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INVEX INEQUALITY SYSTEM WITH APPLICATIONS TO FRACTIONAL MINIMAX OPTIMIZATION

ANULEKHA DHARA AND APARNA MEHRA

Dedicated to Late Professor M. C. Puri, IIT Delhi

Abstract. Two new theorems of alternative for invex infine inequality systems are proved. One of these theorems is applied to characterize optimal solution of nonsmooth scalar-valued fractional minimax programming problem with inequality, equality and abstract constraints. An existence theorem and Karush-Kuhn-Tucker (KKT) type optimality conditions are established.

1. Introduction

There has been an urge to find a class of functions for which Karush Kuhn-Tucker (KKT) necessary optimality conditions become sufficient too. This urge led several authors in the past to study convexity and its various generalizations. Among them, the most significant is the concept of invexity, introduced by Hanson [7] for differentiable functions. It was shown by Ben-Israel and Mond [4] that a differentiable function is invex if and only if every stationary point of the function is its global minimizer. This property of invex functions make their study significant. Later, Reliand [11] extended invexity to locally Lipschitz nonsmooth vector-valued functions. However, the notion of invexity is suitable for optimization problems involving inequality constraints only but not for optimization problems with equality constraints. This motivated Sach et al. [12] to define a new class of nonsmooth infine functions which forms a subclass of invex functions and is also appropriate for optimization problems with equality constraints. Invex functions are also useful in deriving theorems of alternative for various systems of inequalities. For more details, one can refer to [5, 10, 12]. These theorems are used as principal tools in developing necessary optimality conditions for constrained optimization problems. These observations make the study of nonsmooth locally Lipschitz invex infine vector functions important and interesting.

The aim of this paper is twofold. First, we prove two theorems of alternative for systems involving infinite inequalities, finite equalities and abstract constraints under suitable V-invex infine hypotheses. Second, as an application of one of these theorems, we derive necessary and sufficient optimality conditions for nonsmooth fractional minimax problems.

The rest of the paper is organized as follows. In section 2, we introduce the notion of V-invex infine function using Clarke generalized gradient [6]. Example is

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given to support the definition. Section 3 is devoted to prove the main theorem of the paper, that is, the theorem of alternative involving V-invex infine inequalities, equalities and abstract constraints. Relaxing the assumptions of this theorem, we obtain another theorem of alternative for a different system in the same setup. We use these theorems to obtain conditions which ensure the existence of solution for minimax optimization problem in section 4, while in section 5, we present necessary and sufficient optimality conditions for fractional minimax programming problems. Some concluding remarks are given in section 6.

2. Vector invex infine function

In this section, we introduce a new class of nonsmooth locally Lipschitz V-invex infine functions. We begin with some basic definitions and notations that will be used throughout the paper.

Let $f : \Re^n \longrightarrow \Re$ be a locally Lipschitz function at $x_0 \in \Re^n$, that is, there exists a constant $L > 0$ such that for any x, x' in some neighborhood of x_0,

$$|f(x) - f(x')| \leqq L\|x - x'\|.$$

For $x_0 \in \Re^n$, the Clarke directional derivative of f at x_0 in the direction d [6] is defined as

$$f^o(x_0, d) = \lim_{x \to x_0} \sup_{\lambda \downarrow 0} \frac{f(x + \lambda d) - f(x)}{\lambda}$$

and the Clarke subdifferential of f at x_0 is defined as

$$\partial f(x_0) = \{\xi \in \Re^n : f^o(x_0, d) \geqq \langle \xi, d\rangle, \ \forall d \in \Re^n\},$$

where $\langle ., . \rangle$ denotes the inner product in $\Re^n$. It is well known that for any $d \in \Re^n$,

$$f^o(x_0, d) = \max_{\xi \in \partial f(x_0)} \langle \xi, d\rangle$$

and $\partial f(x_0)$ is a nonempty compact convex subset of $\Re^n$.

Let S be a closed subset of $\Re^n$ and $x_0 \in S$. The Clarke tangent cone to S at x_0 is given by

$$T_S(x_0) = \{d \in \Re^n : d_S{}^o(x_0, d) = 0\},$$

where $d_S(x_0)$ is the distance metric defined as

$$d_S(x_0) = \inf_{x \in S} \|x - x_0\|.$$

The Clarke normal cone to S at x_0 is given by

$$N_S(x_0) = \{v \in \Re^n : \ \langle v, d\rangle \ \leqq 0, \ \forall \, d \in T_S(x_0)\}.$$

In the following, we assume that $f = (f_1, f_2, \ldots, f_p)$ is a nonsmooth vector-valued function defined on $\Re^n$ such that each f_i is locally Lipschitz real-valued function.

The generalized gradient of f at x_0, [6], is the set

$$\partial f(x_0) = \partial f_1(x_0) \times \partial f_2(x_0) \times \ldots \times \partial f_p(x_0).$$

We recall below definitions of invex and infine vector-valued functions from [12].

Definition 2.1. *f is said to be invex at $x_0 \in S$ if for any $x \in S$ and $A = (\xi_1, \xi_2, \ldots, \xi_p) \in \partial f(x_0)$, $\xi_i \in \partial f_i(x_0)$, $i = 1, 2, \ldots, p$, there exists $\eta \in T_S(x_0)$ such that*

$$f_i(x) - f_i(x_0) \geqq \langle \xi_i, \eta(x, A) \rangle.$$

Definition 2.2. *f is said to be infine at $x_0 \in S$ if for any $x \in S$ and $A = (\xi_1, \xi_2, \dots, \xi_p) \in \partial f(x_0),\ \xi_i \in \partial f_i(x_0),\ i = 1, 2, \dots, p$, there exists $\eta \in T_S(x_0)$ such that*

$$f_i(x) - f_i(x_0) = \langle \xi_i, \eta(x, A) \rangle.$$

Observe that, in the above definitions, vector η depends on vector x and the matrix A. We now introduce the notion of V-invexity and V-infineness.

Definition 2.3. *f is said to be V-invex at $x_0 \in S$ if for any $x \in S$ and $A = (\xi_1, \xi_2, \dots, \xi_p) \in \partial f(x_0),\ \xi_i \in \partial f_i(x_0),\ i = 1, 2, \dots, p$, there exist $\eta \in T_S(x_0)$ and $\theta_i \in \Re_+ \backslash \{0\}$ such that*

$$f_i(x) - f_i(x_0) \geqq \theta_i(x, \xi_i) \langle \xi_i, \eta(x, A) \rangle.$$

Definition 2.4. *f is said to be V-infine at $x_0 \in S$ if for any $x \in S$ and $A = (\xi_1, \xi_2, \dots, \xi_p) \in \partial f(x_0),\ \xi_i \in \partial f_i(x_0),\ i = 1, 2, \dots, p$, there exist $\eta \in T_S(x_0)$ and $\theta_i \in \Re \backslash \{0\}$ such that*

$$f_i(x) - f_i(x_0) = \theta_i(x, \xi_i) \langle \xi_i, \eta(x, A) \rangle.$$

In definitions 2.3 and 2.4, vector η depends on x and A while θ_i depends on x and ξ_i. Following example illustrates the notion of V-invexity.

Example 2.5. *Consider $f_1(x) = \begin{cases} x, & x \geqq 0 \\ 2x, & x < 0 \end{cases}$ and $f_2(x) = x^2 + |x|$ defined over $\Re$. Let $x_0 = 0$. Then $\partial f_1(x_0) = [1, 2]$ and $\partial f_2(x_0) = [-1, 1]$. The vector function (f_1, f_2) is not invex at x_0 with respect to a common η in the sense of definition 2.1, for if, $A = [1, -1]^T$ and $x = -1/2$, we cannot find η for which (f_1, f_2) is invex. However, (f_1, f_2) is V-invex in the sense of definition 2.3. For $x \geqq 0,\ A = [\xi_1, \xi_2]^T,\ \xi_1 \in [1, 2]$ and $\xi_2 \in [-1, 1]$, we can take $\eta = x/2,\ \theta_1 = \theta_2 = 1$. For $x < 0,\ A = [\xi_1, \xi_2]^T,\ \xi_1 \in [1, 2]$ and $\xi_2 \in [-1/2, 1]$, take $\eta = 2x,\ \theta_1 = \theta_2 = 1$ whereas for $\xi_2 \in [-1, -1/2)$, take $\eta = 2x,\ \theta_1 = 1$ and $\theta_2 = (1 - x)/2$. Observe that in all the cases θ_1 and θ_2 are positive.*

Remark 2.6. If $\theta_i(x,\ \xi_i) = 1,\ \forall\, \xi_i \in \partial f_i(x_0),\ i = 1, 2, \dots, p$, then definitions 2.3 and 2.4 reduce to invex and infine functions respectively, of Sach et al. [12]. Moreover, if η does not depend on the matrix A and θ_i too is independent of the vector ξ_i then the notion of V-invexity introduced here coincides with the V-invexity defined by Bector et al. [3].

Remark 2.7. The above definitions could also be given using the generalized Jacobian, $Jf(x_0)$, of f at x_0, [6], which is defined as the convex hull of all limits of the form

$$A = \lim_{x_i \to x_0; x_i \in D_f} f'(x_i)$$

where D_f is the set of all x where $f'(x)$ exists. However, it is well known that $Jf(x_0) \subset \partial f(x_0)$ and there are functions f for which $Jf(x_0) \neq \partial f(x_0)$. Thus, if the concepts of invexity and infineness are defined in terms of $Jf(x_0)$, instead of $\partial f(x_0)$, we would be restricting the class of functions. Therefore, we consider the two definitions with $\partial f(x_0)$ so as to study wider class of optimization problems.

Definition 2.8. *Let* $f = (f_1, f_2, \ldots, f_p), g = (g_1, g_2, \ldots, g_q)$ *and* $h = (h_1, h_2, \ldots, h_r)$ *be vector-valued functions defined on* $\Re^n$ *such that each* $f_i, g_j, h_k, i = 1, 2, \ldots, p, j = 1, 2, \ldots, q$ *and* $k = 1, 2, \ldots, r$ *is locally Lipschitz real-valued function. Then* $((f, g); h)$ *is said to be V-invex infine at* $x_0 \in S$ *if for any* $x \in S$ *and* $A \in \partial f(x_0) \times \partial g(x_0) \times \partial h(x_0)$*, there exists* $\eta \in T_S(x_0)$ *such that* (f, g) *is V-invex and* h *is V-infine at* x_0 *with respect to* $\eta \equiv \eta(x, A)$*.*

From now onwards we will be writing η instead of $\eta(x, A)$ and θ_i instead of $\theta_i(x,\ \xi_i)$ without any ambiguity.

3. Theorems of alternative

Theorems of alternative play pivotal role in deriving necessary optimality conditions of Karush Kuhn-Tucker type for nonsmooth nonconvex optimization problems. Various types of theorems of alternative for various systems of inequalities have been studied in the past. Here, we would like to mention few of them related to the present work. Jeyakumar et al. [8] discussed the theorems by considering systems of infinite inequalities involving convexlike and concavelike functions. Brandão et al. [5] studied the theorem for system of finite inequalities and abstract constraints involving nonsmooth invex functions. Later, Luu et al. [10] used standard separation theorem of convex sets and Motzkin's theorem of alternative of Schmitendorf [13] to discuss theorem of alternative for system with infinite inequalities only. More recently, Sach et al. [12] derived theorems with systems consisting of finite inequality constraints, equality constraints and abstract constraints under invex infine conditions. In this section, two new theorems of alternative for the systems involving infinite inequalities, finite equalities and abstract constraints are derived under V-invex infine hypotheses. It can be observed that the related theorems of Luu et al. (Theorem 3.3, [10]) and Sach et al. (Theorem 4.1 and Theorem 4.2, [12]) are particular cases of the theorems of alternative presented in this section.

Before proving the theorems, we define the following notations. Let Y be a compact subset of $\Re^l$ and S be a nonempty closed subset of $\Re^n$. Let $F(., y),\ y \in Y$ be locally Lipschitz real-valued function, $g = (g_1, g_2, \ldots, g_m)$ and $h = (h_1, h_2, \ldots, h_p)$ be vector-valued functions with locally Lipschitz components defined over $\Re^n$. Let $J = \{1, 2, \ldots, m\}$ and $K = \{1, 2, \ldots, p\}$ be the index sets. For $x_0 \in S$, denote

$$Y_0 = \{y_0 \in Y : F(x_0, y_0) = \sup_{y \in Y} F(x_0, y)\},$$
$$J_0 = \{j \in J : g_j(x_0) = 0\}.$$

Further, assume that for each $x \in S$, the function $y \longrightarrow F(x, .)$ is upper semicontinuous on Y. Upper semicontinuity of $F(x, .)$ along with compactness of Y ensures that Y_0 is nonempty.

Theorem 3.1. *Assume the following:*

(1) *The system*

$$(\exists x \in \Re^n)(\forall y \in Y)\ F(x, y) \leqq 0,\ g(x) \leqq 0,\ h(x) = 0,\ x \in S \tag{3.1}$$

has a solution x_0*.*

(2) *Jourani's constraint qualification* (JCQ) *(see,* [9]*) is satisfied at* x_0*, that is,* $\forall\,(\beta_{J_0},\ \gamma_K),\ \beta_j \geqq 0,\ j \in J_0$ *and* $\gamma_k \in \Re,\ k \in K$*, not all zeroes,*

$$0 \notin \sum_{j \in J_0} \beta_j \partial g_j(x_0) + \sum_{k \in K} \gamma_k \partial h_k(x_0) + N_S(x_0).$$

(3) *For each* $y \in Y_0$*,* $((F(.,y),\ g_{J_0}(.));\ h_K(.))$ *be V-invex infine at* x_0 *on* S*.*

Then either the system

(I) $(\exists x \in \Re^n)(\forall y \in Y)$

$$F(x,y) < 0,\ g(x) \leqq 0,\ h(x) = 0,\ x \in S \tag{3.2}$$

has a solution

or,

(II) $\exists$ *vectors* $y_1, y_2, \ldots, y_{n+1}$ *in* Y_0*,* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \geq 0,\ \beta_j \geqq 0,\ j \in J_0,\ \gamma_k \in \Re,\ k \in K$ *such that* $\forall x \in S$

$$\sum_{i=1}^{n+1} \lambda_i F(x, y_i) + \sum_{j \in J_0} \beta_j g_j(x) + \sum_{k \in K} \gamma_k h_k(x) \geqq 0$$

but never both.

Proof. Observe that systems (I) and (II) cannot hold together. Assume that system (I) has no solution. Consider the optimization problem

$$(P) \qquad \begin{array}{lll} \textit{minimize} & \phi(x), & \phi(x) = \sup_{y \in Y} F(x,y) \\ \textit{subject to} & g_j(x) \leqq 0, & j \in J, \\ & h_k(x) = 0, & k \in K, \\ & x \in S. & \end{array}$$

Let $S_1 = \{x \in \Re^n : g_J(x) \leqq 0,\ h_K(x) = 0,\ x \in S\}$ denote the feasible solution set of the problem (P). Since (3.1) has a solution so, S_1 is nonempty. Also, (3.2) has no solution, hence, $\sup_{y \in Y} F(x,y) \geqq 0,\ \forall x \in S_1$. Therefore, $\phi(x) \geqq 0,\ \forall\ x \in S_1$. Moreover, by assumption 1, x_0 is a solution of (3.1) which implies $x_0 \in S_1$ and $F(x_0, y) \leqq 0$ thereby implying that $\phi(x_0) = 0$, that is, $\phi(.)$ attains its minimum at x_0. Therefore, x_0 is a local minimizer of (P).

Invoking the necessary optimality conditions from Clarke [6], there exists $\lambda_0' \geqq 0,\ \beta_j' \geqq 0\ (j \in J_0),\ \gamma_k' \in \Re\ (k \in K)$, not all zeroes, such that

$$0 \in \lambda_0' \partial \phi(x_0) + \sum_{j \in J_0} \beta_j' \partial g_j(x_0) + \sum_{k \in K} \gamma_k' \partial h_k(x_0) + N_S(x_0).$$

(JCQ) ensures that $\lambda_0' \neq 0$. In particular, for $\lambda_0' = 1$,

$$0 \in \partial \phi(x_0) + \sum_{j \in J_0} \beta_j' \partial g_j(x_0) + \sum_{k \in K} \gamma_k' \partial h_k(x_0) + N_S(x_0),$$

implying that

$$0 \in co\Big\{ \bigcup_{y \in Y_0} \partial F(x_0, y) \Big\} + \sum_{j \in J_0} \beta'_j \partial g_j(x_0) + \sum_{k \in K} \gamma'_k \partial h_k(x_0) + N_S(x_0).$$

Thus, there exist $y_i \in Y_0$ and $\lambda'_i \geqq 0, i = 1, 2, \ldots, n+1, \sum_{i=1}^{n+1} \lambda'_i = 1$ such that

$$0 \in \sum_{i=1}^{n+1} \lambda'_i \partial F(x_0, y_i) + \sum_{j \in J_0} \beta'_j \partial g_j(x_0) + \sum_{k \in K} \gamma'_k \partial h_k(x_0) + N_S(x_0). \tag{3.3}$$

Consequently,

$$-\Big(\sum_{i=1}^{n+1} \lambda'_i \xi_{y_i} + \sum_{j \in J_0} \beta'_j \bar{\xi}_j + \sum_{k \in K} \gamma'_k \bar{\bar{\xi}}_k \Big) \in N_S(x_0)$$

for some $\xi_{y_i} \in \partial F(x_0, y_i),\ i = 1, 2, \ldots, n+1,\ \bar{\xi}_j \in \partial g_j(x_0),\ j \in J_0$ and $\bar{\bar{\xi}}_k \in \partial h_k(x_0),\ k \in K$.

Applying V-invexity infineness of $((F(., y), g_{J_0}(.)); h_K(.))$ at x_0, we have that for any $x \in S$, there exist vectors $\eta \in T_S(x_0),\ \theta_{y_i},\ \varphi_j \in \Re_+ \backslash \{0\},\ i = 1, 2, \ldots, n+1,\ j \in J_0$ and $\psi_k \in \Re \backslash \{0\},\ k \in K$, such that

$$\sum_{i=1}^{n+1} \lambda_i [F(x, y_i) - F(x_0, y_i)] + \sum_{j \in J_0} \beta_j [g_j(x) - g_j(x_0)] + \sum_{k \in K} \gamma_k [h_k(x) - h_k(x_0)] \geqq 0,$$

where $\lambda_i = \lambda'_i / \theta_{y_i},\ i = 1, 2, \ldots, n+1,\ \beta_j = \beta'_j / \varphi_j,\ j \in J_0$ and $\gamma_k = \gamma'_k / \psi_k,\ k \in K$. Therefore, for any $x \in S$, we obtain,

$$\sum_{i=1}^{n+1} \lambda_i F(x, y_i) + \sum_{j \in J_0} \beta_j g_j(x) + \sum_{k \in K} \gamma_k h_k(x) \geqq$$
$$\sum_{i=1}^{n+1} \lambda_i F(x_0, y_i) + \sum_{j \in J_0} \beta_j g_j(x_0) + \sum_{k \in K} \gamma_k h_k(x_0) = 0.$$

This proves the result. □

We illustrate the above theorem by an example.

Example 3.2. *Consider $F(x, y) = -(1/2)y_1^2 - (1/2)y_2^2 + y_1 x_1 + y_2 x_2$ where $y \in Y = [-1, 1] \times [-1, 1]$, $g(x) = x_1 + (1/2)x_2$ and $h(x) = -x_1 + (1/2)x_2$ defined over $\Re^2$. The assumptions of the theorem are satisfied at the point $x_0 = (0, 0)$ with $Y_0 = \{(0, 0)\}$ and $\eta = x$. We observe that system (I) is not satisfied but system (II) is solvable for $\lambda > 0$, $\beta = 0$ and $\gamma = 0$.*

Assumptions 1 and 2 of the above theorem can be replaced by existence of a solution of an unconstrained optimization problem. For this, we define a scalar-valued function

$$\Theta(.) = \max\left\{ \sup_{y\in Y} F(.,y),\ \max_{j\in J} g_j(.),\ \max_{k\in K} |h_k(.)| \right\}$$

on S.

Theorem 3.3. *Assume the following:*

$1'$. *Function $\Theta(.)$ attains its minimum on S at a point $x_0 \in S$.*

$2'$. *For each $y \in Y_1$, $((F(.,y),\ g_{J_1}(.));\ h_{K_1}(.))$ is V-invex infine at x_0 on S, where*

$$Y_1 = \{y_0 \in Y : F(x_0, y_0) = \Theta(x_0)\},$$
$$J_1 = \{j \in J : g_j(x_0) = \Theta(x_0)\},$$
$$K_1 = \{k \in K : |h_k(x_0)| = \Theta(x_0)\}.$$

Then either the system

(I$'$) $(\exists x \in \Re^n)(\forall y \in Y)$

$$F(x,y) \leqq 0,\ g(x) \leqq 0,\ h(x) = 0,\ x \in S$$

has a solution

or,

(II$'$) $\exists$ *vectors $y_1, y_2, \ldots, y_{n+1}$ in Y_1, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \geqq 0$, $\beta_j \geqq 0$, $j \in J_1$, $\gamma_k \in \Re$, $k \in K_1$, not all zeroes, and a scalar $\epsilon > 0$ such that $\forall x \in S$*

$$\sum_{i=1}^{n+1} \lambda_i F(x, y_i) + \sum_{j\in J_1} \beta_j g_j(x) + \sum_{k\in K_1} \gamma_k h_k(x) > \epsilon$$

but never both.

Proof. Observe that systems (I$'$) and (II$'$) cannot hold together. Assume that system (I$'$) has no solution. Then, $\Theta(x) > 0$, $\forall x \in S$. Since $\Theta(.)$ attains minimum at $x_0 \in S$, there exists $\epsilon' > 0$ such that $\Theta(x_0) > \epsilon'$. By optimality of x_0, we have,

$$0 \in \partial\Theta(x_0) + N_S(x_0).$$

Following on the lines of proof of Theorem 4.1 of Sach et al. [12] and using Levin's theorem (see, chapter 2, Clarke [6]), we get the existence of vectors $y_1, y_2, \ldots, y_{n+1}$ in Y_1 and multipliers $(\lambda', \beta', \gamma')$ where $(\lambda', \beta') \geqq 0$ and $\gamma' \in \Re$, not all zeroes, such that

$$0 \in \sum_{i=1}^{n+1} \lambda'_i \partial F(x_0, y_i) + \sum_{j\in J_1} \beta'_j \partial g_j(x_0) + \sum_{k\in K_1} \gamma'_k \partial h_k(x_0) + N_S(x_0).$$

Applying V-invexity infineness of $((F(.,y), g_{J_1}(.)); h_{K_1}(.))$ on S, we obtain

$$\begin{aligned}\sum_{i=1}^{n+1}\lambda_i F(x,y_i) \quad &+ \quad \sum_{j\in J_1}\beta_j g_j(x) + \sum_{k\in K_1}\gamma_k h_k(x)\\ &\geqq \sum_{i=1}^{n+1}\lambda_i F(x_0,y_i) + \sum_{j\in J_1}\beta_j g_j(x_0) + \sum_{k\in K_1}\gamma_k h_k(x_0)\\ &= \Theta(x_0)\Big\{\sum_{i=1}^{n+1}\lambda_i + \sum_{j\in J_1}\beta_j + \sum_{k\in K_1}\gamma_k a_k\Big\} > \epsilon,\end{aligned}$$

where λ_i, β_j and γ_k are same as defined in Theorem 3.1, $a_k = \mathrm{sign}(h_k(x_0))$ and $\epsilon = \epsilon'\Big\{\sum_{i=1}^{n+1}\lambda_i + \sum_{j\in J_1}\beta_j + \sum_{k\in K_1}\gamma_k a_k\Big\}$. □

Example 3.4. *Consider $F(x,y) = 1/2y^2 - yx$, where $y \in Y = [-1,1]$, $g(x) = x^3 - 1$ and $h(x) = |x| - 1$ defined over $\Re$. We observe that assumption 1 of Theorem 3.1 does not hold. Moreover, Jourani's constraint qualification is also not satisfied. The function Θ, given by*

$$\Theta(x) = \begin{cases} |x| + 1/2, & x \leqq 1.4311\\ x^3 - 1, & x > 1.4311\end{cases},$$

attains minimum at $x_0 = 0$. Observe that $Y_1 = \{-1,1\}$ while J_1 and K_1 are empty. Also, system (I') does not hold but system (II') is solvable for any $\epsilon > 0$ with $\lambda_1 = \lambda_2 = \epsilon$.

4. Existence theorem for minimax problem

In this section, we investigate the existence of a solution of nonsmooth minimax problem (P) using theorem of alternative proved in the previous section.

Definition 4.1. *$(x_0, y_0) \in S_1 \times Y$ is solution of the minimax problem (P) if it satisfies the following*

$$\min_{x\in S_1} F(x,y_0) = \sup_{y\in Y}\min_{x\in S_1} F(x,y) = \min_{x\in S_1}\sup_{y\in Y} F(x,y) = \sup_{y\in Y} F(x_0,y).$$

We now recall the definition of concavelike function as given by Jeyakumar et al. [8].

Definition 4.2. *$F(x,.)$ is said to be concavelike on Y if for any $\beta \in (0,1)$, $y_1, y_2 \in Y$ there exists $y_3(\beta,\ y_1,\ y_2) \in Y$ such that for any $x \in S_1$*

$$\beta F(x,y_1) + (1-\beta)F(x,y_2) \leqq F(x,y_3).$$

Following theorem ensures the existence of a solution of the problem (P) under invex infine and concavelike assumptions.

Theorem 4.3. *Suppose there exists $x_0 \in S_1$ such that $\sup_{y\in Y} F(x_0,y) = v = \min_{x\in S_1}\sup_{y\in Y} F(x,y)$, and let (P) satisfy (JCQ) at x_0. Assume that for each $y \in Y_0$, $((F(.,y),\ g_{J_0}(.));\ h_K(.))$ is V-invex infine at x_0 on S, and for each $x \in S_1$, $F(x,.)$ is concavelike on Y. Then there exists $y_0 \in Y$ such that (x_0,y_0) is a solution of (P).*

Proof. We know that

$$\min_{x\in S_1}\sup_{y\in Y} F(x,y) \geqq \sup_{y\in Y}\min_{x\in S_1} F(x,y). \tag{4.1}$$

Thus, we have to prove that

$$\sup_{y\in Y}\min_{x\in S_1} F(x,y) \geqq \min_{x\in S_1}\sup_{y\in Y} F(x,y).$$

Define $F_1(x,y) = F(x,y) - v$. Then $((F_1(.,y), g_{J_0}(.)); h_K(.))$ is V-invex infine at x_0 on S. Also the system

$$(\exists x \in \Re^n)(\forall y \in Y)\; F_1(x,y) \leqq 0, x \in S_1$$

has a solution x_0. Further, the following system

$$(\exists x \in S_1)(\forall y \in Y) F_1(x,y) < 0$$

is inconsistent. For, if this system is consistent, then there exists $\bar{x} \in S_1$ such that for any $y \in Y,\ F_1(\bar{x},y) < 0$ which implies that

$$\sup_{y\in Y} F(\bar{x},y) < v = \sup_{y\in Y} F(x_0,y). \tag{4.2}$$

But $v = \min_{x\in S_1}\sup_{y\in Y} F(x,y) = \sup_{y\in Y} F(x_0,y) \leqq \sup_{y\in Y} F(x,y)$ for any $x \in S_1$ and hence, in particular, holds true for $\bar{x} \in S_1$ which contradicts the inequality (4.2).

It follows from Theorem 3.1 that there exist vectors $y_1, y_2, \ldots, y_{n+1}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \geq 0$ (we can take $\sum_{i=1}^{n+1}\lambda_i = 1$) such that

$$\sum_{i=1}^{n+1}\lambda_i F_1(x,y_i) \geqq 0,\ \forall x \in S_1,$$

where $y_i \in Y_0 = \{\bar{y} \in Y : F(x_0,\bar{y}) = \sup_{y\in Y} F(x_0,y)\}$, implying

$$\min_{x\in S_1}\sum_{i=1}^{n+1}\lambda_i F(x,y_i) \geqq v = \min_{x\in S_1}\sup_{y\in Y} F(x,y). \tag{4.3}$$

Since $F(x,.)$ is concavelike on Y, by induction, there exists $y_0 \in Y$ such that for any $x \in S_1$,

$$\sum_{i=1}^{n+1}\lambda_i F(x,y_i) \leqq F(x,y_0).$$

Therefore,

$$\min_{x\in S_1}\sum_{i=1}^{n+1}\lambda_i F(x,y_i) \leqq \sup_{y\in Y}\min_{x\in S_1} F(x,y). \tag{4.4}$$

Combining (4.1), (4.3) and (4.4), we obtain

$$\min_{x\in S_1} F(x,y_0) = \sup_{y\in Y}\min_{x\in S_1} F(x,y) = \min_{x\in S_1}\sup_{y\in Y} F(x,y) = \sup_{y\in Y} F(x_0,y).$$

Hence, the invex concavelike minimax problem (P) has a solution. □

5. Invex fractional minimax problem

This section is devoted to study an important class of fractional minimax optimization problems. Such problems arise naturally in modeling various conflicting situations, for example, in the formulation of rational approximation problems with respect to Chebyshev norm, in continuous rational games, in multiobjective programming, in parametric estimation problems, in minimum risk problems, to name the few. For a detailed discussion on minimax optimization problems and their applications, readers can refer to [1, 14, 15] and references cited therein. Optimality conditions for fractional minimax programming problems were earlier studied by Jeyakumar et al. [8] and Bector et al. [2]. In this section, we establish necessary and sufficient optimality conditions for fractional minimax problem in terms of Clarke generalized gradient under nonsmooth invex infine constraints.

We first state the conditions under which the ratio of two invex functions with respect to a common η is also invex.

Consider the functions $\phi(.)$ and $\psi(.)$ defined on S. Then their ratio is invex at $x_0 \in S$ if any of the following conditions are satisfied:

(1) $\phi \geqq 0$, $\psi > 0$ and $(\phi, -\psi)$ is invex at x_0;
(2) $\phi \geqq 0$, $\psi < 0$ and $(-\phi, -\psi)$ is invex at x_0;
(3) $\phi \leqq 0$, $\psi > 0$ and (ϕ, ψ) is invex at x_0;
(4) $\phi \leqq 0$, $\psi < 0$ and $(-\phi, \psi)$ is invex at x_0;

Observe that the ratio of the two invex functions with respect to same $\eta \in T_S(x_0)$ is invex with respect to the vector $\bar{\eta} \in T_S(x_0)$, given by $\bar{\eta} = \ \psi(x_0)\ (\eta/\psi)$.

We consider the following fractional minimax problem

$$(FP) \qquad \begin{array}{rl} \textit{minimize} & \sup_{y\in Y}\{\phi(x,y)/\psi(x,y)\} \\ \textit{subject to} & g_j(x) \leqq 0,\ j = 1,2,\ldots,m, \\ & h_k(x) = 0,\ k = 1,2,\ldots,p, \\ & x \in S, \end{array}$$

where Y is a compact subset of $\Re^m$ and S is a closed subset of $\Re^n$. Assume $\phi,\ \psi,\ g_j,\ h_k$ are locally Lipschitz functions on S and $\phi(x,y) \geqq 0,\ \psi(x,y) > 0,\ \forall (x,y) \in S \times Y$. For each $x \in S$, $y \longrightarrow \phi(x,.)/\psi(x,.)$ is upper semicontinuous on Y. Let S_1 denote the feasible set of (FP) and $F(x,y) = \phi(x,y)/\psi(x,y)$. Therefore, the fractional minimax problem (FP) is equivalent to the ordinary minimax problem $(P1)$

$$(P1) \qquad \begin{array}{lll} \textit{minimize} & f(x), & f(x) = \sup_{y \in Y} F(x,y) \\ \textit{subject to} & g_j(x) \leqq 0, & j = 1,2,\dots,m, \\ & h_k(x) = 0, & k = 1,2,\dots,p, \\ & x \in S. & \end{array}$$

This problem is identical to the problem (P) studied in sections 3 and 4.

Theorem 5.1 (Necessary Optimality Conditions). *If* $(x_0, y_0) \in S \times Y$ *is an optimal solution of the problem* (FP) *and* (JCQ) *holds at* x_0, *then there exist vectors* $y_1, y_2, \dots, y_{n+1}$ *in* Y_0, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1}) \geq 0$, $\beta_j \geqq 0$, $j \in J_0$, $\gamma_k \in \Re$, $k \in K$ *such that*

$$0 \in \sum_{i=1}^{n+1} \lambda_i/\psi(x_0, y_i)\,[\partial\phi(x_0, y_i) - F(x_0, y_i)\partial\psi(x_0, y_i)] + \sum_{j=1}^{m} \beta_j \partial g_j(x_0) + \sum_{k=1}^{p} \gamma_k \partial h_k(x_0) + N_S(x_0),$$

$$\beta_j g_j(x_0) = 0,\ j \in J.$$

Proof. Since (x_0, y_0) is a solution of (FP), x_0 is a solution of the problem $(P1)$. Repeating the arguments of the theorem of alternative (Theorem 3.1), we have, from (3.3)

$$0 \in \sum_{i=1}^{n+1} \lambda_i \partial F(x_0, y_i) + \sum_{j \in J_0} \beta_j \partial g_j(x_0) + \sum_{k \in K} \gamma_k \partial h_k(x_0) + N_S(x_0)$$

which can be rewritten as

$$0 \in \sum_{i=1}^{n+1} \lambda_i \partial F(x_0, y_i) + \sum_{j=1}^{m} \beta_j \partial g_j(x_0) + \sum_{k=1}^{p} \gamma_k \partial h_k(x_0) + N_S(x_0),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n+1}) \geq 0$, $\beta_j \geqq 0\ (j \in J_0)$, $\beta_j = 0\ (j \in J \backslash J_0)$, $\gamma_k \in \Re\ (k \in K)$. Applying Clarke's [6] quotient rule of subdifferentiability to $F(x_0, y_i)$, $i = 1, 2, \dots, n+1$, we have,

$$0 \in \sum_{i=1}^{n+1} \lambda_i/\psi(x_0, y_i)\,[\partial\phi(x_0, y_i) - F(x_0, y_i)\partial\psi(x_0, y_i)] + \sum_{j=1}^{m} \beta_j \partial g_j(x_0) + \sum_{k=1}^{p} \gamma_k \partial h_k(x_0) + N_S(x_0).$$

Also, it is observed that $\beta_j g_j(x_0) = 0,\ j \in J.$ □

Theorem 5.2 (Sufficient Optimality Conditions). *Assume that there exists* $(x_0, y_1, y_2, \dots, y_{n+1}, \ \lambda^0, \beta^0, \gamma^0)$ *such that* $x_0 \in S_1$, $y_i \in Y_0$, $i = 1, 2, \dots, n+1$, $\lambda^0 = (\lambda_1^0, \lambda_2^0, \dots, \lambda_{n+1}^0) \geq 0$, $\beta^0 = (\beta_1^0, \beta_2^0, \dots, \beta_m^0) \geqq 0$, $\gamma^0 = (\gamma_1^0, \gamma_2^0, \dots, \gamma_p^0) \in \Re^p$

satisfying

$$(5.1)\ 0 \in \sum_{i=1}^{n+1} \lambda_i^0/\psi(x_0,y_i)[\partial\phi(x_0,y_i) - F(x_0,y_i)\partial\psi(x_0,y_i)] + \sum_{j=1}^{m} \beta_j^0 \partial g_j(x_0) + \sum_{k=1}^{p} \gamma_k^0 \partial h_k(x_0) + N_S(x_0)$$

$$\beta_j g_j(x_0) = 0,\ j \in J.$$

Let $(\phi,\ -\psi)$ *be invex at* x_0 *on* S, $((\phi(.,y)/\psi(.,y),\ g(.));\ h(.))$ *be V-invex infine on* S. *Then there exists* $y_0 \in Y$ *such that* (x_0, y_0) *is an optimal solution for the fractional minimax problem* (FP).

Proof. From (5.1) we have,

$$0 = \sum_{i=1}^{n+1} \lambda_i^0/\psi(x_0,y_i)[\xi_{y_i}^0 - F(x_0,y_i)\bar{\xi}_{y_i}^0] + \sum_{j=1}^{m} \beta_j^0 \bar{\bar{\xi}}_j^0 + \sum_{k=1}^{p} \gamma_k^0 \tilde{\xi}_k^0 + z^0,$$

where $\xi_{y_i}^0 \in \partial\phi(x_0,y_i)$, $\bar{\xi}_{y_i}^0 \in \partial\psi(x_0,y_i)$, $i = 1,2,\ldots,n+1$, $y_i \in Y_0$, $\bar{\bar{\xi}}_j^0 \in \partial g_j(x_0)$, $j \in J$, $\tilde{\xi}_k^0 \in \partial h_k(x_0)$, $k \in K$ and $z^0 \in N_S(x_0)$. Therefore, for $\eta \in T_S(x_0)$,

$$(5.2)\qquad \sum_{i=1}^{n+1} \lambda_i^0/\psi(x_0,y_i)\langle \xi_{y_i}^0 - F(x_0,y_i)\bar{\xi}_{y_i}^0, \eta\rangle + \sum_{j=1}^{m} \beta_j^0 \langle \bar{\bar{\xi}}_j^0, \eta\rangle + \sum_{k=1}^{p} \gamma_k^0 \langle \tilde{\xi}_k^0, \eta\rangle + \langle z^0, \eta\rangle\ = 0.$$

Now, suppose that x_0 is not a solution of $(P1)$. Then there exists $\bar{x} \in S_1$ such that

$$\sup_{y\in Y} F(\bar{x}, y) < \sup_{y \in Y} F(x_0, y) = F(x_0, y_i),\ i = 1,2,\ldots,n+1$$

which implies

$$\phi(\bar{x}, y_i) - F(x_0,y_i)\psi(\bar{x},y_i) < 0,\ i = 1,2,\ldots,n+1.$$

By invexity of $\phi(.,y_i) - F(x_0,y_i)\psi(.,y_i)$, we have,

$$\langle \xi_{y_i}^0 - F(x_0,y_i)\bar{\xi}_{y_i}^0, \eta\rangle\ < 0$$

which implies

$$\sum_{i=1}^{n+1} \lambda_i^0/\psi(x_0,y_i)\langle \xi_{y_i}^0 - F(x_0,y_i)\bar{\xi}_{y_i}^0, \eta\rangle\ < 0.$$

For $x \in S_1$, $\beta_j^0 g_j(x) \leqq \beta_j^0 g_j(x_0)$ and $\gamma_k^0 h_k(x) = \gamma_k^0 h_k(x_0)$, which by V-invexity infineness leads to

$$\sum_{j=1}^{m} \beta_j^0 \langle \bar{\bar{\xi}}_j^0, \eta\rangle\ \leqq 0$$

and

$$\sum_{k=1}^{p} \gamma_k^0 \langle \tilde{\xi}_k^0, \eta \rangle = 0.$$

Further,

$$\langle z^0, \eta > \leqq 0.$$

Combining the above arguments, we obtain

$$\sum_{i=1}^{n+1} \lambda_i^0/\psi(x_0, y_i)\langle \xi_{y_i}^0 - F(x_0, y_i)\bar{\xi}_{y_i}^0, \eta\rangle + \sum_{j=1}^{m} \beta_j^0 \langle \bar{\bar{\xi}}_j^0, \eta\rangle + \sum_{k=1}^{p} \gamma_k^0 \langle \tilde{\xi}_k^0, \eta\rangle + \langle z^0, \eta\rangle < 0$$

which contradicts (5.2). Therefore, x_0 is the solution of the problem $(P1)$. Hence, there exists $y_0 \in Y$ such that (x_0, y_0) is a solution of (FP). □

6. Concluding remarks

As noted by Sach et al. [12], we also observe that to study applications of invexity ideas in optimization problems, explicit formulae of η is not required. The existence of vector η is sufficient enough to obtain the optimality conditions. This observation motivated us to introduce the notion of V-invex infine functions for nonsmooth case where η depends on x and A while θ_i depends on x and ξ_i. Also, V-invexity infineness is weaker as compared to the earlier ideas of invexity and/or infineness. We established an existence theorem and optimality conditions for fractional minimax programming problem involving functions belonging to the newly defined class. Applications of theorems of alternative established in this paper can also be explored for various optimization problems involving different solution concepts.

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Anulekha Dhara
Department of Mathematics, Indian Institute of Technology Delhi
Hauz Khas, New Delhi - 110016, India
E-mail address: anulekha_iitd@yahoo.co.in

Aparna Mehra
Department of Mathematics, Indian Institute of Technology Delhi
Hauz Khas, New Delhi - 110016, India
E-mail address: apmehra@maths.iitd.ac.in