = p_n'(x_0, f_n(x_0))(\cdot, 0), \quad \xi = p_n'(x_0, f_n(x_0))(0, 1),$$

we have that

$$1 = 1 + A(x - x_0) + \xi(f_n(x) - f_n(x_0)). \tag{2}$$

Also, for $\tau > 0$ small enough we have

$$1 < p_n(x_0, f_n(x_0) - \tau) = p_n(x_0, f_n(x_0)) + p_n'(x_0, f_n(x_0))(0, -\tau) = 1 - \tau\xi,$$

and hence $\xi < 0$. Thus (2) yields that

$$f_n(x) = f_n(x_0) + \frac{1}{\xi} A(x - x)$$

for all $x \in X$ close to x_0, We thus proved that f_n is affine in the vicinity of x_0. □

Remark 4. We do not know if our Theorem can be extended to some larger subclasses of weak Asplund spaces. In particular, we would like to clarify the situation in Banach spaces whose dual norm is strictly convex, and in subspaces of Asplund gererated spaces. Note that both classes have, of course, fragmentable dual unit ball, see [6, page 99] and [6, Theorems 5.1.10 (i), 5.2.3].

References

[1] L. Cheng and S. Chen, *Smooth Approximaton of Convex Functions in Banach Spaces*, to appear.

[2] L. Cheng, S. Shi, and E. S. Lee, *Generic Fréchet Differentiability of Convex Functions functions on Non-Asplund Spaces*, J. Math. Anal. Appl. 214 (1997), 367-377.

[3] L. Cheng, S. Shi, B. Wang and E. S. Lee, *Generic Fréchet differentiability of convex functions functions dominated by a lower semi-continuous convex function*, J. Math. Anal. Appl. 225 (1998), 389-400.

[4] L. Cheng and C. Wu, *The w^*-Fréchet differentiability property, the Radon-Nikodym property and w^*-Asplund spaces*, Acta Math. Sinica 46 (2003), 385-390 (Chinese).

[5] R. Deville, G. Godefroy and V. Zizler, Smoothness and Renormings in Banach Spaces, Longman, Harlow, New York, 1993.
[6] M. J. Fabian, Gateaux Differentiability of Convex Functions and Topology, Weak Asplund Spaces, John Wiley & Sons, New York 1997.
[7] J. P. Moreno, *On weak* Mazur intersection property and Fréchet differentiable norms on dense open sets*, Bull. Sci. Math. 122 (1998), 93-105.
[8] R. R. Phelps, Convex Functions, Monotone Operators and Differentiability, Lect. Notes in Math. Vol. 1364, Springer-Verlag, New York, Berlin, 1989.
[9] D. Preiss, R. R. Phelps and I. Namioka, *Smooth Banach space and monotone or usco mappings*, Israel J. Math. 72 (1990), 257-279.

Manuscript received November 24, 2004

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