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FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR SEMIGROUPS IN METRIC SPACES WITH UNIFORM NORMAL STRUCTURE

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Abstract. In the present paper, we introduce the property $(*)$ for a semigroup $T = \{T(t) : t \in G\}$ of selfmappings on a metric space (X, d). For example, each semigroup $T = \{T(t) : t \in G\}$ on X has the property $(*)$ in the case when (X, d) is a complete bounded metric space with property (P). See, e.g. [1] for the concept of property (P). The purpose of this paper is to establish a fixed point theorem for an asymptotically regular semigroup with property $(*)$ in a metric space with uniform normal structure.

1. Introduction

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be a Lipschitizan mapping if for each integer $n \geq 1$ there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

A Lipschitizian mapping T is said to be uniformly $k-$Lipschitzian if $k_n = k$ for all $n \geq 1$ and nonexpansive if $k_n = 1$ for all $n \geq 1$, respectively. Moreover, a mapping $T : C \to C$ is called asymptotically regular [11], if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0 \text{ for all } x \in C.$$

Edelstein and O'Brien [4] proved that if T is nonexpansive then the averaged mappings $T_a = aI + (1 - a)T$ where $a \in (0, 1)$ and I is the identity operator of X are asymptotically regular on C, i.e.,

$$\lim_{n\to\infty} \|T_a^{n+1}x - T_a^n x\| = 0 \text{ for all } x \in C.$$

Recently Kuczumow [12] proved the following result.

Theorem 1.1 (cf. [12, Theorem 3.2]). *Let X be a Banach space with $r_X(1) > 0$ and with the nonstrict Opial property, C a nonempty weakly compact convex subset of X and $T = \{T(t) : t \in G\}$ an asymptotically regular semigroup with*

$$\liminf_{t\to\infty} |T(t)| = k < 1 + r_X(1),$$

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where $r_X(\cdot)$ is the Opial's modulus of X and $|T(t)|$ is the exact Lipschitzian constant of $T(t)$. Then there exists z in C such that $T(t)z = z$ for all $t \in G$.

It is well known that Kirk [11] proved the following theorem: if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive selfmapping T on C has a fixed point. A nonempty convex subset C of a normed linear space is said to have normal structure if each bounded convex subset K of C consisting of more than one point contains a nondiameter point, i.e. an $x \in K$ such that

$$\sup\{\|x - y\| : y \in K\} < \sup\{\|u - v\| : u, v \in K\} = \operatorname{diam} K.$$

Four years later, in 1969 Kijima and Takahashi [10] eatablished the metric space version of this classical fixed point theorem. Subsequently many authors successfully generalized certain fixed point theorems and structure properties of Banach spaces to metric spaces. For example, Khamsi [9] defined normal and uniform normal structure for metric spaces and proved that if (X, d) is a complete bounded metric space with uniform normal structure, then it has the fixed point property for nonexpansive mappings and a kind of intersection property which extends a result of Maluta [16] to metric spaces.

In view of these facts, we naturally put forth an open question whether one may establish the existence result on the fixed points of asymptotically regular semigroups in metric spaces. To solve this problem, we now recall the new and novel method of Lim and Xu [14].

Definition 1.2 ([14])**.** A metric space (X, d) is said to have property (P) if given any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{z_j : j \geq n\}$ such that $\limsup_{n\to\infty} d(z, x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, x_n)$.

They then proved a fixed point theorem for uniformly Lipschitzian mappings in a complete bounded metric space with both property (P) and uniform normal structure which extends Khamsi's theorem and is also the metric space version of Casini and Maluta's theorem [2].

Lim and Xu's method [14] gives us a new idea for establishing the existence result on the fixed points of asymptotically regular semigroups in metric spaces.

In this paper we introduce the property $(*)$ for a semigroup $T = \{T(t) : t \in G\}$ of selfmappings on a metric space (X, d). For example, each semigroup $T = \{T(t) : t \in G\}$ on X has the property $(*)$ in the case when (X, d) is a complete bounded metric space with property (P). The purpose of the present paper is to establish a fixed point theorem for asymptotically regular semigroups with property $(*)$ in a complete bounded metric space with uniform normal structure.

2. Preliminaries

Throughout this paper, (X, d) will denote a metric space. Let G be an unbounded subset of $[0, \infty)$ such that

$$t + h \in G \text{ for all } t, h \in G, \text{ and}$$

$$t - h \in G \text{ for all } t, h \in G \text{ with } t \geq h.$$

For example, we may let $G = [0, \infty)$ or $G = N$, the set of nonnegative integers. Let $T = \{T(t) : t \in G\}$ be a one-parameter family of selfmappings on X. Then T is called a (one-parameter) semigroup on X if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in X$;

(ii) $T(s+t)x = T(s)T(t)x$ for all $s, t \in G$ and $x \in X$;

(iii) for each $x \in X$, a mapping $t \to T(t)x$ from G into X is continuous when G has the relative topology of $[0, \infty)$;

(iv) for each $t \in G$, $T(t) : X \to X$ is continuous.

A semigroup $T = \{T(t) : t \in G\}$ on X is said to be asymptotically regular at a point $x \in X$ if

$$\lim_{t\to\infty} d(T(t+h)x, T(t)x) = 0$$

for each $h \in G$. If T is asymptotically regular at each $x \in X$, then T is called an asymptotically regular semigroup on X.

For each $t \in G$, let us write

$$k(t) = \sup\{d(T(t)x, T(t)y)/d(x, y) : x, y \in X \text{ and } x \neq y\}$$

which is called the exact Lipschitzian constant of $T(t)$.

A semigroup $T = \{T(t) : t \in G\}$ on X is called a uniformly Lipschitzian (or uniformly $k-$Lipschitzian) semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty.$$

The simplest uniformly Lischitzian semegroup is a semigroup of iterates of a mapping $T : X \to X$ with

$$\sup\{k_n : n \in N\} = k < \infty,$$

where

$$k_n = \sup\{d(T^n x, T^n y)/d(x, y) : x, y \in X \text{ and } x \neq y\}.$$

In this case T is called a uniformly $k-$Lipschitzian mapping. In a natural way this kind of semigroup appears in the problem of stability of the fixed point property for nonexpansive mappings. In [12] one also can find the interesting construction of the uniformly Lipschitzian mapping with

$$\liminf_{n\to\infty} k_n < \limsup_{n\to\infty} k_n.$$

It is remarkable that the concept of asymptotic regularity is due to F. E. Browder and W. V. Petryshyn [1]. In [15] and [17] one can find two very interesting examples of asymptotically regular mappings without fixed points. By the Ishikawa result [8] (see also [4]) in the problem of stability of the fixed point property for nonexpansive mappings in Banach spaces, it is sufficient to consider asymptotically regular and nonexpansive mappings which become asymptotically regular and uniformly Lipschitzian mappings in each equivalent norm to the original norm. In addition, we remind the reader that recently many authors (e.g. [12, 6]) have deeply investigated the existence of fixed points for asymptotically regular semigroups of selfmappings in Banach spaces under various conditions.

Let F be a nonempty family of subsets of X. Following Khamsi [9], we say that F defines a convexity structure on X if F is stable by intersection and that F has

Property (R) if any decreasing sequence $\{C_n\}$ of closed bounded nonempty subsets of X with $C_n \in F$ has a nonvoid intersection. Recall that a subset of X is said to be admissible (cf. [3]) if it is an intersection of closed balls. We denote by $A(X)$ the family of all asmissible subsets of X. It is easy to see that $A(X)$ defines a convexity structure in X. Thuoughout the rest of this paper, we will assume that any other convexity structure F on X contains $A(X)$.

Let M be a bounded subset of X. Following Lim and Xu [14], we adopt the following notations:

$$B(x,r) \text{ is the closed ball centered at } x \text{ with radius } r,$$
$$r(x,M) = \sup\{d(x,y) : y \in M\} \text{ for } x \in X,$$
$$\delta(M) = \sup\{r(x,M) : x \in M\},$$
$$R(M) = \inf\{r(x,M) : x \in M\}.$$

For a bounded subset A of X, we define the admissible hull of A, denoted by ad(A), as the intersection of all those admissible subsets of X which contain A, i.e.,

$$\mathrm{ad}(A) = \bigcap\{B : A \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.$$

Proposition 2.1 ([14])**.** *For a point $x \in X$ and a bounded subset A of X, we have*

$$r(x, ad(A)) = r(x, A).$$

Definition 2.2 ([9])**.** A metric space (X,d) is said to have normal (resp. uniform normal) structure if there exists a convexity structure F on X such that $R(A) < \delta(A)$ (resp. $R(A) < c \cdot \delta(A)$ for some constant $c \in (0,1)$) for all $A \in F$ which is bounded and consists of more than one point. It is also said that F is normal (resp. uniformly normal).

If we define normal structure coefficient $\tilde{N}(X)$ of X (with respect to a given convexity structure F) as the number

$$\sup\left\{\frac{R(A)}{\delta(A)}\right\},$$

where the supremum is taken over all bounded $A \in F$ with $\delta(A) > 0$, then X has uniform normal structure if and only if $\tilde{N}(X) < 1$.

Khamsi proved the following result that will be very useful in the proof of our main theorem in Section 3.

Proposition 2.3 ([9])**.** *Let X be a complete bounded metric space and F be a convexity structure of X with uniform normal structure. Then F has the property (R).*

We now introduce the following property for a semigroup of selfmappings on a metric space (X,d).

Definition 2.4. Let (X,d) be a metric space and $T = \{T(t) : t \in G\}$ be a semigroup on X. Let us write the set

$$\omega(\infty) = \{\{t_n\} : \{t_n\} \subset G \text{ and } t_n \uparrow \infty\}.$$

T is said to have property $(*)$ if for each $x \in X$ and each $\{t_n\} \in \omega(\infty)$, the following conditions are satisfied:

(i) the sequence $\{T(t_n)x\}$ is bounded;

(ii) for any sequence $\{z_n\}$ in $\mathrm{ad}\{T(t_n)x : n \geq 1\}$ there exists some $z \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\}$ such that

$$\limsup_{n\to\infty} d(z, T(t_n)x) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, T(t_n)x).$$

Remark 2.5. If X is a complete bounded metric space with property (P), then each semigroup $T = \{T(t) : t \in G\}$ on X has property $(*)$.

Remark 2.6. If X has property (R), then $\bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\} \neq \emptyset$. Also if X is a weakly compact convex subset of a normed linear space, then admissible hulls are closed convex and $\bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\} \neq \emptyset$ by the weak compactness of X. Also X possesses property (P). This fact follows directly from the weak lower semicontinuity of the functional $\limsup_{n\to\infty} \|x_n - x\|$. Therefore each semigroup $T = \{T(t) : t \in G\}$ on X has property $(*)$. Indeed it is easy to see that $\{T(t_n)x\}$ is bounded for each $x \in X$ and each $\{t_n\} \subset \omega(\infty)$. Clearly, $\mathrm{ad}\{T(t_n)x : n \geq 1\} \subset X$ is bounded. Hence for each $\{z_n\}$ in $\mathrm{ad}\{T(t_n)x : n \geq 1\}$, by the property (P) of X, we know that there exists $z \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{z_j : j \geq n\}$ such that

$$\limsup_{n\to\infty} d(z, T(t_n)x) \leq \limsup_{j\to\infty} \limsup_{t_n\to\infty} d(z_j, T(t_n)x).$$

3. Main result

The following lemma will play an important role in the proof of our main result in the sequel.

Lemma 3.1. *Let (X, d) be a complete bounded metric space with uniform normal structure and $T = \{T(t) : t \in G\}$ be a semigroup on X with property $(*)$. Then for each $x \in X$, each $\{t_n\} \in \omega(\infty)$ and for any constant $\bar{c} > \tilde{N}(X)$, the normal structure coefficient with respect to the given convexity structure F, there exists some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ satisfying the properties:*

(i) $\limsup_{n\to\infty} d(z, T(t_n)x) \leq \bar{c} \cdot A(\{T(t_n)x\})$ *where*

$$A(\{T(t_n)x\}) = \limsup_{n\to\infty} \{d(T(t_i)x, T(t_j)x), i, j \geq n\}$$

is the asymptotic diameter of $\{T(t_n)x\}$;

(ii) $d(z, y) \leq \limsup_{n\to\infty} d(T(t_n)x, y)$ *for all $y \in X$.*

Proof. For each integer $n \geq 1$, let $A_n = \mathrm{ad}(\{T(t_j)x : j \geq n\})$. Then $\{A_n\}$ is a decreasing sequence of admissible subsets of X hence $A := \bigcap_{n=1}^{\infty} A_n \neq \emptyset$ by Proposition 2.3. From Proposition 2.1, it is not difficult to see that $\delta(A_n) = \delta(\{T(t_i)x : i \geq n\})$. Indeed observe that

$$\begin{aligned}
\delta(A_n) = \sup\{r(y, A_n) : y \in A_n\} &= \sup_{y\in A_n} \sup_{j\geq n} d(y, T(t_j)x) \\
&= \sup_{j\geq n} \sup_{y\in A_n} d(y, T(t_j)x) = \sup_{j\geq n} r(T(t_j)x, A_n) \\
&= \sup_{j\geq n} \sup_{i\geq n} d(T(t_j)x, T(t_i)x) \\
&= \delta(\{T(t_i)x : i \geq n\}).
\end{aligned}$$

On the other hand for any $z \in A$ and any $y \in X$, we have

$$\sup_{j\geq n} d(y, T(t_j)x) = r(y, A_n) \geq r(y, A) \geq d(y, z).$$

Therefore,

$$d(y, z) \leq \limsup_{n\to\infty} d(y, T(t_n)x)$$

from which (ii) follows.

We now claim that for each $n \geq 1$, there exists $z_n \in A_n$ such that

$$r(z_n, A_n) \leq \bar{c}\delta(\{T(t_j)x : j \geq n\}). \tag{3.1}$$

Indeed if $\delta(\{T(t_j)x : j \geq n\}) = 0$, then using $\delta(A_n) = \delta(\{T(t_j)x : j \geq n\})$, we conclude that (3.1) holds. Without loss of generality, we may assume that $\delta(\{T(t_j)x : j \geq 0\}) > 0$. Then for $\bar{c} > \tilde{N}(X)$, we choose $\varepsilon > 0$ so small satisfying the following:

$$\tilde{N}(X)\delta(\{T(t_j)x : j \geq n\}) + \varepsilon \leq \bar{c}\delta(\{T(t_j)x : j \geq n\}). \tag{3.2}$$

By the definition of $R(A_n)$, one can find $z_n \in A_n$ such that

$$\begin{aligned} r(z_n, A_n) < R(A_n) + \varepsilon &\leq \tilde{N}(X)\delta(A_n) + \varepsilon \\ &= \tilde{N}(X)\delta(\{T(t_j)x : j \geq n\}) + \varepsilon \\ &\leq \bar{c}\delta(\{T(t_j)x : j \geq n\}). \end{aligned}$$

This shows that (3.1) holds. Obviously it follows from (3.1) that for each $n \geq 1$,

$$\limsup_{j\to\infty} r(z_n, x_j) \leq \bar{c}\delta(\{T(t_j)x : j \geq n\})$$

which implies

$$\limsup_{n\to\infty} \limsup_{j\to\infty} r(z_n, T(t_j)x) \leq \bar{c} \cdot A(\{T(t_n)x\}) \tag{3.3}$$

where $A(\{T(t_j)x\}) = \limsup_{n\to\infty}\{d(T(t_i)x, T(t_j)x) : i, j \geq n\}$. Noticing

$$z_n \in A_n \subset \text{ad}\{T(t_i)x : j \geq 1\} \quad \text{for each } n \geq 1,$$

we know that property $(*)$ yields a point $z \in \bigcap_{n=1}^{\infty}\text{ad}\{z_j : j \geq n\}$ such that

$$\limsup_{j\to\infty} d(z, T(t_j)x) \leq \limsup_{n\to\infty} \limsup_{j\to\infty} r(z_n, T(t_j)x).$$

Since $\{z_j : j \geq n\} \subset A_n$, $z \in A = \bigcap_{n=1}^{\infty}\text{ad}\{T(t_j)x : j \geq n\}$ and satisfies

$$\limsup_{j\to\infty} d(z, T(t_j)x) \leq \bar{c} \cdot A(\{T(t_j)x\}$$

by (3.3). Therefore (i) holds. □

We are now in a position to prove the main result of this section.

Theorem 3.2. *Let (X, d) be a complete bounded metric space with uniform normal structure and let $T = \{T(t) : t \in G\}$ be an asymptotically regular semigroup on X with property $(*)$ and satisfying*

$$(\liminf_{t\to\infty} k(t)) \cdot (\limsup_{t\to\infty} k(t)) < \tilde{N}(X)^{-1}.$$

Then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

Proof. At first, we write

$$k = \liminf_{t\to\infty} k(t) \text{ and } \hat{k} = \limsup_{t\to\infty} k(t).$$

Let us choose a constant $\bar{c}$ such that $\tilde{N}(X) < \bar{c} < 1$ and $k \cdot \hat{k} < \bar{c}^{-1/2}$. We can select a sequence $\{t_n\} \in \omega(\infty)$ such that $\{t_{n+1} - t_n\} \in \omega(\infty)$ and $\lim_{n\to\infty} k(t_n) = k$. Indeed it is easy to choose $\{t_n\} \subset \omega(\infty)$ such that $\lim_{n\to\infty} k(t_n) = k$. Then take $t_{n_1} = t_1, t_{n_2} = t_2$ and $\gamma = t_2 - t_1$. Since $\{t_n\}$ increases monotonely to $+\infty$, there is $n_3 > n_2$ such that

$$t_{n_3} - t_{n_2} > \max(3\gamma, t_{n_3} - t_{n_2}).$$

Similarly, for n_3, there exists $n_4 > n_3$ such that

$$t_{n_4} - t_{n_3} > \max(3\gamma, t_{n_3} - t_{n_2}).$$

Repeating this process, we can obtain a subsequence $\{t_{n_i}\} \subset \{t_n\}$ such that $\{t_{n_{i+1}} - t_{n_i}\}_{i=1}^{\infty} \in \omega(\infty)$ and $\lim_{i\to\infty} k(t_{n_i}) = \lim_{n\to\infty} k(t_n) = k$. Replacing $\{t_{n_i}\}$ by $\{t_n\}$, we get the above statement.

Observe that

$$\{d(T(t_j)x, T(t_i)x) : i, j \geq n\} = \{d(T(t_j)x, T(t_i)x), j > i \geq n\} \bigcup \{0\}$$

for each $n \in N$ and $x \in X$, and

$$\{k(t_j - t_i) : j > i \geq n\} \subset \{k(t) : G \ni t \geq t_{n+1} - t_n\}.$$

(Indeed, for any $j > i \geq n$, we have

$$t_j - t_i \geq t_j - t_{j-1} \geq t_{j-1} - t_{j-2} \geq \cdots \geq t_{i+1} - t_i \geq t_{n+1} - t_n.$$

Hence $k(t_j - t_i) \in \{k(t) : G \ni t \geq t_{n+1} - t_n\}$.)

Now fix an $x_0 \in X$. Then by Lemma 3.1, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty} \subset X$ such that for each integer $l \geq 0$,

(a) $x_{l+1} \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{T(t_i)x_l : i \geq n\}$;

(b) $\limsup_{n\to\infty} d(T(t_n)x_l, x_{l+1}) \leq \bar{c} \cdot A(\{T(t_n)x_l\})$,
where $A(\{T(t_n)x_l\}) = \limsup_{n\to\infty} \{d(T(t_i)x_l, T(t_j)x_l) : i, j \geq n\}$;

(c) $d(x_{l+1}, y) \leq \limsup_{t_n\to\infty} d(T(t_n)x_l, y)$ for all $y \in X$.

Write for each $l \geq 0$,

$$D_l = \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \text{ and } \theta = \bar{c} \cdot k\hat{k} < 1.$$

Observe that for each $i > j \geq 1$,

$$\begin{aligned} d(T(t_i)x_l, T(t_j)x_l) &= d(T(t_j)x_l, T(t_j)T(t_i - t_j)x_l) \\ &\leq k(t_j) \cdot d(x_l, T(t_i - t_j)x_l) \\ &\leq k(t_j) \cdot \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_i - t_j)x_l) \quad \text{(by (c))}. \end{aligned} \tag{3.4}$$

By virtue of the asymptotic regularity of $T = \{T(t) : t \in G\}$ on X, we see that

$$\lim_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_i - t_j)x_{l-1}) = 0,$$

which implies

$$\begin{aligned}
\limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_i - t_j)x_l) &\le \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_i - t_j)x_{l-1}) \\
&\quad + \limsup_{n\to\infty} d(T(t_n + t_i - t_j)x_{l-1}, T(t_n + t_i - t_j)x_l) \\
&\le \limsup_{n\to\infty} d(T(t_i - t_j)T(t_n)x_{l-1}, T(t_i - t_j)x_l) \qquad (3.5) \\
&\le k(t_i - t_j) \cdot \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) \\
&\le k(t_i - t_j) \cdot D_{l-1}.
\end{aligned}$$

Then it follows from (3.4) and (3.5) that for each $i > j \ge 1$,

$$d(T(t_i)x_l, T(t_j)x_l) \le k(t_j) \cdot k(t_i - t_j) \cdot D_{l-1}, \qquad (3.6)$$

which implies for each $n \ge 1$,

$$\begin{aligned}
\sup\{d(T(t_i)x_l, T(t_j)x_l) : i, j \ge n\} &= \sup\{d(T(t_i)x_l, T(t_j)x_l) : i > j \ge n\} \\
&\le \sup\{k(t_j) \cdot k(t_i - t_j) \cdot D_{l-1} : i > j \ge n\} \\
&\le D_{l-1} \cdot \sup\{k(t_j) : j \ge n\} \cdot \sup\{k(t_i - t_j) : i > j \ge n\} \qquad (3.7) \\
&\le D_{l-1} \cdot \sup\{k(t_j) : j \ge n\} \cdot \sup\{k(t) : G \ni t \ge t_{n+1} - t_n\}.
\end{aligned}$$

Hence by using (b) and (3.7), we have

$$\begin{aligned}
D_l &= \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \le \bar{c}A(\{T(t_n)x_l\}) \\
&= \bar{c} \cdot \limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i, j \ge n\} \\
&\le \bar{c} \cdot D_{l-1} \cdot \limsup_{n\to\infty} k(t_n) \cdot \limsup_{n\to\infty}\{k(t) : G \ni t \ge t_{n+1} - t_n\} \qquad (3.8) \\
&\le \bar{c} \cdot D_{l-1} \cdot \lim_{n\to\infty} k(t_n) \cdot \limsup_{s\to\infty}\{k(t) : G \ni t \ge s\} \\
&= \bar{c} \cdot k\hat{k} \cdot D_{l-1} = \theta D_{l-1} \le \cdots \\
&= \theta^l D_0.
\end{aligned}$$

Hence by the asymptotic regularity of T on X, we have for each integer $n \ge 1$,

$$\begin{aligned}
d(x_{l+1}, x_l) &\le d(x_{l+1}, T(t_n)x_l) + d(x_l, T(t_n)x_l) \\
&\le d(x_{l+1}, T(t_n)x_l) + \limsup_{m\to\infty} d(T(t_m)x_{l-1}, T(t_n)x_l) \\
&\le d(x_{l+1}, T(t_n)x_l) + \limsup_{m\to\infty} d(T(t_m)x_{l-1}, T(t_m + t_n)x_{l-1}) \qquad (3.9) \\
&\quad + \limsup_{m\to\infty} d(T(t_n + t_m)x_{l-1}, T(t_n)x_l) \\
&\le d(x_{l+1}, T(t_n)x_l) + k(t_n) \cdot \limsup_{m\to\infty} d(T(t_m)x_{l-1}, x_l) \\
&= d(x_{l+1}, T(t_n)x_l) + k(t_n) \cdot D_{l-1},
\end{aligned}$$

which implies

$$\begin{aligned}
d(x_{l+1}, x_l) &\le \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) + D_{l-1} \cdot \limsup_{n\to\infty} k(t_n) \\
&= D_l + kD_{l-1}.
\end{aligned}$$

It follows from (3.8) that

$$d(x_{l+1}, x_l) \leq D_l + kD_{l-1} \leq (\theta^l + k\theta^{l-1})D_0 \leq \theta^{l-1} \cdot 2D_0 \max\{\theta, k\}.$$

Clearly $\{x_l\}$ is a Cauchy sequence and hence is convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then for each $s \in G$, by the continuity of $T(s)$ we have

$$\lim_{l\to\infty} d(T(s)x_l, T(s)z) = 0.$$

On the other hand, from (3.9) we have actually proven the following inequality:

$$d(T(t_n)x_l, x_l) \leq k(t_n) \cdot D_{l-1} \leq k(t_n) \cdot \theta^{l-1} D_0.$$

Since $\lim_{n\to\infty} k(t_n) = k$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} d(z, T(t_n)z) \leq & d(z, x_l) + \limsup_{n\to\infty} d(x_l, T(t_n)x_l) \\ & + \limsup_{n\to\infty} d(T(t_n)x_l, T(t_n)z) \\ \leq & d(z, x_l) + \limsup_{n\to\infty} k(t_n) \cdot \theta^{l-1} D_0 \\ & + \limsup_{n\to\infty} k(t_n) \cdot d(x_l, z) \\ \leq & (1 + k) \cdot d(z, x_l) + k\theta^{l-1} D_0 \to 0 \quad as \quad l \to \infty,\end{aligned}$$

i.e., $\lim_{n\to\infty} d(z, T(t_n)z) = 0$. Hence for each $s \in G$, by the continuity of $T(s)$, we deduce

$$\lim_{n\to\infty} d(T(s + t_n)z, T(s)z) = 0.$$

Note that for each $s \in G$,

$$\begin{aligned}d(z, T(s)z) \leq & d(z, x_{l+1}) + d(x_{l+1}, T(t_n)x_l) + d(T(t_n)x_l, T(s)z) \\ \leq & d(z, x_{l+1}) + d(x_{l+1}, T(t_n)x_l) + d(T(t_n)x_l, T(t_n + s)x_l) \\ & + d(T(s + t_n)x_l, T(s)z) \\ \leq & d(z, x_{l+1}) + d(x_{l+1}, T(t_n)x_l) + d(T(t_n)x_l, T(t_n + s)x_l) \\ & + d(T(s + t_n)x_l, T(s + t_n)z) + d(T(s + t_n)z, T(s)z) \\ \leq & d(z, x_{l+1}) + d(x_{l+1}, T(t_n)x_l) + d(T(t_n)x_l, T(t_n + s)x_l) \\ & + k(t_n) \cdot d(T(s)x_l, T(s)z) + d(T(s + t_n)z, T(s)z).\end{aligned} \tag{3.10}$$

By taking the superior limit in both sides of (3.10) as $n \to \infty$, we have

$$\begin{aligned}d(z, T(s)z) \leq & d(z, x_{l+1}) + \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) + k \cdot d(T(s)x_l, T(s)z) \\ \leq & d(z, x_{l+1}) + D_l + k \cdot d(T(s)x_l, T(s)z) \\ \leq & d(z, x_{l+1}) + \theta^l \cdot D_0 + k \cdot d(T(s)x_l, T(s)z).\end{aligned} \tag{3.11}$$

Then by taking the limit in both sides of (3.11) as $l \to \infty$, we have $d(z, T(s)z) = 0$, i.e., $T(s)z = z$ for each $s \in G$. □

From Remark 2.5 and Theorem 3.2, we immediately obtain the following result.

Corollary 3.3. *Let (X, d) be a complete bounded metric space with both property (P) and uniform normal structure and let $T = \{T(t) : t \in G\}$ be an asymptotically regular semigroup on X satisfying*

$$(\liminf_{t\to\infty} k(t)) \cdot (\limsup_{t\to\infty} k(t)) < \tilde{N}(X)^{-1}.$$

Then there exists some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

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