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NONOCCURRENCE OF GAP FOR NONCONVEX NONAUTONOMOUS VARIATIONAL PROBLEMS

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Abstract. In our recent work we established nonoccurrence of the Lavrentiev phenomenon for two classes of unconstrained nonconvex variational problems. For the first class of integrands we showed the existence of a minimizing sequence of Lipschitzian functions while for the second class we established that an infimum on the full admissible class is equal to the infimum on a set of Lipschitzian functions with the same Lipschitzian constant. In this paper we generalize these results for classes of constrained variational problems.

1. Introduction

The Lavrentiev phenomenon in the calculus of variations was discovered in [10]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem to possess an infimum on the dense subclass of C^1 admissible functions that is strictly greater than its minimum value on the admissible class. Since this classical work the Lavrentiev phenomenon is of great interest [1-4, 6-8, 11, 12, 16]. Mania [12] simplified the original example of Lavrentiev. Ball and Mizel [3, 4] demonstrated that the Lavrentiev phenomenon can occur with fully regular integrands. Nonoccurrence of the Lavrentiev phenomenon was studied in [1, 2, 7, 8, 11, 16]. Clarke and Vinter [7] showed that the Lavrentiev phenomenon cannot occur when a variational integrand $f(t, x, u)$ is independent of t. Sychev and Mizel [16] considered a class of integrands $f(t, x, u)$ which are convex with respect to the last variable. For this class of integrands they established that the Lavrentiev phenomenon does not occur. In [17] we established nonoccurrence of Lavrentiev phenomenon for two classes of unconstrained nonconvex nonautonomous variational problems with integrands $f(t, x, u)$. For the first class of integrands we showed the existence of a minimizing sequence of Lipschitzian functions while for the second class we established that an infimum on the full admissible class is equal to the infimum on a set of Lipschitzian functions with the same Lipschitzian constant. In this paper we generalize these results for classes of constrained variational problems.

Assume that $(X, ||\cdot||)$ is a Banach space. Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1, \tau_2; X)$ the set of all functions $x : [\tau_1, \tau_2] \to X$ for which there exists a Bochner integrable function $u : [\tau_1, \tau_2] \to X$ such that

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} u(s)ds,\ t \in (\tau_1, \tau_2]$$

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(see, e.g., [5]). It is known that if $x \in W^{1,1}(\tau_1, \tau_2; X)$, then this equation defines a unique Bochner integrable function u which is called the derivative of x and is denoted by x'.

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$.

Let $a, b \in R^1$ satisfy $a < b$. Suppose that $f : [a, b] \times X \times X \to R^1$ is a continuous function such that the following assumptions hold:

(A1)

$$f(t, x, u) \geq \phi(||u||) \text{ for all } (t, x, u) \in [a, b] \times X \times X, \tag{1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is an increasing function such that

$$\lim_{t \to \infty} \phi(t)/t = \infty; \tag{1.2}$$

(A2) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u) - f(t, x_2, u)| \leq \epsilon \max\{f(t, x_1, u), f(t, x_2, u)\}$$

for each $t \in [a, b]$, each $u \in X$ satisfying $||u|| \geq \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$||x_1 - x_2|| \leq \delta, \ ||x_1||, ||x_2|| \leq M;$$

(A3) for each $M, \epsilon > 0$ there exists $\delta > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \epsilon$$

for each $t \in [a, b]$ and each $x_1, x_2, y_1, y_2 \in X$ satisfying

$$||x_i||, ||y_i|| \leq M, \ i = 1, 2 \text{ and } ||x_1 - x_2||, \ ||y_1 - y_2|| \leq \delta.$$

Remark 1.1. If $X = R^n$, then (A3) follows from the continuity of f.

Many examples of integrands which satisfy (A1)-(A3) are given in [17].

For each $z_1, z_2, z_3 \in X$ denote by $\mathcal{A}(z_1, z_2, z_3)$ the set of all functions $v \in W^{1,1}(a, b; X)$ such that

$$v(a) = z_1, \ v(b) = z_2, \ (b - a)^{-1} \int_a^b v(t)dt = z_3 \tag{1.3}$$

and denote by $\mathcal{A}_L(z_1, z_2, z_3)$ the set of all $v \in \mathcal{A}(z_1, z_2, z_3)$ for which there is $M_v > 0$ such that

$$||v'(t)|| \leq M_v \text{ for almost every } t \in [a, b]. \tag{1.4}$$

Clearly for each $v \in W^{1,1}(a, b; X)$ the function $f(t, v(t), v'(t))$, $t \in [a, b]$ is measurable. Set

$$I(v) = \int_a^b f(t, v(t), v'(t))dt, \ v \in W^{1,1}(a, b; X). \tag{1.5}$$

For each $z_1, z_2, z_3 \in X$ we consider the variational problem

$$I(v) \to \min, \ v \in \mathcal{A}(z_1, z_2, z_3). \tag{P}$$

Note that variational problems of this type with the constraint (1.3) arise in continuum mechanics [9, 13-15].

The next theorem is our first main result.

Theorem 1.2. *Let $z_1, z_2, z_3 \in X$. Then*

$$\inf\{I(x) : x \in \mathcal{A}(z_1, z_2, z_3)\} = \inf\{I(x) : x \in \mathcal{A}_L(z_1, z_2, z_3)\}.$$

Theorem 1.2 is proved in Section 3.

Now we present out second main result.

Let $a, b \in R^1$, $a < b$. Suppose that $f : [a,b] \times X \times X \to R^1$ is a continuous function which satisfies the following assumptions:

(B1) There is an increasing function $\phi : [0,\infty) \to [0,\infty)$ such that

$$f(t,x,u) \geq \phi(||u||) \text{ for all } (t,x,u) \in [a,b] \times X \times X, \tag{1.6}$$

$$\lim_{t\to\infty} \phi(t)/t = \infty. \tag{1.7}$$

(B2) For each $M > 0$ there exist positive numbers δ, L and an integrable nonnegative scalar function $\psi_M(t)$, $t \in [a,b]$ such that for each $t \in [a,b]$, each $u \in X$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \leq M,\ ||x_1 - x_2|| \leq \delta$$

the following inequality holds:

$$|f(t,x_1,u) - f(t,x_2,u)| \leq ||x_1 - x_2||L(f(t,x_1,u) + \psi_M(t)).$$

(B3) For each $M > 0$ there is $L > 0$ such that for each $t \in [a,b]$ and each $x_1, x_2, u_1, u_2 \in X$ satisfying $||x_i||, ||u_i|| \leq M$, $i = 1, 2$ the following inequality holds:

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \leq L(||x_1 - x_2|| + ||u_1 - u_2||).$$

Remark 1.3. It is not difficult to see that if (B1)-(B3) hold with each ψ_M bounded, then f satisfies (A1)-(A3). Clearly, (A1) and (B1) are identical.

Many examples of integrands which satisfy (B1)-(B3) are given in [17].

Clearly for each $x \in W^{1,1}(a,b;X)$ the function $f(t,x(t),x'(t))$, $t \in [a,b]$ is measurable.

For each $x \in W^{1,1}(a,b;X)$ set

$$I(x) = \int_a^b f(t,x(t),x'(t))dt.$$

We continue to study the variational problem (P) with $z_1, z_2, z_3 \in X$.

The next theorem is our second main result.

Theorem 1.4. *Let $M > 0$. Then there exists $K > 0$ such that for each $z_1, z_2, z_3 \in X$ satisfying $||z_1||, ||z_2||, ||z_3|| \leq M$ and each $x(\cdot) \in \mathcal{A}(z_1, z_2, z_3)$ the following assertion holds:*

If mes$\{t \in [a,b] : ||x'(t)|| > K\} > 0$, then there exists $y \in \mathcal{A}(z_1, z_2, z_3)$ such that $I(y) < I(x)$ and $||y'(t)|| \leq K$ for almost every $t \in [a,b]$.

In [17] analogs of Theorems 1.2 and 1.4 were established for the variational problem

$$I(v) \to \min,$$
$$v \in W^{1,1}(a,b;X),\ v(a) = z_1,\ v(b) = z_2,$$

where $z_1, z_2 \in X$ and the integrand f satisfies either (A1)-(A3) or (B1)-(B3). There we use the following strategy of the proof. We choose a large positive constant

N_0 and a constant N_1 which is essentially larger than N_0. Then we a consider an admissible function $v \in W^{1,1}(a,b;X)$ which satisfies the constraints $v(a) = z_1$ and $v(a) = z_2$. We define

$$E_1 = \{t \in [a,b] : \ ||v'(t)|| \geq N_1\},$$
$$E_2 = \{t \in [a,b] : \ ||v'(t)|| \leq N_0\},$$
$$E_3 = [a,b] \setminus (E_1 \cup E_2)$$

and

$$h_0 = \int_{E_1} v'(t)dt$$

and assume that $\mathrm{mes}(E_1)$ is positive. Then we define a measurable function $\xi : [a,b] \to X$ by

$$\xi(t) = 0, \ t \in E_1, \ \xi(t) = v'(t), \ t \in E_3,$$
$$\xi(t) = v'(t) + (\mathrm{mes}(E_2))^{-1} h_0, \ t \in E_2$$

and define $u \in W^{1,1}(a,b;X)$ by

$$u(\tau) = \int_0^\tau \xi(t)dt + z_1, \ \tau \in [a,b].$$

It follows from the construction of ξ, u that u satisfies $u(a) = z_1$ and $u(b) = z_2$. Then in [17] we compare $I(v)$ and $I(u)$. It turns out that for the first class of integrands $I(u) \leq I(v) + \epsilon$, where ϵ is a given positive number while for the second class $I(u) < I(v)$. It is not difficult to see that the function u defined above does not necessarily satisfy the third constraint

$$(b-a)^{-1} \int_a^b u(t)dt = z_3.$$

Hence for the classes of constrained variational problems considered in this paper the problem of the construction of ξ and u becomes much more difficult.

The paper is organized as follows. Section 2 contains auxiliary results for Theorem 1.2 which is proved in Section 3. Our second main result (Theorem 1.4) is proved in Section 5. Section 4 contains auxiliary results for Theorem 1.4.

2. Auxiliary results for Theorem 1.2

In this section we assume that the continuous function $f : [a,b] \times X \times X \to R^1$ satisfies (A1)-(A3).

Let $z_1, z_2, z_3 \in X$. Set

$$M_0 = \inf\{I(v) : \ v \in \mathcal{A}(z_1, z_2, z_3)\}. \tag{2.1}$$

Lemma 2.1. *M_0 is a finite number.*

Proof. Clearly $M_0 \geq 0$. Set

$$z_4 = 2z_3 - 2^{-1}(z_1 + z_2) \tag{2.2}$$

and define a function $v : [a,b] \to X$ as follows:

$$v(t) = z_1 + 2(t-a)(b-a)^{-1}(z_4 - z_1), \ t \in [a, 2^{-1}(a+b)], \tag{2.3}$$

$$v(t) = z_4 + [(t - 2^{-1}(b+a))/(2^{-1}(b-a))](z_2 - z_4),\ t \in [2^{-1}(a+b), b].$$

Clearly

$$v \in W^{1,1}(a,b;X),\ v(a) = z_1,\ v(b) = z_2. \tag{2.4}$$

By (2.3)

$$\begin{aligned}\int_a^b v(t)dt &= \int_a^{(a+b)/2} v(t)dt + \int_{(a+b)/2}^b v(t)dt \\ &= 2^{-1}(b-a)z_1 + \left(\int_a^{(a+b)/2} 2(t-a)(b-a)^{-1}dt\right)(z_4 - z_1) \\ &\quad + 2^{-1}(b-a)z_4 + \left[\int_{(a+b)/2}^b (t-(b+a)/2)((b-a)/2)^{-1}dt\right](z_2 - z_4) \\ &= 2^{-1}(b-a)z_1 + 4^{-1}(b-a)(z_4 - z_1) + +2^{-1}(b-a)z_4 + 4^{-1}(z_2 - z_4) \\ &= 2^{-1}(b-a)z_4 + 4^{-1}(z_1 + z_2).\end{aligned}$$

Together with (2.2) this equality implies that

$$(b-a)^{-1}\int_a^b v(t)dt = 2^{-1}z_4 + 4^{-1}(z_1 + z_2) = z_3.$$

Combined with (2.4) this implies that $v \in \mathcal{A}(z_1, z_2, z_3)$. In view of (2.3) the set

$$\{(v(t), v'(t)) :\ t \in [a,b] \setminus \{(a+b)/2\}\}$$

is bounded. It follows from this fact and assumption (A3) that the function $f(t, v(t), v'(t)),\ t \in [a,b] \setminus \{2^{-1}(a+b)\}$ is bounded. Therefore $M_0 \le I(v) < \infty$. Lemma 2.1 is proved. □

Lemma 2.2. *There exists a number $M_1 > 0$ such that for each $v \in \mathcal{A}(z_1, z_2, z_3)$ satisfying $I(v) \le M_0 + 2$ the following inequality holds:*

$$||v(t)|| \le M_1 \text{ for all } t \in [a,b].$$

For the proof of Lemma 2.2 see Lemma 2.1 of [17].

Lemma 2.3 (17, Lemma 2.2)**.** *Let $\epsilon, M > 0$. Then there exist $\Gamma, \delta > 0$ such that*

$$|f(t, x_1, u) - f(t, x_2, u)| \le \epsilon \min\{f(t, x_1, u), f(t, x_2, u)\}$$

for each $t \in [a,b]$, each $u \in X$ satisfying $||u|| \ge \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \le M,\ ||x_1 - x_2|| \le \delta.$$

3. Proof of Theorem 1.2

Set

$$M_0 = \inf\{I(v) :\ v \in \mathcal{A}(z_1, z_2, z_3)\}. \tag{3.1}$$

By Lemma 2.1 M_0 is a finite number. Let $\epsilon \in (0,1)$. In order to prove the theorem it is sufficient to show that for each $v \in \mathcal{A}(z_1, z_2, z_3)$ satisfying $I(v) \le M_0 + 1$ there is $u \in \mathcal{A}_L(z_1, z_2, z_3)$ such that $I(u) \le I(v) + \epsilon$.

By Lemma 2.2 there is $M_1 > 0$ such that

$$||v(t)|| \le M_1,\ t \in [a,b] \tag{3.2}$$

for all $v \in \mathcal{A}(z_1, z_2, z_3)$ satisfying $I(v) \le M_0 + 2$.

Choose a positive number ϵ_0 such that

$$8\epsilon_0(M_0 + 4) < \epsilon \tag{3.3}$$

and a positive number γ_0 such that

$$\gamma_0 < 1 \text{ and } 32\gamma_0(M_0 + 2) < b - a. \tag{3.4}$$

Relation (1.2) implies that there is $N > 1$ such that

$$\phi(t)/t \ge \gamma_0^{-1} \text{ for all } t \ge N. \tag{3.5}$$

In view of Lemma 2.3 there are

$$\delta_0 \in (0, 1),\ N_0 > N \tag{3.6}$$

such that for each $t \in [a,b]$, each $y \in X$ satisfying $||y|| \ge N_0$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \le M_1 + 2,\ ||x_1 - x_2|| \le \delta_0 \tag{3.7}$$

the following inequality holds:

$$|f(t, x_1, y) - f(t, x_2, y)| \le \epsilon_0 \min\{f(t, x_1, y), f(t, x_2, y)\}. \tag{3.8}$$

By (A3) there exists

$$\delta_1 \in (0, \delta_0) \tag{3.9}$$

such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le (8(b - a + 1))^{-1}\epsilon \tag{3.10}$$

for each $t \in [a,b]$ and each $x_1, x_2, y_1, y_2 \in X$ satisfying

$$||x_1||, ||x_2|| \le M_1 + 2,\ ||y_1||, ||y_2|| \le N_0 + 1, \tag{3.11}$$
$$||x_1 - x_2||,\ ||y_1 - y_2|| \le \delta_1.$$

It follows from (A3) that there is

$$M_2 > \sup\{f(t, y, 0):\ t \in [a,b],\ y \in X \text{ and } ||y|| \le M_1 + 1\}. \tag{3.12}$$

Choose a positive number γ_1 such that

$$96 \cdot 32\gamma_1(M_0 + M_1 + 4) < \delta_1 \min\{1, b - a\}. \tag{3.13}$$

By (1.2) there is a number N_1 such that

$$N_1 > N_0 + M_2 + 4 \text{ and } \phi(t)/t \ge \gamma_1^{-1} \text{ for all } t \ge N_1. \tag{3.14}$$

Assume that

$$v \in \mathcal{A}(z_1, z_2, z_3) \text{ and } I(v) \le M_0 + 2. \tag{3.15}$$

It follows from (3.15) and the choice of M_1 that the inequality (3.2) holds. Set

$$E_1 = \{t \in [a,b]:\ ||v'(t)|| \ge N_1\}, \tag{3.16}$$
$$E_2 = \{t \in [a,b]:\ ||v'(t)|| \le N_0\},$$
$$E_3 = [a,b] \setminus (E_1 \cup E_2).$$

Relations (1.1), (1.5), (3.14), (3.15), and (3.16) imply that

$$\left\|\int_{E_1} v'(t)dt\right\| \le \int_{E_1} \|v'(t)\|dt \le \int_{E_1} \gamma_1\phi(\|v'(t)\|)dt \le \gamma_1 \int_{E_1} f(t, v(t), v'(t))dt \le \gamma_1 I(v) \le \gamma_1(M_0 + 2). \tag{3.17}$$

Now we estimate $\mathrm{mes}(E_2)$. It follows from (3.16), the choice of N (see (3.5)), (3.6), (1.1), (1.5), and (3.15) that

$$\begin{aligned}\mathrm{mes}(E_1 \cup E_3) &\le N_0^{-1} \int_{E_1\cup E_3} ||v'(t)||dt \le \gamma_0 N_0^{-1} \int_{E_1\cup E_3} \phi(||v'(t)||)dt \\ &\le \gamma_0 N_0^{-1} \int_{E_1\cup E_3} f(t, v(t), v'(t))dt \le \gamma_0 N_0^{-1} I(v) \\ &\le \gamma_0 I(v) \le \gamma_0(M_0 + 2).\end{aligned} \tag{3.18}$$

Combined with (3.16) and (3.4) this inequality implies that

$$\mathrm{mes}(E_2) \ge (b - a) - \gamma_0(M_0 + 2) \ge (31/32)(b - a). \tag{3.19}$$

Set

$$g_1 = \int_{E_1} v'(t)dt,\ g_2 = \int_a^b \left(\int_{E_1\cap[a,t]} v'(s)ds\right) dt. \tag{3.20}$$

It is not difficult to see that there is $c \in [a, b]$ such that

$$\mathrm{mes}(E_2 \cap [a, c]) = \mathrm{mes}(E_2 \cap [c, b]). \tag{3.21}$$

Set

$$\beta_1 = \int_{E_2\cap[a,c]} (b - s)ds,\ \beta_2 = \int_{E_2\cap[c,b]} (b - s)ds. \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\begin{aligned}\beta_1 - \beta_2 &= \int_{E_2\cap[a,c]} (b - s)ds - \int_{E_2\cap[c,b]} (b - s)ds \\ &\ge \int_{E_2\cap[a,c]} (b - s)ds - (b - c)\ \mathrm{mes}\ (E_2 \cap [c, b]) = \int_{E_2\cap[a,c]} (c - s)ds.\end{aligned} \tag{3.23}$$

By (3.19) and (3.21)

$$\mathrm{mes}(E_2 \cap [a, c]) = \mathrm{mes}(E_2)/2 \ge (31/64)(b - a).$$

This implies that

$$c \ge a + (b - a)/3. \tag{3.24}$$

It follows from (3.16) and (3.19) that

$$\begin{aligned}\mathrm{mes}(E_2 \cap [a, a + (b - a)/4]) &\ge (b - a)/4 - \mathrm{mes}\ (E_1 \cup E_3) \\ &= (b - a)/4 - [b - a - \mathrm{mes}(E_2)] \\ &\ge (b - a)/4 - 32^{-1}(b - a) \\ &\ge 8^{-1}(b - a).\end{aligned} \tag{3.25}$$

Relations (3.23), (3.24), and (3.25) imply that

$$\beta_1 - \beta_2 \geq \int_{E_2 \cap [a,c]} (c-s) \geq \int_{E_2 \cap [a,a+(b-a)/4]} (c-s)ds$$
$$\geq 12^{-1}(b-a) \text{ mes } (E_2 \cap [a, a+(b-a)/4]) \geq (b-a)^2/96. \tag{3.26}$$

Define

$$h_1 = 2(\beta_2 - \beta_1)^{-1} \text{mes}(E_2)^{-1}[\beta_2 g_1 - 2^{-1}\text{mes}(E_2) g_2], \tag{3.27}$$

$$h_2 = 2(\beta_1 - \beta_2)^{-1} \text{mes}(E_2)^{-1}[\beta_1 g_1 - 2^{-1}\text{mes}(E_2) g_2]. \tag{3.28}$$

Clearly h_1, h_2 are well defined. Define a measurable function $\xi : [a,b] \to X$ by

$$\xi(t) = 0,\ t \in E_1,\ \xi(t) = v'(t),\ t \in E_3,$$
$$\xi(t) = v'(t) + h_1,\ t \in E_2 \cap [a,c],$$
$$\xi(t) = v'(t) + h_2,\ t \in E_2 \cap [c,b]. \tag{3.29}$$

Clearly the function ξ is Bochner integrable. Define a function $u : [a,b] \to X$ by

$$u(\tau) = \int_0^\tau \xi(t)dt + z_1,\ \tau \in [a,b]. \tag{3.30}$$

It follows from (3.16), (3.20), (3.21), (3.27), (3.28), and (3.29) that

$$\int_a^b \xi(t)dt = \int_{E_1} \xi(t)dt + \int_{E_2} \xi(t)dt + \int_{E_3} \xi(t)dt = \int_{E_2} \xi(t)dt + \int_{E_3} \xi(t)dt$$
$$= \int_{E_3} v'(t)dt + \int_{E_2} v'(t)dt + \text{mes}(E_2 \cap [a,c])h_1 + \text{mes}(E_2 \cap [c,b])h_2$$
$$= \int_{E_3} v'(t)dt + \int_{E_2} v'(t)dt + 2^{-1}\text{mes}(E_2)(h_1 + h_2)$$
$$= \int_{E_3} v'(t)dt + \int_{E_2} v'(t)dt + g_1 = \int_a^b v'(t)dt.$$

Combined with (3.15) and (3.30) this equality implies that

$$u(b) = z_2. \tag{3.31}$$

Relations (3.16), (3.29) and (3.30) imply that for each $t \in [a,b]$

$$u(t) = z_1 + \int_{[a,t]\cap E_1} \xi(s)ds + \int_{[a,t]\cap E_2} \xi(s)ds + \int_{[a,t]\cap E_3} \xi(s)ds$$
$$= z_1 + \int_{[a,t]\cap E_3} v'(s)ds + \int_{[a,t]\cap E_2} v'(s)ds$$
$$+ \text{mes}(E_2 \cap [a,c] \cap [a,t])h_1 + \text{mes}(E_2 \cap [c,b] \cap [a,t])h_2.$$

This equality, (3.16) and (3.31) imply that for each $t \in [a,c]$

$$u(t) = z_1 + \int_{[a,t]\setminus E_1} v'(s)ds + \text{mes}(E_2 \cap [a,t])h_1 \tag{3.32}$$

and that for each $t \in (c, b]$

$$u(t) = z_1 + \int_{[a,t]\setminus E_1} v'(s)ds + 2^{-1}\ \mathrm{mes}(E_2)h_1 + \mathrm{mes}(E_2 \cap [c,t])h_2. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$\begin{aligned}\int_a^b u(t)dt =&(b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt \\ &+ \left(\int_a^c \mathrm{mes}\ (E_2 \cap [a,t])dt\right)h_1 + 2^{-1}(b-c)\ \mathrm{mes}(E_2)h_1 \\ &+ \left(\int_c^b \mathrm{mes}(E_2 \cap [c,t])dt\right)h_2.\end{aligned} \tag{3.34}$$

By the Fubini theorem

$$\int_a^c \mathrm{mes}(E_2 \cap [a,t])dt = \int_{E_2\cap[a,c]} (c-s)ds, \tag{3.35}$$

$$\int_c^b \mathrm{mes}(E_2 \cap [c,t])dt = \int_{E_2\cap[c,b]} (b-s)ds.$$

In view of (3.20), (3.21), (3.22), (3.27), (3.28), (3.34), and (3.35)

$$\begin{aligned}\int_a^b u(t)dt &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt + \left(\int_{E_2\cap[a,c]} (c-s)ds\right)h_1 \\ &\quad + 2^{-1}(b-c)\mathrm{mes}(E_2)h_1 + \left(\int_{E_2\cap[c,b]} (b-s)ds\right)h_2 \\ &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt + \left(\int_{E_2\cap[a,c]} (b-s)ds\right)h_1 \\ &\quad + \left(\int_{E_2\cap[c,b]} (b-s)ds\right)h_2 \\ &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt + \beta_1 h_1 + \beta_2 h_2 \\ &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt \\ &\quad + 2(\beta_2 - \beta_1)^{-1}\mathrm{mes}(E_2)^{-1}[2^{-1}\mathrm{mes}(E_2)](\beta_2 - \beta_1)g_2 \\ &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]\setminus E_1} v'(s)ds\right)dt + g_2 \\ &= (b-a)z_1 + \int_a^b \left(\int_{[a,t]} v'(s)ds\right)dt = \int_a^b v(t)dt.\end{aligned}$$

This equality, (3.15), (3.30), and (3.31) imply that

$$u \in \mathcal{A}(z_1, z_2, z_3). \tag{3.36}$$

It follows from (3.17), (3.19), (3.20), (3.22), (3.26), (3.27), and (3.28) that

(3.37)
$$\begin{aligned}
||h_1||,\ ||h_2|| &\le (\beta_1 - \beta_2)^{-1} 2 \text{ mes}(E_2)^{-1}(\beta_1 ||g_1|| + \beta_2 ||g_1|| + 2^{-1} \text{ mes } E_2 ||g_2||) \\
&\le 96(b-a)^{-3} 4((b-a)^2 ||g_1|| + (b-a)||g_2||) \\
&\le (b-a)^{-3} 4 \cdot 96((b-a)^2 \gamma_1(M_0+2) + (b-a)^2 \gamma_1(M_0+2)) \\
&= (8 \cdot 96)(b-a)^{-1} \gamma_1(M_0+2).
\end{aligned}$$

Now we show that

$$\|u(t) - v(t)\| \le \delta_1 \text{ for all } t \in [a,b]. \tag{3.38}$$

Let $s \in (a,b]$. By (3.13), (3.15), (3.16), (3.17), (3.29), (3.30), (3.36), and (3.37)

$$\begin{aligned}
\|v(s) - u(s)\| = \left\| \int_a^s [v'(t) - \xi(t)] dt \right\| &\le \left\| \int_{[a,s] \cap E_1} [v'(t) - \xi(t)] dt \right\| \\
&+ \left\| \int_{[a,s] \cap E_2} [v'(t) - \xi(t)] dt \right\| + \left\| \int_{[a,s] \cap E_3} [v'(t) - \xi(t)] dt \right\| \\
&\le \int_{E_1} \|v'(t)\| dt + \|h_1\| \text{mes}(E_2) + \|h_2\| \text{mes}(E_2) \\
&\le \gamma_1(M_0+2) + 2(b-a) 8 \cdot 96 (b-a)^{-1} \gamma_1(M_0+2) \\
&< \gamma_1(M_0+2) 32 \cdot 96 < \delta_1.
\end{aligned}$$

Thus (3.38) holds. It follows from (3.2), (3.6), (3.9), and (3.38) that

$$\|u(t)\| \le M_1 + 1 \text{ for all } t \in [a,b]. \tag{3.39}$$

We estimate $I(u) - I(v)$. In view of (1.5) and (3.16)

$$I(u) - I(v) = \sum_{i=1}^{3} \int_{E_i} [f(t, u(t), u'(t)) - f(t, v(t), v'(t))] dt. \tag{3.40}$$

By (3.12), (3.29), (3.30), and (3.39) for almost every $t \in E_1$

$$f(t, u(t), u'(t)) = f(t, u(t), \xi(t)) = f(t, u(t), 0) < M_2. \tag{3.41}$$

Relations (1.1), (3.16), and (3.14) imply that for almost every $t \in E_1$

$$f(t, v(t), v'(t)) \ge \phi(\|v'(t)\|) \ge N_1 > M_2 + 4.$$

Combined with (3.41) this inequality implies that

$$\int_{E_1} [f(t, u(t), u'(t)) - f(t, v(t), v'(t))] dt \le 0. \tag{3.42}$$

Let $t \in E_2$ and $v'(t), u'(t)$ exist. It follows from (3.16) that

$$\|v'(t)\| \le N_0. \tag{3.43}$$

In view of (3.13), (3.30), (3.29), and (3.37)

(3.44)
$$\|u'(t)-v'(t)\| = \|\xi(t)-v'(t)\| \le \max\{\|h_1\|, \|h_2\|\} \le (8\cdot 96)(b-a)^{-1}\gamma_1(M_0+2) < \delta_1.$$

Relations (3.6), (3.9), (3.43), and (3.44) imply that

(3.45)
$$\|u'(t)\| \le N_0 + 1.$$

By (3.2), (3.38), (3.39), (3.43), (3.44), (3.45), and the choice of δ_1 (see (3.9)-(3.11))

$$|f(t, v(t), v'(t)) - f(t, u(t), u'(t))| \le (8(b-a+1))^{-1}\epsilon.$$

Since this inequality holds for almost every $t \in E_2$ we obtain that

(3.46)
$$\left| \int_{E_2} [f(t, u(t), u'(t)) - f(t, v(t), v'(t))]dt \right| \le 8^{-1}\epsilon.$$

Let $t \in E_3$ and $u'(t)$ and $v'(t)$ exist. By (3.16)

(3.47)
$$\|v'(t)\| \ge N_0.$$

In view of (3.29) and (3.30)

$$|f(t, v(t), v'(t)) - f(t, u(t), u'(t))| = |f(t, v(t), v'(t)) - f(t, u(t), v'(t))|.$$

It follows from this equality, (3.2), (3.9), (3.38), (3.39), (3.47), and the choice of δ_0, N_0 (see (3.6)-(3.8)) that

$$|f(t, v(t), v'(t)) - f(t, u(t), u'(t))| \le \epsilon_0 f(t, v(t), v'(t)).$$

By this inequality which holds for almost every $t \in E_3$, (3.3) and (3.15)

$$\left| \int_{E_3} [f(t, u(t), u'(t)) - f(t, v(t), v'(t))]dt \right| \le \int_{E_3} \epsilon_0 f(t, v(t), v'(t))dt \le \epsilon_0 I(v)$$
$$\le \epsilon_0(M_0 + 2) < \epsilon/8.$$

Combined with (3.42) and (3.46) this inequality implies that $I(u) - I(v) \le \epsilon/2$. This completes the proof of Theorem 1.2.

4. Auxiliary results for Theorem 1.4

In this section we assume that the continuous function $f : [a, b] \times X \times X \to R^1$ satisfies (B1)-(B3).

For each $z_1, z_2, z_3 \in X$ set

$$U(z_1, z_2, z_3) = \inf\{I(x) : \ x \in \mathcal{A}(z_1, z_2, z_3)\}.$$

Lemma 4.1. *Let $M > 0$. Then there is $M_1 > 0$ such that*

$$U(z_1, z_2, z_3) \le M_1 \text{ for each } z_1, z_2, z_3 \in X \text{ satisfying } ||z_1||, ||z_2||, ||z_3|| \le M.$$

Proof. Set

(4.1)
$$M_1 = \sup\{f(s, z, u) : \ s \in [a, b], \ z, u \in X$$
$$\text{and } ||z||, ||u|| \le 8M(1 + (b-a)^{-1})\}(b-a).$$

By (B3) M_1 is finite. Assume that $z_1, z_2, z_3 \in X$ and

(4.2)
$$||z_1||, ||z_2||, ||z_3|| \le M.$$

Define $z_4 \in X$ by (2.2) and define a function $v : [a, b] \to X$ by (2.3). It was shown in the proof of Lemma 2.1 that (2.4) holds. It follows from (2.2) and (2.3) that

$$||v(t)|| \le 3M \text{ for all } t \in [a, b], \tag{4.3}$$

$$||v'(t)|| \le 8M(b - a)^{-1} \text{ for a.e. } t \in [a, b]. \tag{4.4}$$

Relations (4.1), (4.3), and (4.4) imply that for a.e. $t \in [a, b]$

$$f(t, v(t), v'(t)) \le M_1/(b - a).$$

This inequality and (2.4) imply that

$$U(z_1, z_2, z_3) \le I(v) \le M_1.$$

□

Lemma 4.2. *Let $M > 0$. Then there is $M_0 > 0$ such that for each $z_1, z_2, z_3 \in X$ satisfying $||z_1||, ||z_2||, ||z_3|| \le M$ and each $x \in \mathcal{A}(z_1, z_2, z_3)$ satisfying $I(x) \le U(z_1, z_2, z_3) + 1$ the inequality $||x(t)|| \le M_0$ holds for all $t \in [a, b]$.*

For the proof of this lemma see Lemma 5.2 of [17].

5. Proof of Theorem 1.4

Let $M > 0$. By Lemma 4.1 there is $M_1 > 0$ such that

$$U(z_1, z_2, z_3) \le M_1 \text{ for each } z_1, z_2, z_3 \in X \text{ satisfying } ||z_1||, ||z_2||, ||z_3|| \le M. \tag{5.1}$$

In view of Lemma 4.2 there is $M_0 > 0$ such that for each $z_1, z_2, z_3 \in X$ and each $x \in \mathcal{A}(z_1, z_2, z_3)$ satisfying

$$||z_1||, ||z_2||, ||z_3|| \le M, \ I(x) \le U(z_1, z_2, z_3) + 1 \tag{5.2}$$

the following inequality holds:

$$||x(t)|| \le M_0, \ t \in [a, b]. \tag{5.3}$$

By (B2) there are $\delta_0, L_0 > 0$ and an integrable scalar function $\psi_0(t) \ge 0$, $t \in [a, b]$ such that for each $t \in [a, b]$, each $u \in X$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \le M_0 + 8, \ ||x_1 - x_2|| \le \delta_0 \tag{5.4}$$

the following inequality holds:

$$|f(t, x_1, u) - f(t, x_2, u)| \le ||x_1 - x_2|| L_0(f(t, x_1, u) + \psi_0(t)). \tag{5.5}$$

Choose a positive number γ_0 such that

$$\gamma_0 < 1 \text{ and } 64\gamma_0(M_1 + 1) < b - a. \tag{5.6}$$

In view of (B1) and (1.7) there is $K_0 > 1$ such that

$$\phi(t)/t \ge \gamma_0^{-1} \text{ for all } t \ge K_0. \tag{5.7}$$

Set

$$\Delta_0 = \sup\{f(t, z, 0) : \ t \in [a, b], \ z \in X \text{ and } ||z|| \le M_0 + 8\}. \tag{5.8}$$

(B3) implies that Δ_0 is finite.

It follows from (B3) that there is $L_1 > 1$ such that for each $t \in [a,b]$ and each $x_1, x_2, u_1, u_2 \in X$ satisfying

$$||x_1||, ||x_2||, ||u_1||, ||u_2|| \leq K_0 + M_0 + 12 \tag{5.9}$$

the following inequality holds:

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \leq L_1(||x_1 - x_2|| + ||u_1 - u_2||). \tag{5.10}$$

Choose a number $\gamma_1 \in (0,1)$ such that

$$96 \cdot 8\gamma_1(M_1 + 2) < (\min\{1, b-a\}) \min\{1, \delta_0/8\}, \tag{5.11}$$

$$\begin{aligned}\gamma_1 <& (96 \cdot 64 L_1(b - a + 1 + \int_a^b \psi_0(t)dt) + 64 \\ &+ (1 + (b-a)^{-1})96 \cdot 64 L_0(b - a + 1 + M_1 + \int_a^b \psi_0(t)dt))^{-1}.\end{aligned} \tag{5.12}$$

By (B1) and (1.7) there is a number $K > 0$ such that

$$K > 8\Delta_0 + K_0 + 2, \tag{5.13}$$

$$\phi(t)/t \geq \gamma_1^{-1} \text{ for all } t \geq K. \tag{5.14}$$

Assume that

$$z_1, z_2, z_3 \in X,\ ||z_1||, ||z_2||, ||z_3|| \leq M, \tag{5.15}$$

$$x \in \mathcal{A}(z_1, z_2, z_3), \tag{5.16}$$

$$\text{mes}\{t \in [a,b] :\ ||x'(t)|| > K\} > 0. \tag{5.17}$$

We show that there is $u \in \mathcal{A}(z_1, z_2, z_3)$ such that $I(u) < I(x)$ and $||u'(t)|| \leq K$ for almost every $t \in [a,b]$.

We may assume without loss of generality that

$$I(x) \leq U(z_1, z_2, z_3) + 1. \tag{5.18}$$

Relations (5.1), (5.15), and (5.18) imply that

$$I(x) \leq M_1 + 1. \tag{5.19}$$

In view of (5.15), (5.16) and (5.18) and the choice of M_0 (see (5.2), (5.3))

$$||x(t)|| \leq M_0,\ t \in [a,b]. \tag{5.20}$$

Set

$$\begin{aligned} E_1 &= \{t \in [a,b] :\ ||x'(t)|| \geq K\}, \\ E_2 &= \{t \in [a,b] :\ ||x'(t)|| \leq K_0\}, \\ E_3 &= [a,b] \setminus (E_1 \cup E_2). \end{aligned} \tag{5.21}$$

Set

$$g_1 = \int_{E_1} x'(t)dt,\ g_2 = \int_a^b \left(\int_{E_1 \cap [a,t]} x'(s)ds \right) dt, \tag{5.22}$$

$$d = \int_{E_1} ||x'(t)||dt. \tag{5.23}$$

It follows from (5.17) and (5.21) that

$$d > 0. \tag{5.24}$$

Clearly

$$||g_1|| \le d, \ ||g_2|| \le d(b-a). \tag{5.25}$$

By (1.6), (5.14), (5.19), (5.21), and (5.23)

$$d = \int_{E_1} ||x'(t)||dt \le \int_{E_1} \gamma_1 \phi(||x'(t)||)dt \le \gamma_1 \int_a^b \phi(||x'(t)||)dt \tag{5.26}$$
$$\le \gamma_1 \int_a^b f(t, x(t), x'(t))dt \le \gamma_1(M_1 + 1).$$

Now we estimate mes(E_2). It follows from (1.6), (5.7), (5.13), (5.19), (5.21), and the inequality $K_0 > 1$ that

$$\text{mes}(E_1 \cup E_3) \le K_0^{-1} \int_{E_1 \cup E_3} ||x'(t)||dt \le K_0^{-1} \int_{E_1 \cup E_3} \gamma_0 \phi(||x'(t)||)dt \tag{5.27}$$
$$\le \gamma_0 K_0^{-1} \int_a^b \phi(||x'(t)||)dt \le \gamma_0 \int_a^b \phi(||x'(t)||)dt$$
$$\le \gamma_0 \int_a^b f(t, x(t), x'(t))dt \le \gamma_0(M_1 + 1).$$

Together with (5.21) this inequality implies that

$$\text{mes}(E_2) \ge b - a - \gamma_0(M_1 + 1). \tag{5.28}$$

Relations (5.6) and (5.28) imply that

$$\text{mes}(E_2) \ge (31/32)(b - a). \tag{5.29}$$

It is not difficult to see that there is $c \in [a, b]$ such that

$$\text{mes}(E_2 \cap [a, c]) = \text{mes}(E_2 \cap [c, b]). \tag{5.30}$$

Set

$$\beta_1 = \int_{E_2 \cap [a,c]} (b - s)ds, \ \beta_2 = \int_{E_2 \cap [c,b]} (b - s)ds. \tag{5.31}$$

As in the proof of Theorem 1.1 (see (3.23)-(3.26)) we can show that

$$\beta_1 - \beta_2 \ge \int_{E_2 \cap [a,c]} (c - s)ds \ge (b - a)^2/96. \tag{5.32}$$

Define

$$h_1 = 2(\beta_2 - \beta_1)^{-1}\text{mes}(E_2)^{-1}[\beta_2 g_1 - 2^{-1}\text{mes}(E_2)g_2], \tag{5.33}$$

$$h_2 = 2(\beta_1 - \beta_2)^{-1}\text{mes}(E_2)^{-1}[\beta_1 g_1 - 2^{-1}\text{mes}(E_2)g_2]. \tag{5.34}$$

Clearly h_1, h_2 are well defined. Define a measurable function $\xi : [a, b] \to X$ by

$$\xi(t) = 0, \ t \in E_1, \ \xi(t) = x'(t), \ t \in E_3, \tag{5.35}$$

$$\xi(t) = x'(t) + h_1,\ t \in E_2 \cap [a, c],$$
$$\xi(t) = x'(t) + h_2,\ t \in E_2 \cap [c, b].$$

Clearly the function ξ is Bochner integrable. Define a function $u : [a, b] \to X$ by

$$u(\tau) = \int_0^\tau \xi(t)dt + z_1,\ \tau \in [a, b]. \tag{5.36}$$

As in the proof of Theorem 1.2 we can show that

$$u \in \mathcal{A}(z_1, z_2, z_3)$$

(see (3.31)-(3.36)). In view of (5.25), (5.29), (5.31), (5.32), (5.35), and (5.36) for almost every $t \in E_2$

$$\begin{aligned} ||x'(t) - u'(t)|| &= ||x'(t) - \xi(t)|| \le \max\{||h_1||, ||h_2||\} \\ &\le 96(b-a)^{-2} 4(b-a)^{-1}[(b-a)^2||g_1|| + (b-a)||g_2||] \\ &\le 4 \cdot 96(b-a)^{-3} 2(b-a)^2 d = 8 \cdot 96(b-a)^{-1} d. \end{aligned} \tag{5.37}$$

Combined with (5.11), (5.21), and (5.26) this relation implies that for almost every $t \in E_2$

$$\begin{aligned} ||u'(t)|| &\le ||x'(t)|| + 8 \cdot 96(b-a)^{-1} d \\ &\le K_0 + 8 \cdot 96(b-a)^{-1}\gamma_1(M_1 + 1) \\ &\le K_0 + 1. \end{aligned} \tag{5.38}$$

Relations (5.13), (5.21), (5.35), (5.36), and (5.38) imply that

$$||u'(t)|| \le K \text{ for almost every } t \in [a, b]. \tag{5.39}$$

We show that $I(u) < I(x)$. Let $s \in (a, b]$. It follows from (5.16), (5.21), (5.23), (5.35), (5.36), (5.37), and the inclusion $u \in \mathcal{A}(z_1, z_2, z_3)$ that

$$\begin{aligned} ||x(s) - u(s)|| &= ||\int_a^s [x'(t) - u'(t)]dt|| = ||\int_a^s [x'(t) - \xi(t)]dt|| \\ &\le ||\int_{[a,s]\cap E_1} [x'(t) - \xi(t)]dt|| + ||\int_{[a,s]\cap E_2} [x'(t) - \xi(t)]dt|| \\ &\quad + ||\int_{[a,s]\cap E_3} [x'(t) - \xi(t)]dt|| \\ &\le \int_{E_1} ||x'(t)||dt + ||\int_{[a,s]\cap E_2} [x'(t) - \xi(t)]dt|| \\ &\le \int_{E_1} ||x'(t)||dt + (b-a)(||h_1|| + ||h_2||) \\ &\le d + 16 \cdot 96d \le 32 \cdot 96d. \end{aligned}$$

Therefore

$$||x(s) - u(s)|| \le 32 \cdot 96d \text{ for all } s \in [a, b]. \tag{5.40}$$

In view of (5.21)

$$I(u) - I(x) = \sum_{i=1}^{3} \int_{E_i} [f(t, u(t), u'(t)) - f(t, x(t), x'(t))]dt. \tag{5.41}$$

By (5.11), (5.20), (5.26), (5.35), (5.36), and (5.40) for almost every $t \in E_1$

$$\begin{aligned} f(t, u(t), u'(t)) &= f(t, u(t), 0) \\ &\le \sup\{f(t, z, 0) : \ z \in X \text{ and } ||z|| \le M_0 + 32 \cdot 96\gamma_1(M_1 + 1)\} \\ &\le \sup\{f(t, z, 0) : \ z \in X \text{ and } ||z|| \le M_0 + 8\}. \end{aligned}$$

Combined with (5.8) this inequality implies that for almost every $t \in E_1$

$$f(t, u(t), u'(t)) \le \Delta_0. \tag{5.42}$$

It follows from (B1), (1.6), (5.13), (5.14), and (5.21) that for almost every $t \in E_1$

$$f(t, x(t), x'(t)) \ge \phi(||x'(t)||) \ge ||x'(t)|| \ge K > 8\Delta_0.$$

Together with (5.42) this inequality implies that for almost every $t \in E_1$

$$f(t, x(t), x'(t)) - f(t, u(t), u'(t)) \ge 3f(t, x(t), x'(t))/4. \tag{5.43}$$

The inequality (5.43) implies that

$$\int_{E_1} [f(t, u(t), u'(t)) - f(t, x(t), x'(t))]dt \le -(3/4) \int_{E_1} f(t, x(t), x'(t))dt. \tag{5.44}$$

By (5.11), (5.20), (5.26), and (5.40) for all $t \in [a, b]$

$$||u(t)|| \le ||x(t)|| + 32 \cdot 96d \le M_0 + 32 \cdot 96\gamma_1(M_1 + 1) \le M_0 + 8. \tag{5.45}$$

It follows from (5.20), (5.21), (5.38), (5.45), and the choice of L_1 (see (5.9), (5.10)) that for almost every $t \in E_2$

$$|f(t, x(t), x'(t)) - f(t, u(t), u'(t))| \le L_1(||x(t) - u(t)|| + ||x'(t) - u'(t)||).$$

Combined with (5.37) and (5.40) this inequality implies that for almost every $t \in E_2$

$$\begin{aligned} |f(t, x(t), x'(t)) - f(t, u(t), u'(t))| &\le L_1(32 \cdot 96d + 8 \cdot 96(b - a)^{-1}a) \\ &\le L_1 32 \cdot 96d(1 + (b - a)^{-1}). \end{aligned}$$

Therefore

$$\left| \int_{E_2} [f(t, x(t), x'(t)) - f(t, u(t), u'(t))]dt \right| \le L_1 32 \cdot 96(1 + b - a). \tag{5.46}$$

Relations (5.1), (5.26), and (5.40) imply that for all $t \in [a, b]$

$$||x(t) - u(t)|| \le 32 \cdot 96d \le 32 \cdot 96\gamma_1(M_1 + 1) \le \delta_0.$$

By this inequality, (5.20), (5.35), (5.36), (5.45), and the choice of δ_0, L_0, ψ_0 (see (5.4) and (5.5)) for all $t \in E_3$

$$\begin{aligned} |f(t, x(t), x'(t)) - f(t, u(t), u'(t))| &= |f(t, x(t), x'(t)) - f(t, u(t), x'(t))| \\ &\le L_0(f(t, x(t), x'(t)) + \psi_0(t))||x(t) - u(t)||. \end{aligned}$$

Together with (5.40) this inequality implies that for almost all $t \in E_3$

$$\big|f(t, x(t), x'(t)) - f(t, u(t), u'(t))\big| \le 32 \cdot 96dL_0(f(t, x(t), x'(t)) + \psi_0(t)).$$

Therefore combined with (5.19) this inequality implies that

(5.47)
$$\left|\int_{E_3}[f(t,x(t),x'(t))-f(t,u(t),u'(t))dt\right| \le L_0 32\cdot 96 d(I(x)+\int_a^b \psi_0(t)dt)$$
$$\le 32\cdot 96 L_0 d(M_1+1+\int_a^b \psi_0(t)dt).$$

In view of (5.4), (5.41), (5.46), and (5.47)

$$(5.48)\qquad I(u)-I(x)\le(-3/4)\int_{E_1} f(t,x(t),x'(t))dt+32\cdot 96L_1 d(1+b-a)$$
$$+32\cdot 96 L_0 d(M_1+1+\int_a^b \psi_0(t)dt).$$

It follows from (B1), (1.7), (5.14) and (5.21) that for all $t\in E_1$
$$f(t,x(t),x'(t))\ge \phi(||x'(t)||)\ge \gamma_1^{-1}||x'(t)||.$$

Combined with (5.23) this inequality implies that

$$(5.49)\qquad \int_{E_1} f(t,x(t),x'(t))dt \ge \gamma_1^{-1}\int_{E_1}||x'(t)||dt=\gamma_1^{-1}d.$$

By (5.12), (5.48), and (5.49)

$$I(u)-I(x)\le -2^{-1}\gamma_1^{-1}d+32\cdot 96L_1 d(1+b-a)+32\cdot 96L_0 d(M_1+1+\int_a^b \psi_0(t)dt)$$
$$d(-\gamma_1^{-1}/2+32\cdot 96L_1(1+b-a)+32\cdot 96L_0(M_1+1+\int_a^b \psi_0(t)dt))<0.$$

□

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