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RELATIONSHIPS BETWEEN APPROXIMATE JACOBIANS AND CODERIVATIVES

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Abstract. Relationships between the concept of approximate Jacobian for vector-valued functions in finite-dimensional Euclidean spaces, which was introduced by V. Jeyakumar and D. T. Luc, and the concept of coderivative, which was introduced by B. Mordukhovich, are discussed in this paper. Our investigation shows clearly that coderivative and approximate Jacobian are very different concepts. They have a little in common. From the papers cited in the list of references one can note that these concepts require different methods of study, and they give results of quite different forms.

1. Introduction

The role of *set-valued derivatives* of functions and multifunctions has been recognized widely in the literature (see [1] and [20]).

Coderivative in the sense of Mordukhovich (see [13] and [20]) is one type of set-valued derivatives. As shown by Mordukhovich and other authors, it is very useful for the development of nonsmooth analysis and its applications. Coderivatives allow one to characterize the openness, metric regularity, and Lipschitzian properties of functions and multifunctions (see [12]). For the applications of coderivatives in stability and sensitivity analysis of optimization problems and variational systems we refer to [14]–[16]. In order to define coderivative, one uses the (nonconvex) normal cone in the sense of Mordukhovich [10]. Basic definitions and calculus rules concerning coderivatives in finite-dimensional Euclidean spaces can be found in [13]. An infinite-dimensional version of the coderivative theory and its applications was given in [17, 18].

The concept of approximate Jacobian and the corresponding notion of generalized subdifferential were introduced by Jeyakumar and Luc in [3] and [4]. Using this concept one can obtain new types of open mapping theorems [5, 6], Lagrange multiplier rules [22], and sufficient conditions for the metric regularity and for the Aubin property of implicit multifunctions [7] which are applicable for continuous, non-Lipschitzian systems.

It is of interest to study the relationships between the concept of coderivative and the concept of approximate Jacobian. Note that some remarks on the relationships between the Mordukhovich subdifferential and the Jeyakumar-Luc (J-L, for

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brevity) generalized subdifferential have been given in [21] and [22]. The aim of this paper is to study the question in a broader context. After giving some preliminaries in Section 2, in Section 3 we discuss the relationships between the Mordukhovich subdifferential and the J-L generalized subdifferential. Then, in Section 4, we investigate the relationships between coderivatives and approximate Jacobians. Some examples, which help us to compare the concept of coderivative with the concept of approximate Jacobian, are given in Section 5.

For an Euclidean space Z, the symbol $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ and B_Z denote, respectively, the norm, the inner product and the closed unit ball in Z. The closed ball with center a and radius δ is denoted by $B(a,\delta)$. For a subset $M \subset Z$, we denote by $\text{int}M$, $\overline{M}$, $\text{co}M$, and $\text{cone}M$ the interior, the closure, the convex hull and the cone generated by M, respectively. For simplicity of notation, the closure of the last two sets are denoted, respectively, by $\overline{\text{co}}M$ and $\overline{\text{cone}}M$. The negative dual cone of M is denoted by M^*, that is $M^* = \{w \in Z : \langle w, z\rangle \le 0\ \forall z \in M\}$. The distance from $a \in Z$ to $M \subset Z$ is denoted by $d(a, M)$. By convention, $d(a,\emptyset) = +\infty$. If A is a linear operator then A^* stands for the conjugate of A. The space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ (which is identified with the set of $(m\times n)-$matrices) is denoted by $L(\mathbb{R}^n,\mathbb{R}^m)$.

2. Definitions and preliminaries

We first recall some facts from [2] and [13] which will be needed in the sequel.

For a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, let

$$\text{gph}F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ :\ y \in F(x)\}.$$

The Kuratowski-Painlevé upper limit of F as $x \to \overline{x}$ is a subset of $\mathbb{R}^m$ defined by setting

$$\limsup_{x\to\overline{x}} F(x) = \{y \in \mathbb{R}^m \ :\ \exists \text{ sequences } x^k \to \overline{x},\ y_k \to y, \text{ with } y_k \in F(x^k)\ \forall k = 1, 2, \ldots\}.$$

Let $\Omega \subset \mathbb{R}^n$. Denote

$$P(x,\Omega) = \{\omega \in \overline{\Omega}\ :\ \|x - \omega\| = d(x,\Omega)\}.$$

The *Mordukhovich normal cone* to Ω at $\overline{x} \in \overline{\Omega}$ is defined by the formula

$$N(\overline{x},\Omega) = \limsup_{x\to\overline{x}}[\text{cone}(x - P(x,\Omega))]. \tag{2.1}$$

If $\overline{x} \notin \overline{\Omega}$, then one puts $N(\overline{x},\Omega) = \emptyset$. In general, $N(\overline{x},\Omega)$ is a nonconvex cone. So it is not a dual of any tangent object.

The *Clarke tangent cone* $T_C(\overline{x},\Omega)$ to Ω at $\overline{x} \in \overline{\Omega}$ is defined by the formula

$$T_C(\overline{x},\Omega) = \{u \in \mathbb{R}^n \ :\ \forall x^k(\in\Omega) \to \overline{x},\ \ \forall t_k \downarrow 0,\ \exists u_k \to u \text{ such that } x^k + t_k u_k \in \Omega \text{ for all } k\}.$$

The set $N_C(\overline{x},\Omega) := (T_C(\overline{x},\Omega))^*$ is called the *Clarke normal cone* to Ω at $\overline{x}$. Relation between the Clarke normal cone and the Mordukhovich normal cone (see [2, Prop. 2.5.7]) is as follows

$$N_C(\overline{x},\Omega) = \overline{\text{co}}N(\overline{x},\Omega). \tag{2.2}$$

The *Bouligand (contingent) tangent cone* to Ω at $x \in \overline{\Omega}$ is defined by

$$\widehat{T}(x,\Omega) = \{u \in \mathbb{R}^n \ : \ \exists u_k \to u,\ \exists t_k \downarrow 0 \ \text{ such that } \ x + t_k u_k \in \Omega \ \text{ for all } k\}.$$

The negative dual cone to $\widehat{T}(x,\Omega)$ is denoted by $\widehat{N}(x,\Omega)$. If $x \notin \overline{\Omega}$ then one puts $\widehat{N}(x,\Omega) = \emptyset$. It is well known (see [p. 254]13) that

$$\widehat{N}(x,\Omega) = \big\{x^* \in \mathbb{R}^n \ : \ \limsup_{y(\in\Omega)\to x} \frac{\langle x^*, y - x\rangle}{\|y - x\|} \le 0\big\},$$

where $y(\in \Omega) \to x$ means $y \to x$ and $y \in \Omega$.

Proposition 2.1 ([8])**.** *For any $\Omega \subset \mathbb{R}^n$, any $\overline{x} \in \overline{\Omega}$, one has*

$$N(\overline{x},\Omega) = \limsup_{x\to\overline{x}} \widehat{N}(x,\Omega). \tag{2.3}$$

Proposition 2.2. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $(\overline{x},\overline{y}) \in \overline{gphF}$. The multifunction $D^*F(\overline{x},\overline{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ defined by*

$$D^*F(\overline{x},\overline{y})(y^*) = \{x^* \in \mathbb{R}^n \ : \ (x^*, -y^*) \in N((\overline{x},\overline{y}), gphF)\}$$

is called the coderivative of F at $(\overline{x},\overline{y})$. By convention, $D^*F(\overline{x},\overline{y})(y^*) = \emptyset$ for all $(\overline{x},\overline{y}) \notin \overline{gphF}$ and $y^* \in \mathbb{R}^m$. When F is single-valued, one writes $D^*F(\overline{x})$ instead of $D^*F(\overline{x},\overline{y})$, where $\overline{y} = F(\overline{x})$. The corresponding Clarke coderivative is

$$D^*_C F(\overline{x},\overline{y})(y^*) = \{x^* \in \mathbb{R}^n \ : \ (x^*, -y^*) \in N_C((\overline{x},\overline{y}), gphF)\}.$$

The graph of $D^*_C F(\overline{x},\overline{y})(\cdot)$ is a closed convex cone in the product space $\mathbb{R}^m \times \mathbb{R}^n$. If F has convex graph then the Clarke coderivative and the Mordukhovich coderivative coincide, i.e.,

$$D^*F(\overline{x},\overline{y})(y^*) = D^*_C F(\overline{x},\overline{y})(y^*) \quad \text{for all } \ (\overline{x},\overline{y}) \in \text{gph}F,\ y^* \in \mathbb{R}^m.$$

If F is a strictly differentiable vector-valued function, then the two coderivatives also coincide. Namely, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is strictly differentiable at $\overline{x}$, then

$$D^*f(\overline{x})(y^*) = D^*_C f(\overline{x})(y^*) \quad \text{for all } \ y^* \in \mathbb{R}^m.$$

Recall that $f : \mathbb{R}^m \to \mathbb{R}^m$ is said to be *strictly differentiable* at $\overline{x}$ if there exists $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{x\to\overline{x},\ t\downarrow 0} \frac{f(x+tu) - f(x) - tAu}{t} = 0 \quad \forall u \in \mathbb{R}^n,$$

provided that the convergence is uniform for u in compact sets. Except for the just described two situations, the graph of the Mordukhovich coderivative is often smaller than the graph of the Clarke coderivative.

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let

$$\text{dom}\varphi = \{x \in \mathbb{R}^n \ : \ -\infty < \varphi(x) < +\infty\}.$$

The formula

$$F(x) = E_\varphi(x) := \{\mu \in \mathbb{R} \ : \ \mu \ge \varphi(x)\}$$

defines the *epigraphical multifunction* of φ. Clearly,

$$\text{gph}F = \text{epi}\varphi := \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \ : \ \mu \ge \varphi(x)\}.$$

Definition 2.3. Let $\overline{x} \in \text{dom}\varphi$. The set

$$\begin{aligned}\partial_M\varphi(\overline{x}) &:= D^*E_\varphi(\overline{x},\varphi(\overline{x}))(1)\\ &= \{x^* \in \mathbb{R}^n \;:\; (x^*,-1) \in N((\overline{x},\varphi(\overline{x})),\text{epi}\varphi)\}\end{aligned}$$

is called the *Mordukhovich subdifferential* of φ at $\overline{x}$, and the set

$$\begin{aligned}\partial_M^\infty\varphi(\overline{x}) &:= D^*E_\varphi(\overline{x},\varphi(\overline{x}))(0)\\ &= \{x^* \in \mathbb{R}^n \;:\; (x^*,0) \in N((\overline{x},\varphi(\overline{x})),\text{epi}\varphi)\}\end{aligned}$$

is called the *Mordukhovich singular subdifferential* of φ at $\overline{x}$. If $\overline{x} \notin \text{dom}\varphi$ then we put $\partial_M\varphi(\overline{x}) = \partial_M^\infty\varphi(\overline{x}) = \emptyset$. The Clarke subdifferential $\partial_C\varphi(\overline{x})$ and the Clarke singular subdifferential $\partial_C^\infty\varphi(\overline{x})$ are defined similarly; instead of D^* (resp. $N(\cdot)$) one considers D_C^* (resp. $N_C(\cdot)$).

If φ is strictly differentiable at $\overline{x}$, then $\partial_C\varphi(\overline{x}) = \partial_M\varphi(\overline{x}) = \{\nabla\partial\varphi(\overline{x})\}$. For any lower semicontinuous function φ and for any $\overline{x} \in \text{dom}\varphi$, from (2.2) it follows that

$$\partial_C\varphi(\overline{x}) = \overline{\text{co}}[\partial_M\varphi(\overline{x}) + \partial_M^\infty\varphi(\overline{x})]. \tag{2.4}$$

Under some mild conditions, the Clarke subdifferential $\partial_C\varphi(\overline{x})$ can be computed via the Clarke-Rockafellar directional derivatives. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then the Clarke-Rockafellar directional derivative $\varphi^\uparrow(\overline{x},u)$ of φ at $\overline{x}$ in direction u is defined [2, p. 97] by setting

$$\varphi^\uparrow(\overline{x},u) = \lim_{\varepsilon\downarrow 0}\ \limsup_{x\to\overline{x},\ t\downarrow 0}\ \inf_{u'\in u+\varepsilon B_{\mathbb{R}^n}} \frac{\varphi(x+tu')-\varphi(x)}{t}. \tag{2.5}$$

Proposition 2.4 (See [2, p. 97])**.** *One has $\partial_C\varphi(\overline{x}) = \emptyset$ if and only if $\varphi^\uparrow(\overline{x},0) = -\infty$. Otherwise, one has*

$$\partial_C\varphi(\overline{x}) = \{x^* \in \mathbb{R}^n \;:\; \varphi^\uparrow(\overline{x},u) \geq \langle x^*,u\rangle \;\; \forall u \in \mathbb{R}^n\} \tag{2.6}$$

and

$$\varphi^\uparrow(\overline{x},u) = \sup\{\langle x^*,u\rangle \;:\; x^* \in \partial_C\varphi(\overline{x})\} \quad \forall u \in \mathbb{R}^n. \tag{2.7}$$

If φ is locally Lipschitz at $\overline{x}$, then

$$\varphi^\uparrow(\overline{x},u) = \varphi^o(\overline{x},u)$$

for every $u \in \mathbb{R}^n$, where

$$\varphi^o(\overline{x},u) = \limsup_{x\to\overline{x},\ t\downarrow 0} \frac{\varphi(x+tu)-\varphi(x)}{t}$$

is the generalized directional derivative of φ at $\overline{x}$ in direction u in the sense of Clarke [2]. Since $\partial_C^\infty\varphi(\overline{x}) = \{0\}$ (see [2, Proposition 2.9.7]) and $\partial_M^\infty\varphi(\overline{x}) \subset \partial_C^\infty\varphi(\overline{x})$, we deduce that $\partial_M^\infty\varphi(\overline{x}) = \{0\}$.

We now recall the concept of approximate Jacobian and the corresponding notion of generalized subdifferential introduced by Jeyakumar and Luc (see [3, 4]).

Definition 2.5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous vector-valued function. A closed subset $Jf(\overline{x}) \subset L(\mathbb{R}^n,\mathbb{R}^m)$ is called an *approximate Jacobian* of f at $\overline{x} \in \mathbb{R}^n$ if

$$(y^*\circ f)^+(\overline{x},u) \leq \sup_{A\in Jf(\overline{x})} \langle y^*,Au\rangle, \quad \forall u \in \mathbb{R}^n,\ \forall y^* \in \mathbb{R}^m, \tag{2.8}$$

where $(y^* \circ f)(x) = \langle y^*, f(x)\rangle$ is the composite function of y^* and f, and

$$(y^* \circ f)^+(\overline{x}, u) = \limsup_{t\downarrow 0} \frac{(y^* \circ f)(\overline{x} + tu) - (y^* \circ f)(\overline{x})}{t}$$

is the *upper Dini directional derivative* of $y^* \circ f$ at $\overline{x}$ in direction u. An approximate Jacobian of f at $\overline{x}$ is said to be *minimal* if it contains no proper (closed) subset which is also an approximate Jacobian of f at $\overline{x}$.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at $\overline{x}$ with the Fréchet derivative $f'(\overline{x})$, then $Jf(\overline{x}) = \{f'(\overline{x})\}$ is an approximate Jacobian of f at $\overline{x}$.

Definition 2.6. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. If $J\varphi(\overline{x})$ is an approximate Jacobian of φ at $\overline{x}$ then one writes $\partial\varphi(\overline{x})$ for $J\varphi(\overline{x})$ and calls $\partial\varphi(\overline{x})$ a *J-L subdifferential* of φ at $\overline{x}$. A J-L subdifferential of φ at $\overline{x}$ is said to be *minimal* if it contains no proper (closed) subset which is also a J-L subdifferential of f at $\overline{x}$.

Note that the function φ considered in Example 3.1 below does not have any minimal J-L subdifferential at $\overline{x} = 0$.

If $f = \varphi$, a real-valued function, then (2.8) is equivalent to the following pair of conditions:

$$\limsup_{t\downarrow 0} \frac{\varphi(\overline{x} + tu) - \varphi(\overline{x})}{t} \le \sup_{x^* \in \partial\varphi(\overline{x})} \langle x^*, u\rangle \quad \forall u \in \mathbb{R}^n \tag{2.9}$$

and

$$\liminf_{t\downarrow 0} \frac{\varphi(\overline{x} + tu) - \varphi(\overline{x})}{t} \ge \inf_{x^* \in \partial\varphi(\overline{x})} \langle x^*, u\rangle \quad \forall u \in \mathbb{R}^n. \tag{2.10}$$

In the next section we will deal with Mordukhovich and J-L subdifferentials of real-valued functions. Coderivatives and approximate Jacobians of vector-valued functions will be studied in Sections 4 and 5.

3. Mordukhovich subdifferentials and J-L subdifferentials

In this section we study the following question: *Is any Mordukhovich subdifferential a J-L subdifferential?*

Let us begin with a well-known example (see [5]).

Example 3.1. Let $\varphi(x) = x^{1/3}$, $x \in \mathbb{R}$. Then $\partial\varphi(0) = [\alpha, +\infty)$, where $\alpha \in \mathbb{R}$ is an arbitrarily chosen number, is a J-L subdifferential of φ at 0. Indeed, substituting $\overline{x} = 0$, $u = 1$ and $u = -1$ into (2.9) and (2.10) we see that both conditions are satisfied. Using (2.3) we get

$$N((0,0), \mathrm{epi}\varphi) = \{(x^*, 0) \in \mathbb{R}^2 \;:\; x^* \ge 0\}.$$

Therefore $\partial_M\varphi(0) = \emptyset$ and $\partial_M^\infty\varphi(0) = [0, +\infty)$. So $\partial_M\varphi(0)$ is not a J-L subdifferential of φ at 0.

This example shows that the above question should be formulated as follows:

Question 1: *If a Mordukhovich subdifferential is nonempty, is it a J-L subdifferential?*

The next three examples and Proposition 3.5 below are in favor of a positive answer to Question 1.

Example 3.2. Let $\varphi(x) = |x|$ for all $x \in \mathbb{R}$. Note that φ is a convex, Lipschitzian function on R. Using (2.1) or (2.3) we obtain $N((0,0), \text{epi}\varphi) = \{(x^*, y^*) \in \mathbb{R}^2 \ : \ |x^*| \le -y^*\}$. Hence $\partial_M \varphi(0) = [-1, 1]$. Since $\partial\varphi(0) := \{-1, 1\}$ is a J-L subdifferential of φ at 0, we conclude that $\partial_M \varphi(0)$ is a J-L subdifferential of φ at 0, but it is not a minimal subdifferential. (Note that $\partial\varphi(0) = \{-1, 1\}$ is a minimal J-L subdifferential of φ at 0).

Example 3.3. Let $\varphi(x) = -|x|$ for all $x \in \mathbb{R}$. Note that φ is a concave, Lipschitzian function on $\mathbb{R}$. Using (2.1) or (2.3) we have

$$N((0,0), \text{epi}\varphi) = \{(x^*, y^*) \in \mathbb{R}^2 \ : \ |x^*| = |y^*|\} \cup \{(x^*, y^*) \in \mathbb{R}^2 \ : \ x^* \ge |y^*|\}.$$

Hence $\partial_M \varphi(0) = \{-1, 1\}$. It is easily verified that $\partial\varphi(0) := \{-1, 1\}$ is a minimal J-L subdifferential of φ at 0. Thus the Mordukhovich subdifferential of φ at 0 is a minimal J-L subdifferential of φ at 0.

Example 3.4. Let $\varphi(x) = 0$ for $x \in (-\infty, 0]$ and $\varphi(x) = x^{1/2}$ for $x \in (0, +\infty)$. Note that φ is a nonconvex, nonconcave, non-Lipschitzian function on R. Using (2.3) we can show that

$$N((0,0), \text{epi}\varphi) = \{(x^*, y^*) \in \mathbb{R}^2 \ : \ x^* \ge 0, \ y^* \le 0\}.$$

So $\partial_M \varphi(0) = \partial_M^\infty \varphi(0) = [0, +\infty)$. Direct verification shows that conditions (2.9) and (2.10) are satisfied with $\partial\varphi(0) := [0, +\infty)$. Thus $\partial_M \varphi(0)$ is a J-L subdifferential of φ at 0. It is easy to see that this J-L subdifferential is not minimal.

Proposition 3.5. *If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $\overline{x}$, then $\partial_M \varphi(\overline{x})$ is a J-L subdifferential of φ at $\overline{x}$.*

Proof. By (2.4), (2.6) and (2.7) we have

$$\begin{aligned}\limsup_{t \downarrow 0} \frac{\varphi(\overline{x} + tu) - \varphi(\overline{x})}{t} &\le \varphi^o(\overline{x}, u) \\ &= \max\{\langle x^*, u\rangle \ : \ x^* \in \partial_C \varphi(\overline{x})\} \\ &= \max\{\langle x^*, u\rangle \ : \ x^* \in \partial_M \varphi(\overline{x})\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\liminf_{t \downarrow 0} \frac{\varphi(\overline{x} + tu) - \varphi(\overline{x})}{t} &\ge \liminf_{x \to \overline{x}, \ t \downarrow 0} \frac{\varphi(x + tu) - \varphi(x)}{t} \\ &= -\varphi^o(\overline{x}, -u) \\ &= -\max\{\langle x^*, -u\rangle \ : \ x^* \in \partial_C \varphi(\overline{x})\} \\ &= \min\{\langle x^*, u\rangle \ : \ x^* \in \partial_M \varphi(\overline{x})\}.\end{aligned}$$

The properties (2.9) and (2.10) have been established for $\partial\varphi(\overline{x}) := \partial_M \varphi(\overline{x})$. So $\partial_M \varphi(\overline{x})$ is a J-L subdifferential of φ at $\overline{x}$. □

The next example gives a negative answer for Question 1.

Example 3.6. Let $\varphi(x) = x^2 \sin(1/x)$ for $x \in (-\infty, 0)$ and $\varphi(x) = -x^{1/3}$ for $x \in [0, +\infty)$. Then φ is a continuous function which is not locally Lipschitz at 0. We claim that $\partial_M \varphi(0) \neq \emptyset$, but it is not a J-L subdifferential of φ at 0. Indeed, it is easy to see that $\widehat{N}((0,0), \text{epi}\varphi) = \{0\}$. Since

$$\text{epi}\varphi = \{(x, y) \ : \ y - x^2 \sin(1/x) \ge 0, \ x < 0\} \cup \{(x, y) \ : \ y + x^{1/3} \ge 0, \ x \ge 0\},$$

by the formula for computing the Bouligand tangent cone for inequality systems defined by differentiable functions (see, for instance, [1, p. 124]) we have

$$\widehat{T}((x,\varphi(x)),\text{epi}\varphi) = \{(v_1,v_2)\in\mathbb{R}^2 \ : \ (-2x\sin(1/x)+\cos(1/x))v_1+v_2\geq 0\} \quad \text{if } x<0,$$

$$\widehat{T}((x,\varphi(x)),\text{epi}\varphi) = \{(v_1,v_2)\in\mathbb{R}^2 \ : \ \frac{1}{3}x^{-2/3}v_1+v_2\geq 0\} \quad \text{if } x>0.$$

Therefore

$$\widehat{N}((x,\varphi(x)),\text{epi}\varphi) = \{\lambda^*(2x\sin(1/x)-\cos(1/x),-1) \ : \ \lambda^*\geq 0\} \quad \text{if } x<0,$$

$$\widehat{N}((x,\varphi(x)),\text{epi}\varphi) = \{\lambda^*(-\frac{1}{3}x^{-2/3},-1) \ : \ \lambda^*\geq 0\} \quad \text{if } x>0.$$

Applying (2.3) we deduce that

$$N((0,0),\text{epi}\varphi) = \{(x^*,y^*)\in\mathbb{R}^2 \ : \ -|x^*|\geq y^*\}\cup(-\infty,0]\times\{0\}.$$

Then $\partial_M\varphi(0)=[-1,1]$. Note that (2.10), where $\overline{x}:=0$, fails to hold for $u=1$ because the left-hand-side is $-\infty$, while the right-hand-side is -1. Thus $\partial_M\varphi(0)$ is a nonempty convex compact set which is *not* a J-L subdifferential of φ at 0.

It is worthy observing that, in Examples 3.1 and 3.6, the set $\partial\varphi(0):=\partial_M\varphi(0)\cup\partial_M^\infty\varphi(0)$ is a J-L subdifferential of φ at 0 (despite to the fact that $\partial_M\varphi(0)$ does not have the property). One may wish to know *whether it is true that for any continuous function $\varphi:\mathbb{R}^n\to\mathbb{R}$ and for any $\overline{x}\in\mathbb{R}^n$, the union of the Mordukhovich subdifferential and the Mordukhovich singular subdifferential*

$$\partial\varphi(\overline{x}):=\partial_M\varphi(\overline{x})\cup\partial_M^\infty\varphi(\overline{x})$$

is a J-L subdifferential of φ at $\overline{x}$? We leave this question as unresolved.

4. Coderivatives and approximate Jacobians

Coderivatives are homogeneous multifunctions. But for the coderivative $D^*f(\overline{x})(\cdot)$ of a continuous vector-valued function $f:\mathbb{R}^n\to\mathbb{R}^m$ at $\overline{x}\in\mathbb{R}^n$ it may happen that there does not exist any closed subset $\Delta\subset L(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$D^*f(\overline{x})(y^*)=\{A^*y^* \ : \ A\in\Delta\}. \tag{4.1}$$

So it is impossible to compare the concept of coderivative with the concept of approximate Jacobian. To overcome this difficulty, we introduce the following definition.

Definition 4.1. A nonempty closed set $\Delta\subset L(\mathbb{R}^n,\mathbb{R}^m)$ of linear operators is said to be a *representative* of the coderivative mapping $D^*f(\overline{x})(\cdot)$ if

$$\sup_{x^*\in D^*f(\overline{x})(y^*)}\langle x^*,u\rangle=\sup_{A\in\Delta}\langle A^*y^*,u\rangle \quad \forall u\in\mathbb{R}^n,\ \forall y^*\in\mathbb{R}^m. \tag{4.2}$$

From the separation theorem it follows that (4.2) is equivalent to the condition

$$\overline{\text{co}}D^*f(\overline{x})(y^*)=\overline{\text{co}}\{A^*y^* \ : \ A\in\Delta\} \quad \forall y^*\in\mathbb{R}^m. \tag{4.3}$$

If f is strictly differentiable at $\overline{x}$, then $\Delta:=\{f'(\overline{x})\}$ is a representative of the coderivative mapping $D^*f(\overline{x})(\cdot)$.

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at $\overline{x}$, i.e., there exists $\ell > 0$ such that $\|f(x') - f(x)\| \le \ell \|x' - x\|$ for all x, x' in a neighborhood of $\overline{x}$, then the compact set

$$J_B f(\overline{x}) = \{\lim_{k\to\infty} f'(x^k) : \{x^k\} \subset \Omega_f, x^k \to \overline{x}\},$$

called the $B-$*derivative*, is an approximate Jacobian of f at $\overline{x}$. Here

$$\Omega_f = \{x \in \mathbb{R}^n : \exists \text{the Fréchet derivative } f'(x) \text{ of } f \text{ at } x\}.$$

Note that the larger set

$$J_C f(\overline{x}) := \text{co}\{\lim_{k\to\infty} f'(x^k) : \{x^k\} \subset \Omega_f, x^k \to \overline{x}\},$$

which is the *Clarke generalized Jacobian* of f at $\overline{x}$, is also an approximate Jacobian of f at $\overline{x}$. In the case $m = 1$, one has $J_C f(\overline{x}) = \partial_C f(\overline{x})$ (see [2]).

Proposition 4.2. *If a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz at $\overline{x}$, then the set $\Delta := J_B f(\overline{x})$ is a representative of the coderivative mapping $D^* f(\overline{x})(\cdot)$.*

Proof. According to [13, formula (2.23)], we have

$$\{A^* y^* \ : \ A \in J_C f(\overline{x})\} = \text{co} D^* f(\overline{x})(y^*) \quad \forall y^* \in \mathbb{R}^m.$$

Since $J_C f(\overline{x}) = \text{co} J_B f(\overline{x})$, it follows that

$$\text{co} D^* f(\overline{x})(y^*) = \{A^* y^* \ : \ A \in \text{co} J_B f(\overline{x})\}.$$

Hence (4.3) is valid if we choose $J f(\overline{x}) = J_B f(\overline{x})$. This shows that $\Delta = J_B f(\overline{x})$ is a representative of the coderivative mapping $D^* f(\overline{x})(\cdot)$. □

Proposition 4.3. *If f is locally Lipschitz at $\overline{x}$ and if Δ is a representative of the coderivative mapping $D^* f(\overline{x})(\cdot)$, then $J f(\overline{x}) := \Delta$ is an approximate Jacobian of f at $\overline{x}$.*

Proof. Let $y^* \in \mathbb{R}^m$ be given arbitrarily. According to [[13], Proposition 2.11], we have

$$D^* f(\overline{x})(y^*) = \partial_M (y^* \circ f)(\overline{x}). \tag{4.4}$$

Since $y^* \circ f$ is locally Lipschitz at $\overline{x}$, it holds

$$(y^* \circ f)^o(\overline{x}, u) = \sup\{\langle x^*, u\rangle \ : \ x^* \in \partial_C (y^* \circ f)(\overline{x})\} \quad \forall u \in \mathbb{R}^n.$$

Combining this with (2.4) and (4.4) gives

$$(y^* \circ f)^o(x, u) = \sup\{\langle x^*, u\rangle \ : \ x^* \in D^* f(\overline{x})(y^*)\} = \sup\{\langle A^* y^*, u\rangle \ : \ A \in \Delta\}.$$

Therefore

$$(y^* \circ f)^+(\overline{x}, u) \le (y^* \circ f)^o(x, u) = \sup\{\langle y^*, Au\rangle \ : \ A \in \Delta\}.$$

Since this holds for every $y^* \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$, we conclude that $J f(\overline{x}) = \Delta$ is an approximate Jacobian of f at $\overline{x}$. □

In connection with Proposition 4.3 it is natural to raise the following question.
Question 2: *Is it true that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous vector-valued function and Δ is a representative for the coderivative mapping $D^* f(\overline{x})(\cdot) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, then $J f(\overline{x}) := \Delta$ is an approximate Jacobian of f at $\overline{x}$?*

Combining the next proposition with Proposition 4.3 we get an *affirmative answer* for Question 2.

Proposition 4.4. *If the coderivative mapping $D^*f(\overline{x})(\cdot):\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of a continuous function $f:\mathbb{R}^n\to\mathbb{R}^m$ admits a representative $Jf(\overline{x})\subset L(\mathbb{R}^n,\mathbb{R}^m)$, then f is locally Lipschitz at $\overline{x}$.*

Proof. From (4.3) it follows that $\overline{\mathrm{co}}D^*f(\overline{x})(0)=\{0\}$. Hence $D^*f(\overline{x})(0)=\{0\}$. By [11, Proposition 2.8], this implies that the multifunction $x\mapsto\{f(x)\}$ is pseudo-Lipschitz around $(\overline{x},f(\overline{x}))$. Since f is a single-valued mapping, f is locally Lipschitz at $\overline{x}$. □

5. More examples

Let us consider some more examples where we compute the Mordukhovich subdifferentials and coderivatives of nonsmooth functions.

Example 5.1. Let $f:\mathbb{R}\to\mathbb{R}^2$ be defined by the formula $f(x)=(|x|^{1/2},-|x|)$ for all $x\in\mathbb{R}$. Then f is a continuous function which is not locally Lipschitz at 0, and $\mathrm{gph}\,f=\{(x,|x|^{1/2},-|x|)\ :\ x\in\mathbb{R}\}$. Using (2.3) and the formula for the normal cone $\widehat{N}(x,\Omega)$ recalled in Section 2, we can show that

$$N((0,0,0),\mathrm{gph}f)=\widehat{N}((0,0,0),\mathrm{gph}f)=\mathbb{R}\times(-\infty,0]\times\mathbb{R}.$$

Hence, for every $y^*=(y_1^*,y_2^*)\in\mathbb{R}^2$,

$$D^*f(0)(y^*)=\begin{cases}\mathbb{R} & \text{if } y_1^*\ge 0,\\ \emptyset & \text{if } y_1^*<0.\end{cases}$$

Since f is not locally Lipschitz at $\overline{x}=0$, Proposition 4.4 shows that the coderivative mapping $D^*f(0)(\cdot)$ has no representative in the form of a set of linear operators. A direct calculation shows that, for every $y^*=(y_1^*,y_2^*)\in\mathbb{R}^2$ and $u\in\mathbb{R}$, it holds

$$(y^*\circ f)^+(0,u)=\begin{cases}+\infty & \text{if } y_1^*>0,\ u\ne 0\\ -|u|y_2^* & \text{if } y_1^*=0\\ -\infty & \text{if } y_1^*<0,\ u\ne 0\\ 0 & \text{if } y_1^*<0,\ u=0.\end{cases}$$

If we choose $Jf(0)=(-\infty,0]\times\mathbb{R}$, $\overline{x}=0$, and let $Au=(\alpha u,\beta u)$ for every $A=(\alpha,\beta)\in Jf(0)$, $u\in\mathbb{R}$, then (2.8) is not fulfilled because $\sup_{A\in Jf(0)}\langle y^*,Au\rangle=0$ if $y_1^*>0,\ u>0,\ y_2^*=0$, while $(y^*\circ f)^+(0,u)=+\infty$. Similarly, if we chose $Jf(0)=[0,+\infty)\times\mathbb{R}$ and $\overline{x}=0$, then (2.8) does not hold because $\sup_{A\in Jf(0)}\langle y^*,Au\rangle=0$ if $y_1^*>0,\ u<0,\ y_2^*=0$, while $(y^*\circ f)^+(0,u)=+\infty$. Thus, the chosen sets $Jf(0)$ are not approximate Jacobians for f at 0. However, a set like $Jf(0):=\big\{(-\infty,-1]\cup[2,+\infty)\big\}\times\mathbb{R}$ is an approximate Jacobian of f at 0.

Example 5.2. Let $f:\mathbb{R}\to\mathbb{R}^2$ be given by the formula $f(x)=(-|x|^{1/3},x^{1/3})$ for all $x\in\mathbb{R}$. Then f is a continuous function which is not locally Lipschitz at 0, and $\mathrm{gph}\,f=\{(x,-|x|^{1/3},x^{1/3})\,:\,x\in\mathbb{R}\}$. Applying (2.3) and the formula for the normal cone $\widehat{N}(x,\Omega)$ recalled in Section 2, we can show that

$$N((0,0,0),\mathrm{gph}f)=\widehat{N}((0,0,0),\mathrm{gph}f)=\mathbb{R}\times W,$$

where $W = \{y^* = (y_1^*, y_2^*) \in \mathbb{R}^2 \,:\, -y_1^* \le y_2^* \le y_1^*\}$. Therefore, for every $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$,

$$D^*f(0)(y^*) = \begin{cases} \mathbb{R} & \text{if } y_1^* \le y_2^* \le -y_1^* \\ \emptyset & \text{otherwise.} \end{cases}$$

The coderivative mapping $D^*f(0)(\cdot)$ has no representative in the form of a set of linear operators. It is not difficult to show that, for every $y^* = (y_1^*, y_2^*) \in \mathbb{R}^2$ and $u \in \mathbb{R}$, it holds

$$(y^* \circ f)^+(0,u) = \begin{cases} 0 & \text{if } u = 0 \\ 0 & \text{if } y_2^* = y_1^* = 0,\ u \ne 0 \\ 0 & \text{if } y_2^* - y_1^* = 0,\ u > 0 \\ +\infty & \text{if } y_2^* - y_1^* > 0,\ u > 0 \\ -\infty & \text{if } y_2^* - y_1^* < 0,\ u > 0 \\ 0 & \text{if } y_2^* + y_1^* = 0,\ u < 0 \\ -\infty & \text{if } y_2^* + y_1^* > 0,\ u < 0 \\ +\infty & \text{if } y_2^* + y_1^* < 0,\ u < 0. \end{cases}$$

A direct verification using (2.8) shows that the set

$$Jf(0) = \{(\alpha, -\alpha) \,:\, \alpha \le 0\} \cup \{(\alpha, \alpha) \,:\, \alpha \ge 0\}$$

is an approximate Jacobian of f at 0 if we embed $Jf(0)$ into $L(\mathbb{R}, \mathbb{R}^2)$ by setting $Au = (\alpha u, \beta u)$ for any $A = (\alpha, \beta) \in Jf(0)$ and $u \in \mathbb{R}$.

Example 5.3 (See also [11, p. 65])**.** Let $f(x) = |x_1| - |x_2|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $\overline{x} = (0,0)$. This function is neither convex, nor concave. It is not subdifferentially regular (see [13]) at $\overline{x} = (0,0)$. In order to compute the coderivative mapping $D^*f(\overline{x})(\cdot) : \mathbb{R} \rightrightarrows \mathbb{R}^2$ we have to define the normal cone $N(\overline{x}, \mathrm{gph} f)$. Note that

$$\begin{aligned} \mathrm{gph} f &= \{(x_1, x_2, t) \,:\, t = f(x_1, x_2)\} \\ &= \{(x_1, x_2, t) \,:\, x_1 \ge 0,\ x_2 \ge 0,\ t = x_1 - x_2\} \\ &\quad \cup \{(x_1, x_2, t) \,:\, x_1 \ge 0,\ x_2 \le 0,\ t = x_1 + x_2\} \\ &\quad \cup \{(x_1, x_2, t) \,:\, x_1 \le 0,\ x_2 \le 0,\ t = -x_1 + x_2\} \\ &\quad \cup \{(x_1, x_2, t) \,:\, x_1 \le 0,\ x_2 \ge 0,\ t = -x_1 - x_2\}. \end{aligned}$$

Denote the four polyhedral convex sets in the last union by Γ_1, Γ_2, Γ_3, and Γ_4, respectively. Let $z = (x_1, x_2, t) \in \mathrm{gph} f$.

If z belongs to the relative interior of Γ_1 (resp., Γ_2, Γ_3, and Γ_4), then $\widehat{N}(z, \mathrm{gph} f) = \{\lambda(1,-1,-1) \,:\, \lambda \in \mathbb{R}\}$ (resp., $\widehat{N}(z, \mathrm{gph} f) = \{\lambda(1,1,-1) \,:\, \lambda \in \mathbb{R}\}$, $\widehat{N}(z, \mathrm{gph} f) = \{\lambda(-1,1,-1) \,:\, \lambda \in \mathbb{R}\}$, and $\widehat{N}(z, \mathrm{gph} f) = \{\lambda(-1,-1,-1) \,:\, \lambda \in \mathbb{R}\}$).

If $x_1 > 0$ and $x_2 = 0$, then $z \in \Gamma_1 \cap \Gamma_2$. Since

$$\widehat{T}(z, \Gamma_1) = \{(v_1, v_2, \alpha) \in \mathbb{R}^3 \,:\, v_2 \ge 0,\ 0 = v_1 - v_2 - \alpha\},$$

using the Farkas Lemma (see [19, p. 200]) we get

$$\widehat{N}(z, \Gamma_1) = \{(\eta_1, \eta_2, \theta) = -\lambda(0,1,0) - \mu(1,-1,-1) \,:\, \lambda \ge 0,\ \mu \in \mathbb{R}\}.$$

Similarly,

$$\widehat{N}(z, \Gamma_2) = \{(\eta_1, \eta_2, \theta) = -\lambda'(0,-1,0) - \mu'(1,1,-1) \,:\, \lambda' \ge 0,\ \mu' \in \mathbb{R}\}.$$

As $\widehat{N}(z, \mathrm{gph} f) = \widehat{N}(z, \Gamma_1) \cap N(z, \Gamma_2)$, we can deduce that

$$\widehat{N}(z, \mathrm{gph} f) = \{(-\mu, \mu - \lambda, \mu) \ : \ 2\mu \geq \lambda \geq 0\}.$$

It is clear that this Fréchet normal cone does not depend on the position of $z \neq 0$ in the half-line $\Gamma_1 \cap \Gamma_2$.

If $x_1 < 0$ and $x_2 = 0$, then $z \in \Gamma_3 \cap \Gamma_4$. Arguing similarly as the above, we get

$$\widehat{N}(z, \mathrm{gph} f) = \{(\mu, \lambda - \mu, \mu) \ : \ 2\mu \geq \lambda \geq 0\}.$$

If $x_1 = 0$ and $x_2 > 0$, then $z \in \Gamma_1 \cap \Gamma_4$ and

$$\widehat{N}(z, \mathrm{gph} f) = \{(-\lambda - \mu, \mu, \mu) \ : \ -2\mu \geq \lambda \geq 0\}.$$

If $x_1 = 0$ and $x_2 < 0$, then $z \in \Gamma_2 \cap \Gamma_3$ and

$$\widehat{N}(z, \mathrm{gph} f) = \{(-\lambda - \mu, -\mu, \mu) \ : \ -2\mu \geq \lambda \geq 0\}.$$

If $x_1 = 0$ and $x_2 = 0$, then $z = (\overline{x}, 0) \in \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \cap \Gamma_4$. Since

$$\widehat{T}((\overline{x}, 0), \Gamma_1) = \{(v_1, v_2, \alpha) \ : \ v_1 \geq 0, \ v_2 \geq 0, \ 0 = v_1 - v_2 - \alpha\},$$

by the Farkas Lemma we have

$$\widehat{N}((\overline{x}, 0), \Gamma_1) = \{-\lambda_1(1,0,0) - \lambda_2(0,1,0) - \mu(1,-1,-1) \ : \ \lambda_1 \geq 0, \ \lambda_2 \geq 0, \ \mu \in R\}.$$

In a similar way we can find the normal cones $\widehat{N}((\overline{x}, 0), \Gamma_i)$ $(i = 2, 3, 4)$. Then, using the formula

$$\widehat{N}((\overline{x}, 0), \mathrm{gph} f) = \bigcap_{i=1}^{4} \widehat{N}((\overline{x}, 0), \Gamma_i)$$

we can show that $\widehat{N}((\overline{x}, 0), \mathrm{gph} f) = \{(0,0,0)\}$.

Combining all the above results with formula (2.3), we obtain

$$\begin{aligned} N((\overline{x}, 0), \mathrm{gph} f) &= \limsup_{z \to (\overline{x}, 0)} \widehat{N}(z, \mathrm{gph} f) \\ &= \mathrm{cone}\{(1,-1,-1), \ (1,1,-1), \ (-1,1,-1), \ (-1,-1,-1)\} \\ &\quad \cup \{(-\mu, \mu - \lambda, \mu) \ : \ 2\mu \geq \lambda \geq 0\} \\ &\quad \cup \{(\mu, \lambda - \mu, \mu) \ : \ 2\mu \geq \lambda \geq 0\} \\ &\quad \cup \{(-\lambda - \mu, \mu, \mu) \ : \ -2\mu \geq \lambda \geq 0\} \\ &\quad \cup \{(-\lambda - \mu, -\mu, \mu) \ : \ -2\mu \geq \lambda \geq 0\}. \end{aligned}$$

Consequently,

$$D^* f(\overline{x})(y^*) = \begin{cases} \{(y^*, -y^*), \ (y^*, y^*), \ (-y^*, y^*), \ (-y^*, -y^*)\} \\ \cup\{(-\lambda^* + y^*, -y^*) \ : \ 2y^* \geq \lambda^* \geq 0\} \\ \cup\{(-\lambda^* + y^*, y^*) \ : \ 2y^* \geq \lambda^* \geq 0\} \\ \qquad \text{for } \ y^* > 0, \\ \{(y^*, -y^*), \ (y^*, y^*), \ (-y^*, y^*), \ (-y^*, -y^*)\} \\ \cup\{(y^*, -y^* - \lambda^*) \ : \ -2y^* \geq \lambda^* \geq 0\} \\ \cup\{(-y^*, y^* + \lambda^*) \ : \ -2y^* \geq \lambda^* \geq 0\} \\ \qquad \text{for } \ y^* < 0, \\ \{(0,0)\} \quad \text{for } \ y^* = 0. \end{cases}$$

Thus, for every y^*, $D^*f(0)(y^*)$ is a nonempty (usually nonconvex) compact set.

By the same method, we can obtain

$$\begin{aligned} N((\overline{x},0),\mathrm{epi}f) &= \limsup_{z\to(\overline{x},0)} \widehat{N}(z,\mathrm{epi}f) \\ &= \mathrm{cone}\{(1,-1,-1),\ (1,1,-1),\ (-1,1,-1),\ (-1,-1,-1)\} \\ &\quad \cup\{(-\lambda-\mu,\mu,\mu)\ :\ -2\mu\geq\lambda\geq 0\} \\ &\quad \cup\{(-\lambda-\mu,-\mu,\mu)\ :\ -2\mu\geq\lambda\geq 0\}. \end{aligned}$$

Therefore

$$\begin{aligned} \partial_M f(\overline{x}) &= \{x^* : (x^*,-1)\in N((\overline{x},0),\mathrm{epi}f)\} \\ &= \{(1,-1),\ (1,1),\ (-1,1),\ (-1,-1)\} \\ &\quad \cup\{(-\lambda^*+1,-1)\ :\ 2\geq\lambda^*\geq 0\}\cup\{(-\lambda^*+1,1)\ :\ 2\geq\lambda^*\geq 0\} \\ &= \{(\lambda^*,1)\ :\ -1\leq\lambda^*\leq 1\}\cup\{(\lambda^*,-1)\ :\ -1\leq\lambda^*\leq 1\}. \end{aligned}$$

Thus $\partial_M f(\overline{x})$ is a nonconvex compact set. This set is a J-L subdifferential of f at $\overline{x}$. But it is not a minimal J-L subdifferential, because

$$\partial f(\overline{x}) := \{(1,-1),\ (-1,1)\}$$

is also a J-L subdifferential of f at $\overline{x}$ (see [4]).

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