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ITERATIVE CONSTRUCTION OF FIXED POINTS OF NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper we introduce a new iteration process whose rate of convergence is similar to the Picard iteration process and faster than other fixed point iteration processes. We then apply it to deal with the problem of approximation of fixed points of nearly asymptotically nonexpansive mappings. Our iteration process is independent from Mann [Proc. Amer. Math. Soc. 4 (1953), 506-610] and Ishikawa [Proc. Amer. Math. Soc. 4 (1974), 147-150] iteration processes.

1. Introduction

Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. Then T is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a positive number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

A Lipschitzian mapping T is said to be *uniformly k-Lipschitzian* if $k_n = k$ for all $n \in \mathbb{N}$ and *asymptotically nonexpansive* (cf. [13]) if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

It is easy to see that every nonexpansive mapping T (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) is asymptotically nonexpansive with sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in\mathbb{N}} k_n$.

The class of nearly Lipschitzian mappings is an important generalization of the class of Lipschitzian mappings and was introduced by Sahu in [28].

Let C be a nonempty subset of a Banach space X and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is said to be *nearly Lipschitzian* with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n) \text{ for all } x, y \in C. \tag{1.1}$$

The infimum of constants k_n for which (1.1) holds is denoted by $\eta(T^n)$ and called *the nearly Lipschitz constant*.

A nearly Lipschitzian mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be

(i) *nearly nonexpansive* if $\eta(T^n) = 1$ for all $n \in \mathbb{N}$,

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(ii) *nearly asymptotically nonexpansive* if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$,

(iii) *nearly uniformly* k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$,

(iv) *nearly uniform k-contraction* if $\eta(T^n) \leq k < 1$ for all $n \in \mathbb{N}$.

Example 1.1. Let $X = \mathbb{R}$, $C = [0,1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{1}{2} \text{ if } x \in [0, \frac{1}{2}], \\ 0 \text{ if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, T is discontinuous and a non-Lipschitzian mapping. However, it is a nearly nonexpansive mapping and hence nearly asymptotically nonexpansive. Indeed, for a sequence $\{a_n\}$ with $a_1 = \frac{1}{2}$ and $a_n \to 0$, we have

$$\|Tx - Ty\| \leq \|x - y\| + a_1 \text{ for all } x, y \in C$$

and

$$\|T^n x - T^n y\| \leq \|x - y\| + a_n \text{ for all } x, y \in C \text{ and } n \geq 2,$$

since

$$T^n x = \frac{1}{2} \text{ for all } x \in [0,1] \text{ and } n \geq 2.$$

Indeed, Sahu in [28] initiated the fixed theory of nearly Lipschitzian mappings by establishing a nearly contraction principle for demicontinuous nearly contraction mappings and an existence theorem for demicontinuous nearly asymptotically nonexpansive mappings.

On the other hand, the following fixed point iteration processes have been extensively studied by many authors for approximating either fixed points of nonlinear mappings (when these mappings are already known to have fixed points) or solutions of nonlinear operator equations:

(a) The Mann iteration process (see, for example [22], [26]) is defined as follows: For C a convex subset of a linear space X and T a mapping of C into itself, the sequence $\{x_n\}$ is generated from $x_1 \in C$, and is defined by

$$\text{(M)} \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n = M(x_n, \alpha_n, T), n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in [0,1] which satisfies the conditions:

(i) $0 \leq \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(b) The Ishikawa iteration process (see, for example [15], [26]) is defined as follows: with C and T as in (a), the sequence $\{x_n\}$ is generated from $x_1 \in C$, and is defined by

$$\text{(I)} \qquad \begin{cases} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1] which satisfy the conditions:

(i') $0 \leq \alpha_n \leq \beta_n < 1$,

$(ii^{'})$ $\lim_{n\to\infty} \beta_n = 0$,

$(iii^{'})$ $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

(c) The modified Mann iteration process (see, for example [30, 31]) is defined as follows: with C and T as in (a), the sequence $\{x_n\}$ is generated from $x_1 \in C$, and is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, n \in \mathbb{N}, \tag{1.2}$$

where $\{\alpha_n\}$ is a real sequence in [0,1] which satisfies the condition $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$.

It is clear that the process (M) is not a special case of the process (I) because of condition $(i^{'})$.

It is well known that the process (M) does not converge to fixed points of Lipschitz pseudocontractive mapping T even if the domain of T is a compact convex subset of a Hilbert space (see [8]). Indeed, the process (I) was introduced by Ishikawa [15] to deal with the problem of approximation of fixed points of Lipschitz pseudocontractive mappings with compact convex domain in Hilbert spaces.

In recent papers (see, e.g. [6, 10, 34, 36, 37]) the condition $(i^{'})$ $0 \leq \alpha_n \leq \beta_n < 1$ has been replaced by the general condition (i^0) $0 \leq \alpha_n, \beta_n < 1$. With this general setting, the process (I) is a natural generalization of the process (M).

Remark 1.2. If the process (M) is convergent, then the process (I) with condition (i^0) is also convergent under suitable conditions on α_n and β_n.

Let C be a nonempty subset of a linear space X and $T : C \to X$ a mapping with $F(T) = \{x : Tx = x\} \neq \emptyset$. Then these iteration processes are included in the following more general iteration process:

$$x_{n+1} = f(x_n, T), n \in \mathbb{N}. \tag{GI}$$

In fact, by setting $f(x_n, T) = M(x_n, \alpha_n, T)$, (GI) reduces to the Mann process iteration. The iteration process (GI) also includes the modified Mann iteration process.

We say that the iterative sequence $\{x_n\}$ defined by (GI) for the mapping T with $F(T) \neq \emptyset$ has property (D_1) if $\lim_{n\to\infty} \|x_n - p\|$ for all $p \in F(T)$. Also $\{x_n\}$ has property (D_2) if $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

The properties $(D_1) \sim (D_2)$ play important role in the approximation of fixed points of nonexpansive and asymptotically nonexpansive mappings by means of the Mann and Ishikawa iteration processes (see [1, 4, 5, 7, 9, 10, 12, 14, 16, 18, 19, 21, 23, 25, 27, 30, 31, 32, 34, 35]) in Banach spaces under suitable geometric structures.

It is well known that the Picard iteration process is faster than the Mann iteration process for contraction mappings (see Proposition 3.1). This brings us to the following open problem:

Problem 1.3. *Is it possible to develop an iteration process whose rate of convergence is similar to the Picard iteration process and faster than the Mann iteration process for contraction mappings?*

In this paper, motivated by Problem 1.3, we introduce a new iteration process which provides a positive solution to Problem 1.3 and then this new iteration process is applied to deal with the problem of approximation of fixed points of non-Lipschitzian nearly asymptotically nonexpansive mappings $T : C \to C$ in a Banach space X.

More precisely, conditions on C, X and T are discussed in section 3 (see Theorem 3.8) which guarantee that our iterative sequence $\{x_n\}$ enjoys properties $(D_1) \sim (D_2)$. In particular, strong convergence of our iteration process for nearly nonexpansive mappings is discussed in a strictly convex Banach space. Our iteration process is independent of the Ishikawa [15] and Mann [22] iteration processes and is of independent interest.

2. Preliminaries

Let X be a Banach space and let X^* be its dual. The value of $f \in X^*$ at $x \in X$ will be denoted by $\langle x, f\rangle$. Then the multivalued mapping $J : X \to 2^{X^*}$ defined by

$$J(x) := \{f \in X^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}$$

is called the *normalized duality mapping.*

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in $S = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable.* It is well known that if X is smooth, then J is single-valued. The norm of X is said to be *Fréchet differentiable* if for each $x \in S$, this limit is attained uniformly for $y \in S$. In this case

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \le \frac{1}{2}\|x + h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + b(\|h\|) \tag{2.1}$$

for all bounded x, h in X, where $J(x) = \partial\frac{1}{2}\|x\|^2$ is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$, and $b(.)$ is a function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} b(t)/t = 0$.

A Banach space X is said to satisfy *Opial condition* (see [24]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \ y \in X.$$

A Banach space X is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in X and each point $x \in X$,

$$x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\| \text{ imply } x_n \to x.$$

Let X be a Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be *demiclosed at a point* $p \in D(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $z \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tz = p$.

The modulus of convexity of a Banach space X is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|\frac{x + y}{2}\| : \|x\|, \|y\| \le 1, \|x - y\| \ge \varepsilon\}$$

for all $\varepsilon \in [0,2]$. X is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Throughout this paper the set of all weak subsequential limits of $\{x_n\}$ will be denoted by $\omega_w(\{x_n\})$.

Proposition 2.1 (Limaye [20]). *Let $\{x_n\}$ be a bounded sequence in reflexive Banach space X and $A_n = \overline{co}(\{x_n\}_{k\geq n})$. If $\cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty}\overline{co(\{x_n, x_{n+1}, \cdots\})} = \{x\}$, then $x_n \rightharpoonup x$.*

Proposition 2.2 (Dunford and Schwartz [11]). *A subset of a reflexive Banach space X is compact in the weak topology of X iff it is bounded in the strong topology.*

Proposition 2.3 (Bruck [2]). *Let $\{x_n\}$ be a sequence in a weakly compact subset of a Banach space X. Then $\overline{co}(\omega_w(\{x_n\})) = \cap_{n=1}^{\infty}\overline{co}(\{x_k\}_{k\geq n})$.*

Combining Propositions 2.1 and 2.3, we obtain the following useful result:

Proposition 2.4. *Let $\{x_n\}$ be a bounded sequence in reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \rightharpoonup x$.*

Proof. Let $C := \{x_n\}$ be a bounded set. Note that X is reflexive, C is a weakly compact set by Proposition 2.2. Let $A_n = \overline{co}(\{x_k\}_{k\geq n})$. It follows from Proposition 2.3 that $\overline{co}(\omega_w(\{x_n\})) = \cap_{n=1}^{\infty}\overline{co}(\{x_k\}_{k\geq n})$. Thus, $\cap_{n=1}^{\infty} A_n = \cap_{n=1}^{\infty}\overline{co}(\{x_k\}_{k\geq n}) = \{x\}$. Therefore, $x_n \rightharpoonup x$ by Proposition 2.1 □

In what follows, we shall make use of the following lemmas.

Lemma 2.5 (Osilike and Aniagbosor [25]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that*

$$\beta_n \geq 1 \text{ and } \delta_{n+1} \leq \beta_n\delta_n + \gamma_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Lemma 2.6 (Schu [31]). *Let X be a uniformly convex Banach space and let $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$ and $\lim_{n\to\infty}\|(1-\alpha_n)x_n + \alpha_n y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.7 (Kaczor [17]). *Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in X and $x, y \in \omega_w(\{x_n\})$. Suppose $\lim_{n\to\infty}\|tx_n + (1-t)x - y\|$ exists for all $t \in [0,1]$. Then $x = y$.*

Lemma 2.8 (Sahu and Beg [29]). *Let X be a uniformly convex Banach space satisfying the Opial condition, C a nonempty closed convex (not necessarily bounded) subset of X and $T : C \to C$ a uniformly continuous nearly asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.9. *Let C be a nonempty convex subset of a Banach space X and let $G_n : C \to X$ $(n = 1, 2, \dots)$ be mappings with $\bigcap_{n\in\mathbb{N}} F(G_n) \neq \emptyset$ such that*

$$\|G_n x - G_n y\| \leq L_n||x - y|| + \rho_n \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}, \tag{2.2}$$

where $\{L_n\}$ and $\{\rho_n\}$ are sequences of real numbers such that

(i) $L_n \geq 1$ *for each* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty}(L_n - 1) < \infty$,

(ii) $\rho_n \geq 0$ *for each* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} \rho_n < \infty$.

Let $\{x_n\}$ *be a sequence in* C *generated from* $x_1 \in C$ *and defined by*

$$x_{n+1} = G_n x_n \text{ for all } n \in \mathbb{N}.$$

Then we have the following:

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in \bigcap_{n\in\mathbb{N}} F(G_n)$;

(b) *if* X *is uniformly convex, it follows that* $\lim_{n\to\infty} \|tx_n + (1-t)v_1 - v_2\|$ *exists for all* $v_1, v_2 \in \bigcap_{n\in\mathbb{N}} F(G_n)$ *and* $t \in [0, 1]$;

(c) *if* X *is a real uniformly convex Banach space with Fréchet differentiable norm, it follows that* $\lim_{n\to\infty} \langle x_n, J(v_1 - v_2)\rangle$ *exists for all* $v_1, v_2 \in \bigcap_{n\in\mathbb{N}} F(G_n)$.

Proof. (a) Let $v \in \bigcap_{n\in\mathbb{N}} F(G_n)$. Observe that

$$\|x_{n+1} - v\| = \|G_n x_n - v\| \leq L_n\|x_n - v\| + \rho_n \text{ for all } n \in \mathbb{N},$$

so it follows from Lemma 2.5 that $\lim_{n\to\infty} \|x_n - v\|$ exists.

(b) Set

$$a_n(t) := \|tx_n + (1-t)v_1 - v_2\|, \quad n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n(0) = \|v_1 - v_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v_2\|$ exist. It now remains to show that the proposition is true for $t \in (0, 1)$. Define

$$S_{n,m} := G_{n+m-1}G_{n+m-2}\ldots G_n \text{ for all } n, m \in \mathbb{N}.$$

Then $x_{n+m} = S_{n,m}x_n, S_{n,m}v = v$ for all $v \in \bigcap_{n\in\mathbb{N}} F(G_n)$ and

$$\|S_{n,m}x - S_{n,m}y\| \leq \left(\prod_{j=n}^{n+m-1} L_j\right)\left(\|x - y\| + \sum_{i=n}^{n+m-1} \rho_i\right)$$

for all $x, y \in C$.

It is well known (see [3], p. 108) that

$$\|tx + (1-t)y\| \leq 1 - 2t(1-t)\delta_X(\|x - y\|) \tag{2.3}$$

for all $t \in [0,1]$ and for all $x, y \in X$ such that $\|x\| \le 1, \|y\| \le 1$. Set

$$
\begin{aligned}
H_{n,m} &:= \prod_{j=n}^{n+m-1} L_j,\ H_n := \prod_{j=n}^{\infty} L_j;\\
\sigma_{n,m} &:= \sum_{i=n}^{n+m-1} \rho_i,\ \sigma_n := \sum_{i=n}^{\infty} \rho_i;\\
b_{n,m} &:= \Big[S_{n,m}(tx_n + (1-t)v_1) - tS_{n,m}x_n - (1-t)S_{n,m}v_1\Big]||x_n - v_1||;\\
c_{n,m} &:= \Big[S_{n,m}v_1 + S_{n,m}x_n - 2S_{n,m}(tx_n + (1-t)v_1)\Big]\sigma_{n,m};\\
d_{n,m} &:= H_{n,m} \times \Big[t||x_n - v_1|| + \sigma_{n,m}\Big] \times \Big[(1-t)||x_n - v_1|| + \sigma_{n,m}\Big];\\
d_n &:= H_n \times \Big[t||x_n - v_1|| + \sigma_n\Big] \times \Big[(1-t)||x_n - v_1|| + \sigma_n\Big];\\
e_{n,m} &:= \Big[tS_{n,m}v_1 + (1-2t)S_{n,m}(tx_n + (1-t)v_1) - (1-t)S_{n,m}x_n\Big]\sigma_{n,m};\\
u_{n,m} &:= \Big[S_{n,m}v_1 - S_{n,m}(tx_n + (1-t)v_1)\Big] / \Big[H_{n,m}(t\|x_n - v_1\| + \sigma_{n,m})\Big];\\
v_{n,m} &:= \Big[S_{n,m}(tx_n + (1-t)v_1) - S_{n,m}x_n\Big] / \Big[H_{n,m}((1-t)\|x_n - v_1\| + \sigma_{n,m})\Big].
\end{aligned}
$$

Then $\|u_{n,m}\| \le 1$ and $\|v_{n,m}\| \le 1$ and it follows from (2.3) that

$$2t(1-t)\delta_X(\|u_{n,m} - v_{n,m}\|) \le 1 - \|tu_{n,m} + (1-t)v_{n,m}\|. \tag{2.4}$$

Observe that

$$\|u_{n,m} - v_{n,m}\| = \frac{\|b_{n,m} - c_{n,m}\|}{c_{n,m}},$$

and

$$\|tu_{n,m} + (1-t)v_{n,m}\| = \frac{||t(1-t)(||x_n - v_1||)(S_{n,m}v_1 - S_{n,m}x_n) + e_{n,m}\sigma_{n,m}||}{d_{n,m}}.$$

From (2.4), we obtain

$$
\begin{aligned}
&2t(1-t)d_{n,m}\delta_X\left(\frac{||b_{n,m} - c_{n,m}||}{d_{n,m}}\right)\\
&\qquad \le d_{n,m} - \|t(1-t)(\|x_n - v_1\|)(v_1 - x_{n+m}) + e_{n,m}\sigma_{n,m}\|.
\end{aligned}\tag{2.5}
$$

Also notice

$$
\begin{aligned}
d_{n,m} &= H_{n,m}[t(1-t)||x_n - v_1||^2 + (||x_n - v_1|| + \sigma_{n,m})\sigma_{n,m}]\\
&\le H_n d_n
\end{aligned}
$$

so for some constant $K_1 > 0$, it follows from (2.5) that

$$
\begin{aligned}
2d_{n,m}\delta_X\left(\frac{||b_{n,m} - c_{n,m}||}{d_{n,m}}\right) &\le H_n\left(||x_n - v_1||^2 + \frac{K_1\sigma_n}{t(1-t)}\right)\\
&\quad -\|x_n - v_1\|\ \|x_{n+m} - v_1\| + \frac{\|e_{n,m}\|\sigma_n}{t(1-t)}.
\end{aligned}\tag{2.6}
$$

Let $d =: sup\{d_nH_n : n \in \mathbb{N}\}$. Since X is uniformly convex, then $\delta_X(s)/s$ is nondecreasing and hence it follows from (2.6) that

$$2d\delta_X\left(\frac{||b_{n,m}-c_{n,m}||}{d}\right) \le H_n\left(||x_n-v_1||^2+\frac{K_1\sigma_n}{t(1-t)}\right) \tag{2.7}$$
$$-\|x_n-v_1\|\ \|x_{n+m}-v_1\|+\frac{K_2\sigma_n}{t(1-t)}$$

for some constant $K_2 > 0$. Since $\lim_{n\to\infty}\|x_n-v_1\|$ exists we have that $\lim_{n\to\infty}\|x_n-v_1\| = \lim_{n\to\infty}\|x_{n+m}-v_1\|$. Since $\delta_X(0) = 0$ and $\lim_{n\to\infty} H_n = 1$, then the continuity of δ_X yields from (2.7) that $\lim_{m,n\to\infty}||b_{n,m}-c_{n,m}|| = 0$. Since

$$\begin{aligned}||b_{n,m}|| &\le ||b_{n,m}-c_{n,m}||+||c_{n,m}||\\ &\le ||b_{n,m}-c_{n,m}||+K_3\sigma_n\end{aligned}$$

for some constant $K_3 > 0$, it follows that $\lim_{m,n\to\infty}||b_{n,m}|| = 0$. Observe that

$$\begin{aligned}&a_{n+m}(t)\\ &\le \|tx_{n+m}+(1-t)v_1-v_2+[S_{n,m}(tx_n+(1-t)v_1)-tS_{n,m}x_n-(1-t)S_{n,m}v_1]\|\\ &\quad+\|-[S_{n,m}(tx_n+(1-t)v_1)-tS_{n,m}x_n-(1-t)S_{n,m}v_1]\|\\ &=\|S_{n,m}(tx_n+(1-t)v_1)-S_{n,m}v_2\|\\ &\quad+\|S_{n,m}(tx_n+(1-t)v_1)-tS_{n,m}x_n-(1-t)S_{n,m}v_1\|\\ &\le H_{n,m}(a_n(t)+\sigma_{n,m})+\|S_{n,m}(tx_n+(1-t)v_1)-tS_{n,m}x_n-(1-t)S_{n,m}v_1\|\\ &\le H_n(a_n(t)+\sigma_n)+\|S_{n,m}(tx_n+(1-t)v_1)-tS_{n,m}x_n-(1-t)S_{n,m}v_1\|.\end{aligned}$$

Hence $\limsup_{m\to\infty} a_m(t) \le \liminf_{n\to\infty} H_n(a_n(t)+\sigma_n)+\lim_{m,n\to\infty}||b_{n,m}|| = \liminf_{n\to\infty} a_n(t)$, completing the proof of Part (b).

(c) It follows from (2.1) that

$$\begin{aligned}\frac{1}{2}\|v_1-v_2\|^2 &+ t\langle x_n-v_1, J(v_1-v_2)\rangle\\ &\le \frac{1}{2}a_n^2(t) \le \frac{1}{2}\|v_1-v_2\|^2+t\langle x_n-v_1, J(v_1-v_2)\rangle+b(t\|x_n-v_1\|).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\|v_1-v_2\|^2+t\limsup_{n\to\infty}\langle x_n-v_1, J(v_1-v_2)\rangle &\le \frac{1}{2}\lim_{n\to\infty} a_n^2(t)\\ &\le \frac{1}{2}\|v_1-v_2\|^2+t\liminf_{n\to\infty}\langle x_n-v_1, J(v_1-v_2)\rangle+o(t).\end{aligned}$$

Hence $\limsup_{n\to\infty}\langle x_n, J(v_1-v_2)\rangle \le \liminf_{n\to\infty} \langle x_n-v_1, J(v_1-v_2)\rangle+\frac{o(t)}{t}$. On letting $t\to 0^+$, we see that $\lim_{n\to\infty}\langle x_n, J(v_1-v_2)\rangle$ exists. □

Lemma 2.10. *Let X be a reflexive Banach space satisfying the Opial condition, C a nonempty closed convex subset of X and $T : C \to X$ a mapping such that*

(i) $F(T) \neq \emptyset$,

(ii) $I - T$ *is demiclosed at zero.*

Let $\{x_n\}$ *be a sequence in* C *satisfying the following properties:*

(D_1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F(T)$*;*

(D_2) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Then $\{x_n\}$ *converges weakly to a fixed point of* T*.*

Proof. Notice since X is reflexive and $\{x_n\}$ is bounded by (D_1) that $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to p. Since $\{x_{n_j}\} \subset C$ and C is weakly closed, then $p \in C$. From (D_2), $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$ and since $I - T$ is demiclosed at zero, we have $(I - T)p = 0$, so that $p \in F(T)$. To complete the proof, we show that $\{x_n\}$ converges weakly to a fixed point of T, so it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, p. Suppose there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \neq p$. As in the case of p, we must have $q \in C$ and $q \in F(T)$. It follows from (D_1) that $\lim_{n\to\infty} ||x_n - p||$ and $\lim_{n\to\infty} ||x_n - q||$ exist. Since X satisfies the Opial condition, we have

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{j\to\infty} ||x_{n_j} - p|| < \lim_{j\to\infty} ||x_{n_j} - q|| = \lim_{n\to\infty} ||x_n - q||,$$

$$\lim_{n\to\infty} ||x_n - q|| = \lim_{k\to\infty} ||x_{n_k} - q|| < \lim_{k\to\infty} ||x_{n_k} - p|| = \lim_{n\to\infty} ||x_n - p||,$$

which is a contradiction. Hence $p = q$ so $\omega_w(\{x_n\})$ is a singleton. Thus, $\{x_n\}$ converges weakly to p. □

Lemma 2.11. *Let* C *be a nonempty closed convex subset of a real Banach space* X *and* $T : C \to C$ *a mapping such that*

(i) $F(T) \neq \emptyset$,

(ii) $I - T$ *is demiclosed at zero.*

Let $\{x_n\}$ *be a sequence in* C *which satisfies property* (D_2)*. Suppose* $\{x_n\}$ *satisfies one of the following:*

(a) X *is uniformly convex with Fŕechet differentiable norm and*

$$\lim_{n\to\infty} \langle x_n, J(p - q)\rangle \text{ exists for all } p, q \in F(T); \tag{2.8}$$

(b) X *is reflexive,* X^* *has the Kadec-Klee property and* $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ *exists for all* $t \in [0, 1]$ *and for all* $p, q \in \omega_w(\{x_n\})$*.*

Then $\{x_n\}$ *converges weakly to a fixed point of* T*.*

Proof. We show that $\omega_w(\{x_n\})$ has exactly one point. Let $u, v \in \omega_w(\{x_n\})$ with $u \neq v$. Then for some subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, we have $x_{n_i} \rightharpoonup$

u and $x_{n_j} \rightharpoonup v$. By (D_2), $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, which implies by demiclosedness of $I - T$ at zero that $u, v \in \omega_w(\{x_n\}) \subset F(T)$.

(a) From (2.8), we have

$$\langle u, J(p-q)\rangle = d \text{ (say), and } \langle v, J(p-q)\rangle = d,$$

so

$$\langle u - v, J(p-q)\rangle = 0 \text{ for all } p, q \in F(T). \tag{2.9}$$

From (2.9) we obtain that

$$\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0,$$

a contradiction. Hence $\omega_w(\{x_n\})$ is a singleton. Therefore, $\{x_n\}$ converges weakly to a fixed point of T.

(b) By assumption, $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ exists. Lemma 2.7 guarantees that $u = v$, a contradiction. Hence $\omega_w(\{x_n\})$ is a singleton. Therefore, $\{x_n\}$ converges weakly to a fixed point of T. □

3. S-iteration process and convergence analysis

First, we introduce a new iteration process:

For C a convex subset of a linear space X and T a mapping of C into itself, the iterative sequence $\{x_n\}$ of our iteration process is generated from $x_1 \in C$, and is defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n) = \infty. \tag{3.2}$$

The iterative sequence $\{x_n\}$ is well defined since C is convex. We will call it *the S-iteration process*. It is easy to see that neither the process (M) nor the process (I) reduces to our iteration process and vice verse. Thus, the S-iteration process is independent of the Mann [22] and Ishikawa [15] iteration processes.

Throughout the paper we use the following notation for the S-iteration $\{x_n\}$ associated with the mapping T:

$$x_{n+1} = S(x_n, \alpha_n, \beta_n, T),\ n \in \mathbb{N},$$

where $S(x_n, \alpha_n, \beta_n, T) = (1-\alpha_n)Tx_n + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n]$.

Next, we compare the rate of convergence of the Picard, Mann and S-iteration processes for contraction mappings.

Proposition 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction mapping with Lipschitz constant k and a unique fixed point p. For $u_1 = v_1 = w_1 \in C$, define sequences $\{u_n\}, \{v_n\}$ and $\{w_n\}$ in C as follows:*

Picard iteration : $\quad u_{n+1} = Tu_n, n \in \mathbb{N};$

Mann iteration : $v_{n+1} = (1 - \alpha_n)v_n + \alpha_n T v_n, n \in \mathbb{N};$

S-iteration : $w_{n+1} = (1 - \alpha_n)Tw_n + \alpha_n T y_n,$
$y_n = (1 - \beta_n)w_n + \beta_n T w_n, n \in \mathbb{N},$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $(0, 1)$. *Then we have the following:*

(a) $\|u_{n+1} - p\| \leq k\|u_n - p\|$ *for all* $n \in \mathbb{N}$*;*

(b) $\|v_{n+1} - p\| \leq \|v_n - p\|$ *for all* $n \in \mathbb{N}$*;*

(c) $\|w_{n+1} - p\| \leq k\|w_n - p\|$ *for all* $n \in \mathbb{N}$.

Proof. Part (a) is obvious.

(b) Now part (b) follows from

$$\begin{aligned} \|v_{n+1} - p\| &= \|(1 - \alpha_n)(v_n - p) + \alpha_n(Tv_n - p)\| \\ &\leq (1 - \alpha_n)\|v_n - p\| + k\alpha_n\|v_n - p\| \\ &\leq [1 - (1 - k)\alpha_n]\|v_n - p\| \text{ for all } n \in \mathbb{N}. \end{aligned}$$

(c) For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|w_{n+1} - p\| &\leq (1 - \alpha_n)k\|w_n - p\| + \alpha_n k\|y_n - p\| \\ &\leq k[(1 - \alpha_n)\|w_n - p\| + \alpha_n((1 - \beta_n)\|w_n - p\| + k\beta_n\|w_n - p\|)] \\ &= k[1 - (1 - k)\alpha_n\beta_n]\|w_n - p\|. \end{aligned}$$

□

Remark 3.2. The rate convergence of our iteration process is similar to the Picard iteration process, but faster than the Mann iteration process for contraction mappings. This provides a positive answer of Problem 1.3.

Before presenting the main results of this section, we give definitions and a proposition:

Definition 3.3. *Let* C *be a convex subset of a linear space* X *and* $T : C \to C$ *a mapping. Then the modified S-iteration process is a sequence* $\{x_n\}$ *generated from* $x_1 \in C$*, and is defined by*

$$x_{n+1} = S(x_n, \alpha_n, \beta_n, T^n), n \in \mathbb{N}, \tag{3.3}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$.

Definition 3.4. *Let* C *be a nonempty subset of a Banach space* $X, T : C \to C$ *a mapping and* $k > 0$ *a real number. Then* T *will said to be asymptotic* k*-Lipschitzian if there exists a sequence* $\{a_n\}$ *in* $[0, \infty)$ *with* $a_n \to 0$ *such that*

$$\|T^n x - T^n y\| \leq (k + a_n)\|x - y\| \textit{ for all } x, y \in C \textit{ and } n \in \mathbb{N}.$$

We will say an asymptotic k-Lipschitzian is asymptotic k-contraction if $k \in (0, 1)$. We note that every asymptotic k-Lipschitzian with sequence $\{a_n\}$ is uniformly L-Lipschitzian with $L = \sup_{n \in \mathbb{N}}\{k + a_n\}$. It is obvious that an asymptotic 1-Lipschitzian mapping with sequence $\{a_n\}$ is asymptotically nonexpansive.

Proposition 3.5. *Let C be a nonempty bounded subset of a Banach space X and $T : C \to C$ an asymptotic k-contraction mapping with sequence $\{a_n\}$. Then T is a nearly uniformly k-contraction with sequence $\{a_n\}$.*

Proof. By the definition of asymptotic k-contraction mapping,

$$\begin{aligned}\|T^n x - T^n y\| &\le (k + a_n)\|x - y\| \\ &\le k\|x - y\| + a_n diam(C)\end{aligned}$$

for all $x, y \in C$ and $n \in \mathbb{N}$. □

By Proposition 3.5, we have the following implications:

contraction $\Rightarrow$ asymptotic k-contraction $\Rightarrow$ nearly uniformly k-contraction.

The following example shows that a nearly uniformly k-contraction is not necessarily a contraction.

Example 3.6. Let $X = \mathbb{R}, C = [0, 1]$ and $T : C \to C$ be a mapping defined by

$$Tx = \begin{cases} \frac{x}{2} & if \quad x \in [0, 1), \\ 0 & if \quad x = 1. \end{cases}$$

It is obvious that T is a non-Lipschitzian discontinuous mapping. Hence it is not a contraction. However, it is a nearly uniformly $\frac{1}{2}$-contraction with sequence $\{\frac{1}{2^n}\}$. Indeed,

$$\begin{aligned}\|T^n x - T^n y\| &\le \frac{1}{2^n}\|x - y\| + \frac{1}{2^n} \\ &\le \frac{1}{2}\|x - y\| + \frac{1}{2^n} \quad for\ all \quad n \in \mathbb{N}\ and\ x, y \in C.\end{aligned}$$

Our first result shows that the sequence $\{x_n\}$ defined by (3.3) converges strongly to fixed points of nearly uniformly k-contraction mappings in a Banach space.

Theorem 3.7. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a nearly uniformly k-contraction mapping with sequence $\{a_n\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$. For $x_1 \in C$, let $\{x_n\}$ be the sequence defined by (3.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Then the following hold:*

(a) $\|x_{n+1} - p\| \le k\|x_n - p\| + k(1 + k)a_n$ *for all* $n \in \mathbb{N}$;

(b) $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. Let $p \in F(T)$. Then

$$\begin{aligned}\|x_{n+1} - p\| &\le (1 - \alpha_n)\|T^n x_n - p\| + \alpha_n\|T^n y_n - p\| \\ &\le (1 - \alpha_n)k(\|x_n - p\| + a_n) + \alpha_n k(\|y_n - p\| + a_n) \\ &\le k[(1 - \alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| + a_n] \\ &\le k[(1 - \alpha_n)\|x_n - p\| + \alpha_n[(1 - \beta_n)\|x_n - p\| \\ &\quad +\beta_n k(\|x_n - p\| + a_n)] + a_n] \\ &\le k[1 - \alpha_n + \alpha_n[(1 - \beta_n) + k\beta_n]]\|x_n - p\| + k(1 + k)\alpha_n a_n \\ &= k[1 - (1 - k)\alpha_n\beta_n]\|x_n - p\| + k(1 + k)\alpha_n a_n\end{aligned}$$

$$
\begin{aligned}
(3.4) \qquad & \leq k\|x_n - p\| + k(1+k)a_n \\
& \leq \|x_n - p\| + k(1+k)a_n.
\end{aligned}
$$

Using Lemma 1 of Tan and Xu [34], we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. Set $\lim_{n\to\infty} \|x_n - p\| = d > 0$. It follows from (3.4) that

$$d \leq kd,$$

a contradiction. Therefore, $\{x_n\}$ converges strongly to a fixed point of T. □

We now show that the modified S-iteration process converges weakly to fixed points of nearly asymptotically nonexpansive mappings in uniformly convex Banach spaces. We begin with the following theorem which shows that the sequence $\{x_n\}$ defined by (3.3) has properties $(D_1) \sim (D_2)$.

Theorem 3.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $T : C \to C$ a nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n)\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{x_n\}$ be the modified S-iteration defined by (3.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $(0,1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the following hold:*

(a) $\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|y_n - p\|$ *exists for $p \in F(T)$;*

(b) $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$*;*

(c) *if T is uniformly continuous, it follows that $\{x_n\}$ has property (D_2).*

Proof. (a) Let $p \in F(T)$. For each $n \in \mathbb{N}$, define a mapping $G_n : C \to C$ by

$$G_n x = (1-\alpha_n)T^n x + \alpha_n T^n((1-\beta_n)x + \beta_n T^n x), \ x \in C.$$

Hence $x_{n+1} = G_n x_n$ for all $n \in \mathbb{N}$. Set $\eta := \sup_{n\in\mathbb{N}} \eta(T^n)$. Observe that

$$
\begin{aligned}
\|G_n x - G_n y\| &\leq (1-\alpha_n)\|T^n x - T^n y\| \\
&\quad + \alpha_n \|T^n((1-\beta_n)x + \beta_n T^n x) - T^n((1-\beta_n)y + \beta_n T^n y)\| \\
&\leq \eta(T^n)[(1-\alpha_n)(\|x-y\| + a_n) \\
&\quad + \alpha_n(\|(1-\beta_n)(x-y) + \beta_n(T^n x - T^n y)\| + a_n)] \\
&\leq \eta(T^n)[(1-\alpha_n)\|x-y\| + \alpha_n((1-\beta_n)\|x-y\| \\
&\quad + \beta_n \eta(T^n)(\|x-y\| + a_n)) + a_n)] \\
&\leq \eta(T^n)[(1-\alpha_n)\|x-y\| + \alpha_n \eta(T^n)\|x-y\| + (1+\eta(T^n))a_n] \\
&\leq L_n\|x-y\| + \rho_n \ \text{ for all } x, y \in C \text{ and } n \in \mathbb{N},
\end{aligned}
$$

where $L_n = \eta(T^n)^2$ and $\rho_n = \eta(1+\eta)a_n$. Moreover,

$$\sum_{n=1}^{\infty}(L_n - 1) = \sum_{n=1}^{\infty}(\eta(T^n)+1)(\eta(T^n)-1) \leq (1+\eta)\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$$

and $\sum_{n=1}^{\infty} \rho_n < \infty$. It is easily seen that $F(T) \subseteq F(G_n)$. It follows from Lemma 2.9(a) that $\lim_{n\to\infty} \|x_n - p\|$ exists. Set $d := \lim_{n\to\infty} \|x_n - p\|$. Since

$$\|T^n x_n - p\| \leq \eta(T^n)(\|x_n - p\| + a_n) \text{ for all } n \in \mathbb{N},$$

we have that

$$\limsup_{n\to\infty} \|T^n x_n - p\| \le d.$$

Also

$$\begin{aligned} \|y_n - p\| &\le (1-\beta_n)\|x_n - p\| + \beta_n\|T^n x_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n \eta(T^n)(\|x_n - p\| + a_n) \\ &\le \eta(T^n)\|x_n - p\| + \eta(T^n)a_n, \end{aligned}$$

which yields

$$\limsup_{n\to\infty} \|y_n - p\| \le d. \tag{3.5}$$

Hence

$$\limsup_{n\to\infty} \|T^n y_n - p\| \le \limsup_{n\to\infty}(\eta(T^n)(\|y_n - p\| + a_n)) \le d. \tag{3.6}$$

Since

$$d = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1-\alpha_n)(T^n x_n - p) + \alpha_n(T^n y_n - p)\|,$$

it follows from Lemma 2.6 that

$$\lim_{n\to\infty} \|T^n x_n - T^n y_n\| = 0. \tag{3.7}$$

Form (3.3) and (3.7), we have

$$\begin{aligned} \|x_{n+1} - T^n x_n\| &= \alpha_n\|T^n y_n - T^n x_n\| \\ &\le b\|T^n y_n - T^n x_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.8}$$

Hence

$$\|x_{n+1} - T^n y_n\| \le \|x_{n+1} - T^n x_n\| + \|T^n x_n - T^n y_n\| \to 0 \text{ as } n \to \infty.$$

Now

$$\begin{aligned} \|x_{n+1} - p\| &\le \|x_{n+1} - T^n y_n\| + \|T^n y_n - p\| \\ &\le \|x_{n+1} - T^n y_n\| + \eta(T^n)(\|y_n - p\| + a_n), \end{aligned} \tag{3.9}$$

which gives from (3.9) that

$$d \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.10}$$

From (3.5) and (3.10), we obtain

$$d = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)\|. \tag{3.11}$$

(b) Apply Lemma 2.6 in (3.11), and we obtain that $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$.

(c) By (3.8), we have

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Also observe that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \eta(T^{n+1})(\|x_{n+1} - x_n\| \end{aligned}$$

$$+a_{n+1}) + \|T^{n+1}x_n - Tx_n\|. \tag{3.12}$$

Since $\lim_{n\to\infty} \|Tx_n - T^{n+1}x_n\| = 0$ by the uniform continuity of T, it follows from (3.12) that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. □

Theorem 3.9. *Let X be a uniformly convex Banach space X satisfying the Opial condition, C a nonempty closed convex subset of X and $T : C \to C$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and given $x_1 \in C$, let $\{x_n\}$ be the sequence in C defined by (3.3). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Note that X reflexive since it is uniformly convex. Observe that

(i) $F(T) \neq \emptyset$ by assumption,

(ii) $\{x_n\}$ has properties $(D_1) \sim (D_2)$ by Theorem 3.8,

(iii) $I - T$ is demiclosed at zero by Lemma 2.8.

Therefore, $\{x_n\}$ converges weakly to a fixed point of T by Lemma 2.10. □

It is well known that there exist classes of uniformly convex Banach spaces without the Opial condition (e.g., L_p spaces, $p \neq 2$). Therefore, Theorem 3.9 is not true for such Banach spaces. We now show that Theorem 3.9 is valid if the assumption that X satisfies the Opial condition is replaced by either (a) X has Fréchet differentiable norm or (b) X^* has the Kadec-Klee property.

Theorem 3.10. *Let X be a real uniformly convex Banach space with Fréchet differentiable norm, C a nonempty closed convex subset of X and $T : C \to C$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $I - T$ is demiclosed at zero, $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and given $x_1 \in C$, let $\{x_n\}$ be the sequence in C defined by (3.3). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Observe that

(i) $\{x_n\}$ has properties $(D_1) \sim (D_2)$ by Theorem 3.8,

(ii) $I - T$ is demiclosed at zero by assumption,

(iii) $\lim_{n\to\infty} \langle x_n, J(p - q)\rangle$ exists for all $p, q \in F(T)$ by Lemma 2.9(c).

Hence the result follows from Lemma 2.11. □

Theorem 3.11. *Let X be a real uniformly convex Banach space such that X^* has the Kadec-Klee property, C a nonempty closed convex subset of X and $T : C \to C$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $I - T$ is demiclosed at zero, $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$ such that*

$0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and given $x_1 \in C$, let $\{x_n\}$ be the sequence in C defined by (3.3). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. As in proof of Theorem 3.10, we have the following:

(i) $\{x_n\}$ has properties $(D_1) \sim (D_2)$;

(ii) $I - T$ is demiclosed at zero by assumption;

(iii) $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $p, q \in F(T)$ and $t \in [0,1]$ by Lemma 2.9(b).

Hence Theorem 3.11 follows from Lemma 2.11. □

We conclude this paper with strong convergence of the S-iteration process to fixed points of nearly asymptotically nonexpansive mappings in a strictly convex Banach space.

Let X be a Banach space and let C be a nonempty closed convex bounded subset of X with diameter $diam(C) > 0$. For $0 \leq \varepsilon \leq 1$, we define the number $\delta(C,\varepsilon) > 0$ by

$$\delta(C,\varepsilon) = \frac{1}{d}\inf\{\max\{\|x\|,\|y\|\} - \frac{\|x+y\|}{2} : x, y \in C, \|x-y\| \geq d\varepsilon\},$$

where $d := diam(C)$. The following lemmas were proved in Takahashi and Tsukiyama [33].

Lemma 3.12. *Let X be a Banach space and C a nonempty compact convex subset of X with $d = diam(C) > 0$. Let $x, y \in C$ with $\|x - y\| \geq d\varepsilon$ for some $0 \leq \varepsilon \leq 1$. Then*

$$\|\lambda x + (1-\lambda)y\| \leq \max\{\|x\|,\|y\|\} - 2\lambda(1-\lambda)d\delta(C,\varepsilon) \text{ for all } \lambda \in [0,1].$$

Lemma 3.13. *Let X be a strictly convex Banach space and C a nonempty compact convex subset of X with $diam(C) > 0$. If $\lim_{n\to\infty} \delta(C,\varepsilon_n) = 0$, then $\lim_{n\to\infty} \varepsilon_n = 0$.*

Theorem 3.14. *Let C be a nonempty compact convex subset of a strictly convex Banach space X and $T : C \to C$ a uniformly continuous nearly nonexpansive mapping with sequence $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n < \infty$. Let $\{x_n\}$ be the sequence in C generated from $x_1 \in C$, and defined by (3.3), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n)$ exists and $\lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Schauder's fixed point theorem we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$. By (3.3), we have

$$\begin{aligned} \|T^n x_n - p\| &\leq \|x_n - p\| + a_n, \\ \|y_n - p\| &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p)\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n(\|x_n - p\| + a_n) \\ &\leq \|x_n - p\| + a_n, \end{aligned}$$

and

$$\|x_{n+1} - p\| = \|(1-\alpha_n)(T^n x_n - p) + \alpha_n(T^n y_n - p)\|$$

$$\leq (1-\alpha_n)(\|x_n-p\|+a_n)+\alpha_n(\|y_n-p\|+a_n)$$

$$= (1-\alpha_n)\|x_n-p\|+\alpha_n\|y_n-p\|+a_n \tag{3.13}$$

$$\leq \|x_n-p\|+2a_n. \tag{3.14}$$

Apply Lemma 2.5, and we obtain from (3.14) that $\lim_{n\to\infty}\|x_n-p\|$ exists.

Set $d := diam(C)$ and $\varepsilon_n := \|x_n - T^n x_n\|/d$. Then $0 \leq \varepsilon_n \leq 1$. Using (3.13) and Lemma 3.12, we have

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\|(1-\beta_n)(x_n-p)+\beta_n(T^nx_n-p)\|+a_n\\ &\leq (1-\alpha_n)\|x_n-p\|+\alpha_n\max\{\|x_n-p\|,\|T^nx_n-p\|\}\\ &\quad -2\alpha_n\beta_n(1-\beta_n)d\delta(C,\varepsilon_n)+a_n\\ &\leq \|x_n-p\|-2\alpha_n\beta_n(1-\beta_n)d\delta(C,\varepsilon_n)+2a_n\end{aligned}$$

which yields

$$\begin{aligned}2d\sum_{n=1}^{m}\alpha_n\beta_n(1-\beta_n)\delta(C,\varepsilon_n) &\leq \sum_{n=1}^{m}(\|x_n-p\|-\|x_{n+1}-p\|)+2\sum_{n=1}^{m}a_n\\ &= \|x_1-p\|-\|x_{m+1}-p\|+2\sum_{n=1}^{m}a_n\\ &\leq \|x_1-p\|+2\sum_{n=1}^{m}a_n \text{ for all } m\in\mathbb{N}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} a_n < \infty$ we have that $\sum_{n=1}^{\infty}\alpha_n\beta_n(1-\beta_n)\delta(C,\varepsilon_n) < \infty$. The condition $\lim_{n\to\infty}\alpha_n\beta_n(1-\beta_n) \neq 0$ implies that $\lim_{n\to\infty}\delta(C,\varepsilon_n) = 0$. By Lemma 3.13, we have $\lim_{n\to\infty}\varepsilon_n = 0$. Thus,

$$\lim_{n\to\infty}\|x_n-T^nx_n\| = 0. \tag{3.15}$$

Since T is uniformly continuous, it follows from (3.15) that

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{3.16}$$

By the compactness of C, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}x_{n_i} = v. \tag{3.17}$$

Since T is continuous, it follows from (3.16) that $v \in F(T)$. Since $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T)$, we conclude from (3.17) that $\lim_{n\to\infty}x_n = v \in F(T)$. □

References

[1] S. C. Bose, *Weak convergence to the fixed of an asymptotically nonexpansive map*, Proc. Amer. Math. Soc. 68 (1978), 305-308.

[2] R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak w-limit set*, Israel Jour. Math. 29 (1978), 1-16.

[3] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. 32 (1979), 107-116.

[4] S. S. Chang, *On the approximation problem of fixed points for asymptotically nonexpansive mappings*, Indian J. Pure Appl. Math. 32 (2001), 1297-1307.

[5] S. S. Chang, Y. J. Cho and H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc. 38 (2001), 1245-1260.

[6] C. E. Chidume, *Global iteration schemes for strongly pseudocontractive maps*, Proc. Amer. Math. Soc. 126 (1998), 2641-2649.

[7] C. E. Chidume and B. Ali, *Approximation of common fixed points for finite families of nonself asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., in press.

[8] C. E. Chidume and S. Mutangadura, *An example on the Mann iteration method for Lipschitzian pseudocontractions*, Proc. Amer. Math. Soc. 129 (2001), 2359-2363.

[9] C. E. Chidume, E. U. Ofoedu, H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 280 (2003), 364-374.

[10] L. Deng, *Convergence of the Ishikawa iteration process for nonexpansive mappings*, J. Math. Anal. Appl. 199 (1996), 769-775.

[11] N. Dunford and J. T. Schwartz, Linear Operators, Vol. I, Interscience, New York, 1958.

[12] M. Edelstein, *A remark on a theorem of Krasnoselskii*, Amer. Math. Monthly 73 (1966), 509-510.

[13] K. Goebel and W. A. Kirk, *A fixed point theorem of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972), 171-174.

[14] J. S. Jung and D. R. Sahu, *Dual convergences of iteration processes for nonexpansive mappings in Banach spaces*, Czechoslovak Math. J. 53 (2003), 397-404.

[15] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[16] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. 59 (1976), 65-71.

[17] W. Kaczor, *Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups*, J. Math. Anal. Appl. 272 (2002), 565-574.

[18] S. H. Khan and H. Fukharuddin, *Weak and strong convergence of a scheme with errors for two nonexpansive mappings*, Nonlinear Anal. 61 (2005), 1295-1301.

[19] M. A. Krasnoselskii, *Two observations about the method of successive approximations*, Uspehi Math. Nauk 10 (1955), 123-127.

[20] B. V. Limaye, Functional Analysis, New Age International Publishers Limited, New Delhi, 1996.

[21] Z. Q. Liu and S. M. Kang, *Weak and strong convergence for fixed points of asymptotically nonexpansive mappings*, Acta Math. Sin. 20 (2004), 1009-1018

[22] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. 4 (1953), 506-610.

[23] Shin-ya Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. (2004), 37-47

[24] Z. OpiaI, *Weak convergence of successive approximations for nonexpansive mappings*, Bul. Amer. Math. Soc. 73 (1967), 591-597.

[25] M. O. Osilike and S. C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling 32 (2000), 1181-1191.

[26] B. E. Rhoades, *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. 56 (1976), 741-750.

[27] B. E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. 183 (1994), 118-120.

[28] D. R. Sahu, *Fixed points of demicontinuous nearly Lipschitzian mappings in Banach spaces*, Comment. Math. Univ. Carolin. 46 (2005), 653-666.

[29] D. R. Sahu and I. Beg, *Demiclosedness principle for nonlipschitzian nearly asymptotically nonexpansive mappings*, submitted.

[30] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mapping*, J. Math. Anal. 159 (1991), 407-413.

[31] J. Schu, *Weak and strong convergence of fixed points of asymptotically nonexpansive maps*, Bull. Austral. Math. Soc. 43 (1991), 153-159.

[32] N. Shazad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. (2006), 1-10.

[33] W. Takahashi and N. Tsukiyama, *Approximating fixed points of nonexpansive mappings with compact domains*, Comm. Appl. Nonlinear Anal. 7 (2000), 39-47.
[34] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993), 301-308.
[35] K. K. Tan and H. K. Xu, *Fixed point iteration process for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 122 (1994), 733-739.
[36] L. C. Zeng, *A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1998), 245-250.
[37] H. Y. Zhou and Y. T. Jia, *Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption*, Proc. Amer. Math. Soc. 125 (1997), 1705-1709.

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