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STRONG CONVERGENCE TO COMMON FIXED POINTS OF FAMILIES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^{\infty} F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. We consider a sequence $\{x_n\}$ generated by $x \in C$, $x_n = \alpha_n x + (1-\alpha_n)T_n x_n$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset (0,1)$ and then give the conditions of $\{\alpha_n\}$, $\{T_n\}$ and $\mathcal{T}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}$. We also consider a sequence $\{x_n\}$ generated by $x_1 = x \in C$, $x_{n+1} = \alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ and then give the conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{T_n\}$ and $\mathcal{T}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}$. Using these results, we improve and extend well-known strong convergence theorems.

1. Introduction

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$ and let $\mathbf{N}$ be the set of all positive integers. Let C be a nonempty closed convex subset of E. Then, a mapping $T : C \longrightarrow C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \ (\forall x, y \in C).$$

Browder [4] considered a sequence $\{x_n\}$ as follows:

$$x \in C,\ x_n = \alpha_n x + (1-\alpha_n)Tx_n \ (\forall n \in \mathbf{N}), \tag{1.1}$$

where $\{\alpha_n\} \subset (0,1)$ and he proved the first strong convergence theorem in the framework of a Hilbert space. Shioji and Takahashi [28], and Suzuki [30] also proved strong convergence theorems of Browder's type for one-parameter nonexpansive semigroups. Recently, authors [19] obtained a theorem which generalizes the results of [4, 30], simultaneously. In a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable, Shioji and Takahashi [26, 28, 29] and Nakajo [17] proved strong convergence theorems of Browder's type. On the other hand, Halpern [8] considered the following process: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n \ (\forall n \in \mathbf{N}), \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1)$. Wittmann [38] proved a strong convergence theorem of Halpern's type in the framework of a Hilbert space and then, several authors [3, 12, 10, 11, 13, 22, 25] proved strong convergence theorems. In a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable, Shioji and

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Takahashi [24, 27, 28, 29], Kamimura and Takahashi [14, 15], Shimoji and Takahashi [23], Takahashi, Tamura and Toyoda [36] and Kimura, Takahashi and Toyoda [16] and Nakajo [17] proved the strong convergence theorems of Halpern's type.

In this paper, for families $\{T_n\}$ and $\mathcal{T}$ of nonexpansive mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^{\infty} F(T_n)$, we consider a sequence $\{x_n\}$ generated by $x \in C,\ x_n = \alpha_n x + (1-\alpha_n)T_n x_n$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset (0,1)$ and then give the conditions of $\{\alpha_n\}$, $\{T_n\}$ and $\mathcal{T}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}$. We also consider a sequence $\{x_n\}$ generated by $x_1 = x \in C,\ x_{n+1} = \alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n)$ $(\forall n \in \mathbf{N})$, where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ and then give the conditions of $\{\alpha_n\}$, $\{\beta_n\}$, $\{T_n\}$ and $\mathcal{T}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}$. Using these results, we improve and extend well-known strong convergence theorems.

2. Preliminaries

Let E be a Banach space. We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x. Let C be a subset of E and let $T : C \longrightarrow E$. T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ holds for each $x, y \in C$. We denote by $F(T)$ the set of all fixed points of T. We define the modulus δ_E of convexity of E as follows: δ_E is a function of $[0,2]$ into $[0,1]$ such that $\delta_E(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\}$ for every $\varepsilon \in [0,2]$. E is called uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon > 0$. E is called strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space E, we have that if $\|x\| = \|y\| = \|\lambda x + (1-\lambda)y\|$ for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. It is known that a uniformly convex Banach space is strictly convex. Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself. We know that if E is strictly convex, $F(T)$ is closed and convex. Let $G = \{g : [0,\infty) \longrightarrow [0,\infty) : g(0) = 0,\ g : \text{continuous, strictly increasing, convex}\}$. Xu [39] proved the following result.

Lemma 2.1. *Let E be a uniformly convex Banach space. Then, for every bounded subset B of E, there exists $g_B \in G$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g_B(\|x-y\|) \tag{2.1}$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$.

Let E be a Banach space and let E^* be the dual space of E. A set-valued mapping J of E into E^* defined by

$$J(x) = \{x^* \in E^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}\ (\forall x \in E)$$

is called the duality mapping on E. E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for every $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. And the norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, (2.2) is attained uniformly for $x \in S(E)$. It is known that the duality mapping $J : E \longrightarrow 2^{E^*}$ is single valued and norm to weak* uniformly continuous on bounded subsets of E

when E has a uniformly Gâteaux differentiable norm. The following lemma is well known; for example, see [33].

Lemma 2.2. *Let E be a smooth Banach space. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2(x - y, J(y)).$$

Let μ be a continuous, linear functional on l^∞. We call μ a Banach limit when μ satisfies $\|\mu\| = \mu(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $\{a_n\} \in l^\infty$. We know that $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for every $\{a_n\} \in l^\infty$; see [33]. We have the following result from [37]; see also [6] and [33].

Lemma 2.3. *Let C be a convex subset of E whose norm is uniformly Gâteaux differentiable and let $z \in C$. Let $\{x_n\} \subset E$ be a bounded sequence and let μ be a Banach limit. Then, $\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$ if and only if $\mu_n(y - z, J(x_n - z)) \leq 0$ for all $y \in C$.*

Let C be a convex subset of E and let K be a nonempty subset of C. Let P be a retraction of C onto K, that is, $Px = x$ for every $x \in K$. P is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. We know the following result; see[5, 21, 33].

Lemma 2.4. *Let C be a convex subset of a smooth Banach space E and let K be a nonempty subset of C. Let P be a retraction of C onto K. Then, P is sunny and nonexpansive if and only if $(x - Px, J(y - Px)) \leq 0$ for every $x \in C$ and $y \in K$. Hence, there is at most one sunny nonexpansive retraction of C onto K.*

3. Lemmas

Let E be a Banach space and let C be a subset of E. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself such that $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^\infty F(T_n)$, where $F(T_n)$ is the set of all fixed points of T_n and $F(\mathcal{T})$ is the set of all common fixed points of $\mathcal{T}$. Motivated by [19] and [20], we consider the following conditions of $\{T_n\}$ and $\mathcal{T}$:

(I) For each bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ implies $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ for every $T \in \mathcal{T}$.

(II) For every bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_{n+1} - T_n z_n\| = 0$ implies $\lim_{n\to\infty} \|z_n - T_m z_n\| = 0$ for all $m \in \mathbf{N}$.

(III) There exists $\{a_n\} \subset [0, \infty)$ with $\sum_{n=1}^\infty a_n < \infty$ such that for every bounded subset B of C, there exists $M_B > 0$ such that $\|T_n x - T_{n+1}x\| \leq a_n M_B$ holds for all $n \in \mathbf{N}$ and $x \in B$.

We have the following results for nonexpansive mappings.

Lemma 3.1. *Let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = T$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{T\}$ satisfy the condition (I) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = F(T)$;*

(ii) *$\{T_n\}$ with $T_n = T$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{T\}$ satisfy the condition (III) with $a_n = 0$ ($\forall n \in \boldsymbol{N}$).*

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $a \leq b$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = \gamma_n S + (1-\gamma_n)T$ $(\forall n \in \boldsymbol{N})$ and $\mathcal{T} = \{\frac{S+T}{2}\}$ satisfy the condition (I) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(S) \cap F(T)$;*

(ii) *$\{T_n\}$ with $T_n = \gamma_n S + (1-\gamma_n)T$ $(\forall n \in \boldsymbol{N})$ such that $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$ and $\mathcal{T} = \{\frac{S+T}{2}\}$ satisfy the conditions (I) and (III) with $a_n = |\gamma_n - \gamma_{n+1}|$ $(\forall n \in \boldsymbol{N})$ and $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(S) \cap F(T)$.*

Proof. Since E is strictly convex, we have $\cap_{n=1}^{\infty} F(\gamma_n S + (1-\gamma_n)T) = F(\frac{S+T}{2}) = F(S) \cap F(T)$.

(i). Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ and let $z \in \cap_{n=1}^{\infty} F(T_n)$. There exists $g \in G$ such that

$$\begin{aligned}
\|z_n - z\|^2 &\leq \{\|z_n - T_n z_n\| + \|T_n z_n - z\|\}^2 \leq \|z_n - T_n z_n\| \cdot M + \|T_n z_n - z\|^2 \\
&\leq \|z_n - T_n z_n\| \cdot M + \gamma_n \|S z_n - z\|^2 \\
&\quad + (1-\gamma_n)\|T z_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|S z_n - T z_n\|) \\
&\leq \|z_n - T_n z_n\| \cdot M + \|z_n - z\|^2 - \gamma_n(1-\gamma_n) g(\|S z_n - T z_n\|)
\end{aligned}$$

for all $n \in \mathbf{N}$, where $M = \sup_{n\in\mathbf{N}}\{\|z_n - T_n z_n\| + 2\|z_n - z\|\}$. So, we get $\lim_{n\to\infty} \|S z_n - T z_n\| = 0$. Since

$$\|z_n - S z_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - S z_n\| = \|z_n - T_n z_n\| + (1-\gamma_n)\|S z_n - T z_n\|$$

for every $n \in \mathbf{N}$, we obtain $\lim_{n\to\infty} \|z_n - S z_n\| = 0$ and hence, $\lim_{n\to\infty} \|z_n - T z_n\| = 0$. Therefore, $\lim_{n\to\infty} \|z_n - \frac{S+T}{2} z_n\| = 0$.

(ii). By (i), (I) holds. Let $z \in F(S) \cap F(T)$. We have

$$\begin{aligned}
\|T_n x - T_{n+1} x\| &= \|\{\gamma_n S x + (1-\gamma_n) T x\} - \{\gamma_{n+1} S x + (1-\gamma_{n+1}) T x\}\| \\
&\leq |\gamma_n - \gamma_{n+1}| \cdot \|S x - T x\| \leq |\gamma_n - \gamma_{n+1}| \cdot \{2\|x - z\|\}
\end{aligned}$$

for every $n \in \mathbf{N}$ and $x \in C$. So, for each bounded subset B of C, there exists $M_B > 2 \cdot \sup_{x\in B} \|x - z\|$ such that $\|T_n x - T_{n+1} x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\gamma_n - \gamma_{n+1}|$ $(\forall n \in \mathbf{N})$. So, (III) holds. □

An operator $A \subset E \times E$ is called accretive if for (x_1, y_1), $(x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$, where J is the duality mapping of E. An accretive operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $D(A)$ is the domain of A, $R(I + \lambda A)$ is the range of $I + \lambda A$ and $\overline{D(A)}$ is the closure of $D(A)$. An accretive operator A is said to be m-accretive if $R(I + \lambda A) = E$ for every $\lambda > 0$. If A is accretive, then we can define, for each $r > 0$, a mapping $J_r : R(I + rA) \longrightarrow D(A)$ by $J_r = (I + rA)^{-1}$. J_r is called the resolvent of A. We know that J_r is nonexpansive for all $r > 0$ and $A^{-1}0 = F(J_r)$ for every $r > 0$. We also define the Yosida approximation A_r by $A_r = (I - J_r)/r$ for each $r > 0$; see [33, 34] for more details. We have the following result for the resolvents [18].

Lemma 3.3. *Let $A \subset E \times E$ be an accretive operator. Let $r, \lambda > 0$ and $D(A) \subset R(I + \lambda A)$. Then, $\frac{1}{\lambda}\|(I - J_\lambda) J_r x\| \leq \frac{1}{r}\|(I - J_r) x\|$ holds for every $x \in R(I + rA)$.*

We also have the following result for the resolvents [7].

Lemma 3.4. *Let $A \subset E \times E$ be an accretive operator and let $r, \lambda > 0$. For each $x \in R(I + rA) \cap R(I + \lambda A)$, $\|J_\lambda x - J_r x\| \leq \frac{|\lambda - r|}{\lambda}\|x - J_\lambda x\|$ holds.*

We get the following results for the resolvents by Lemmas 3.3 and 3.4.

Lemma 3.5. *Let C be a nonempty closed convex subset of E and let $A \subset E \times E$ be an accretive operator with $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$ and $A^{-1}0 \neq \emptyset$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = J_{\lambda_n}$ $(\forall n \in \boldsymbol{N})$ with $\{\lambda_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} \lambda_n > 0$ and $\mathcal{T} = \{J_1\}$ satisfy the condition (I) and $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = A^{-1}0$;*

(ii) *$\{T_n\}$ with $T_n = J_{\lambda_n}$ $(\forall n \in \boldsymbol{N})$ with $\{\lambda_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} \lambda_n > 0$ and $\sum_{n=1}^\infty |\lambda_n - \lambda_{n+1}| < \infty$ and $\mathcal{T} = \{J_1\}$ satisfy the conditions (I) and (III) with $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \boldsymbol{N})$ and $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = A^{-1}0$;*

(iii) *$\{T_n\}$ with $T_n = J_{\lambda_n}$ $(\forall n \in \boldsymbol{N})$, where $\{\lambda_n\} \subset (0, \infty)$ and $\lim_{n\to\infty} \lambda_n = \infty$ and $\mathcal{T} = \{J_1\}$ satisfy the conditions (I) and (II) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = A^{-1}0$.*

Proof. We know that J_r is a nonexpansive mapping of C into itself for all $r > 0$ and $\cap_{n=1}^\infty F(J_{\lambda_n}) = F(J_1) = A^{-1}0$; see [33].
(i). Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - J_{\lambda_n} z_n\| = 0$. We have

$$\begin{aligned} \|z_n - J_1 z_n\| &\leq \|z_n - J_{\lambda_n} z_n\| + \|J_{\lambda_n} z_n - J_1 J_{\lambda_n} z_n\| + \|J_1 J_{\lambda_n} z_n - J_1 z_n\| \\ &\leq 2\|z_n - J_{\lambda_n} z_n\| + \frac{1}{\lambda_n}\|z_n - J_{\lambda_n} z_n\| \end{aligned}$$

for every $n \in \mathbf{N}$ by Lemma 3.3. From $\inf_{n\in\mathbf{N}} \lambda_n > 0$, we get $\lim_{n\to\infty} \|z_n - J_1 z_n\| = 0$. So, (I) holds.
(ii). From (i), (I) holds. By Lemma 3.4, we have

$$\|J_{\lambda_n} x - J_{\lambda_{n+1}} x\| \leq \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_n}\|x - J_{\lambda_n} x\| \leq \frac{|\lambda_n - \lambda_{n+1}|}{c}\{2\|x - u\|\}$$

for every $n \in \mathbf{N}$ and $x \in C$, where $u \in A^{-1}0$ and $c = \inf_{n\in\mathbf{N}} \lambda_n\,(> 0)$. So, for each bounded subset B of C, there exists $M_B > \frac{2}{c}\sup_{x\in B}\|x - u\|$ such that $\|T_n x - T_{n+1} x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = |\lambda_n - \lambda_{n+1}|$ $(\forall n \in \mathbf{N})$. So, (III) holds.
(iii). As in the proof of (i), (I) holds. Further, let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_{n+1} - J_{\lambda_n} z_n\| = 0$ and fix $m \in \mathbf{N}$. Then, by Lemma 3.3 we have

$$\begin{aligned} \|z_{n+1} - J_{\lambda_m} z_{n+1}\| &\leq \|z_{n+1} - J_{\lambda_n} z_n\| + \|J_{\lambda_n} z_n - J_{\lambda_m} J_{\lambda_n} z_n\| \\ &\quad + \|J_{\lambda_m} J_{\lambda_n} z_n - J_{\lambda_m} z_{n+1}\| \\ &\leq 2\|z_{n+1} - J_{\lambda_n} z_n\| + \frac{\lambda_m}{\lambda_n}\|z_n - J_{\lambda_n} z_n\| \end{aligned}$$

and hence $\|z_{n+1} - J_{\lambda_m} z_{n+1}\| \to 0$. So, (II) holds. □

Let C be a nonempty closed convex subset of E. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself and let $\beta_1, \beta_2, \ldots$ be real numbers such that $0 \leq \beta_i \leq 1$ for every $i \in \mathbf{N}$. Then, for any $n \in \mathbf{N}$, Takahashi [32] (see also [23, 34, 35]) introduced a mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \beta_n S_n U_{n,n+1} + (1-\beta_n)I, \\ U_{n,n-1} &= \beta_{n-1} S_{n-1} U_{n,n} + (1-\beta_{n-1})I, \\ &\vdots \\ U_{n,k} &= \beta_k S_k U_{n,k+1} + (1-\beta_k)I, \\ &\vdots \\ U_{n,2} &= \beta_2 S_2 U_{n,3} + (1-\beta_2)I, \\ W_n = U_{n,1} &= \beta_1 S_1 U_{n,2} + (1-\beta_1)I. \end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$. We know that if E is strictly convex, $\cap_{i=1}^n F(S_i) \neq \emptyset$, $0 < \beta_i < 1$ for every $i = 2, 3, \ldots, n$ and $0 < \beta_1 \leq 1$, then, $F(W_n) = \cap_{i=1}^n F(S_i)$; see [34, 35]. We also have that if E is strictly convex, $\cap_{n=1}^\infty F(S_n) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbf{N}$ for some $b \in (0,1)$, then, $\lim_{n\to\infty} U_{n,k}x$ exists for every $x \in C$ and $k \in \mathbf{N}$; see [23]. So, we can define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. We have that if E is strictly convex, $\cap_{i=1}^\infty F(S_i) \neq \emptyset$ and $0 < \beta_i \leq b < 1$ for every $i \in \mathbf{N}$ for some $b \in (0,1)$, then, $F(W) = \cap_{i=1}^\infty F(S_i)$; see [23]. We know the following results for the W-mappings.

Lemma 3.6. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself with $\cap_{n=1}^\infty F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \boldsymbol{N}$ for some $b \in (0,1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$ for every $n \in \boldsymbol{N}$ and let W be the W-mapping generated by $S_1, S_2, \ldots$ and $\beta_1, \beta_2, \ldots$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = W_n$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{W\}$ satisfy the condition (I) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{n=1}^\infty F(S_n)$;*

(ii) *$\{T_n\}$ with $T_n = W_n$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{W\}$ satisfy the conditions (I) and (III) with $a_n = b^{n+1}$ ($\forall n \in \boldsymbol{N}$) and $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = \cap_{n=1}^\infty F(S_n)$.*

Proof. We have $\cap_{n=1}^\infty F(W_n) = \cap_{n=1}^\infty F(S_n) = F(W) \neq \emptyset$.
(i). Let $z \in \cap_{n=1}^\infty F(S_n)$. We get

$$\begin{aligned} \|W_n x - W_{n+1}x\| &= \|\beta_1 S_1 U_{n,2}x - \beta_1 S_1 U_{n+1,2}x\| \leq \beta_1 \|U_{n,2}x - U_{n+1,2}x\| \\ &= \beta_1 \|\beta_2 S_2 U_{n,3}x - \beta_2 S_2 U_{n+1,3}x\| \\ &\leq \beta_1\beta_2 \|U_{n,3}x - U_{n+1,3}x\| \\ &\leq \cdots \leq \beta_1\beta_2 \ldots \beta_n\beta_{n+1} \|x - S_{n+1}x\| \leq b^{n+1}\{2\|x - z\|\} \end{aligned}$$

for every $n \in \mathbf{N}$ and $x \in C$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$. Let $n \in \mathbf{N}$. We get

$$\begin{aligned}\|z_n - W_{n+m}z_n\| &\leq \|z_n - W_n z_n\| + \|W_n z_n - W_{n+1}z_n\| + \cdots \\ &\quad + \|W_{n+m-1}z_n - W_{n+m}z_n\| \\ &\leq \|z_n - W_n z_n\| + b^{n+1}\{2\|z_n - z\|\} + \cdots + b^{n+m}\{2\|z_n - z\|\} \\ &\leq \|z_n - W_n z_n\| + (b^{n+1} + \cdots + b^{n+m})M \\ &\leq \|z_n - W_n z_n\| + \frac{b^{n+1}(1-b^m)}{1-b}M\end{aligned}$$

for every $m \in \mathbf{N}$, where $M = \sup_{n\in\mathbf{N}}\{2\|z_n - z\|\}$. So, we obtain

$$\|z_n - Wz_n\| = \lim_{m\to\infty} \|z_n - W_{n+m}z_n\| \leq \|z_n - W_n z_n\| + \frac{b^{n+1}}{1-b}M$$

for each $n \in \mathbf{N}$ which implies $\lim_{n\to\infty} \|z_n - Wz_n\| = 0$. So, (I) holds.
(ii). Let $z \in \cap_{n=1}^{\infty} F(S_n)$. As in the proof of (i), we have

$$\|W_n x - W_{n+1}x\| \leq b^{n+1} 2\|x - z\|$$

for every $n \in \mathbf{N}$ and $x \in C$. So, for each bounded subset B of C, there exists $M_B > 2 \cdot \sup_{x\in B} \|x - z\|$ such that $\|T_n x - T_{n+1}x\| \leq a_n M_B$ for all $n \in \mathbf{N}$ and $x \in B$, where $a_n = b^{n+1}$ ($\forall n \in \mathbf{N}$). So, (III) holds. As in the proof of (i), (I) holds. □

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Then, for every $s \in S$ and $f \in B(S)$, we can define $l_s f \in B(S)$ by $(l_s f)(t) = f(st)$ for each $t \in S$. We also denote by l_s^* the adjoint operator of l_s. Let D be a subspace of $B(S)$ containing constants and let μ be an element of D^*, where D^* is its topological dual. A linear functional μ is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be satisfied that for each bounded sequence $\{f_n : n \in \mathbf{N}\}$ of D, the mappings $t \mapsto \inf_n f_n(t)$ and $t \mapsto \sup_n f_n(t)$ are in D. A mean μ on D is said to be monotone convergent if $\mu_t(\lim_{n\to\infty} f_n(t)) = \lim_{n\to\infty} \mu_t(f_n(t))$ for every bounded sequence $\{f_n : n \in \mathbf{N}\}$ of D such that $0 \leq f_1 \leq f_2 \leq \cdots$. We know that if μ is a monotone convergent mean on D and $\{f_n : n \in \mathbf{N}\}$ is a bounded sequence of D, then $\limsup_{n\to\infty} \mu_t(f_n(t)) \leq \mu_t(\limsup_{n\to\infty} f_n(t))$. Let C be a nonempty closed convex subset of E. A family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(st) = T(s)T(t)$ for every $s, t \in S$;
(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for each $s \in S$ and $x, y \in C$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, i.e., $\cap_{t\in S} F(T(t))$. Hirano, Kido and Takahashi [9] proved the following; see also [31].

Lemma 3.7. *Let S be a semigroup. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that for every $x \in C$, $\{T(t)x : t \in S\}$ is contained in a weakly compact convex subset of C. Let D be a subspace of $B(S)$ such that D contains constants and the mapping $t \mapsto (T(t)x, y^*)$ is in D for each $x \in C$ and $y^* \in E^*$. Then, for any mean μ on D and $x \in C$, there exists a unique element $T_\mu x$ in C such that $(T_\mu x, x^*) = \mu_s(T(s)x, x^*)$*

for every $x^* \in E^*$. And T_μ is a nonexpansive mapping of C into itself and $T_\mu x = x$ for all $x \in F(\mathcal{S})$.

Further, Atsushiba, Shioji and Takahashi [2] proved the following; see also [1, 29].

Lemma 3.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let S be a semigroup and let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ containing constants and being invariant under l_s for every $s \in S$ and for each $x \in C$ and $x^* \in E^*$, the function $t \mapsto (T(t)x, x^*)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for all $s \in S$. Let $w \in F(\mathcal{S})$ and $D_r = \{y \in C : \|y - w\| \le r\}$ for $r > 0$. Then, $\lim_{n\to\infty} \sup_{x \in D_r} \|T_{\mu_n} x - T(t)T_{\mu_n} x\| = 0$ for every $r > 0$ and $t \in S$.*

We have the following results for nonexpansive semigroups from Lemmas 3.7 and 3.8.

Lemma 3.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and being invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $x^* \in E^*$, the function $t \mapsto (T(t)x, x^*)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Then, the following hold:*

- (i) *$\{T_n\}$ with $T_n = T_{\mu_n}$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \mathcal{S}$ satisfy the condition (I) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = F(\mathcal{S})$;*
- (ii) *moreover, assume that the mappings $t \mapsto \sup_n f_n(t)$ and $t \mapsto \inf_n f_n(t)$ are in D for every bounded sequence $\{f_n : n \in \boldsymbol{N}\}$ of D and $\{\mu_n\}$ is a sequence of monotone convergent means on D. Then, $\{T_n\}$ with $T_n = T_{\mu_n}$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \mathcal{S}$ satisfy the conditions (I) and (II) with $\cap_{n=1}^\infty F(T_n) = F(\mathcal{T}) = F(\mathcal{S})$.*

Proof. By Lemmas 3.7 and 3.8, we have $\cap_{n=1}^\infty F(T_{\mu_n}) = F(\mathcal{S})$.
(i). Let $\{z_n\}$ in C be a bounded sequence such that $\lim_{n\to\infty} \|z_n - T_{\mu_n} z_n\| = 0$. For all $t \in S$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|z_n - T(t)z_n\| &\le \|z_n - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T(t)T_{\mu_n} z_n\| + \|T(t)T_{\mu_n} z_n - T(t)z_n\| \\ &\le 2\|z_n - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T(t)T_{\mu_n} z_n\|. \end{aligned}$$

From Lemma 3.8, we obtain $\lim_{n\to\infty} \|z_n - T(t)z_n\| = 0$ for every $t \in S$. So, (I) holds.
(ii). As in the proof of (i), (I) holds. Let $\{z_n\} \subset C$ be a bounded sequence such that $\lim_{n\to\infty} \|z_{n+1} - T_{\mu_n} z_n\| = 0$. We have

$$\begin{aligned} \|z_{n+1} - T_{\mu_m} z_{n+1}\| &\le \|z_{n+1} - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T_{\mu_m} T_{\mu_n} z_n\| \\ &\quad + \|T_{\mu_m} T_{\mu_n} z_n - T_{\mu_m} z_{n+1}\| \\ &\le 2\|z_{n+1} - T_{\mu_n} z_n\| + \|T_{\mu_n} z_n - T_{\mu_m} T_{\mu_n} z_n\| \end{aligned}$$

for every $m, n \in \mathbf{N}$. Hence, for each $m \in \mathbf{N}$, we get

$$\begin{aligned} \limsup_{n\to\infty} \|z_{n+1} - T_{\mu_m} z_{n+1}\|^2 &\le \limsup_{n\to\infty} \|T_{\mu_n} z_n - T_{\mu_m} T_{\mu_n} z_n\|^2 \\ &= \limsup_{n\to\infty} (\mu_m)_t (T(t)(T_{\mu_n} z_n) - T_{\mu_n} z_n, x_n^*) \end{aligned}$$

$$\leq \ (\mu_m)_t\Big(\limsup_{n\to\infty}(T(t)(T_{\mu_n}z_n) - T_{\mu_n}z_n, x_n^*)\Big) \leq 0$$

by Lemma 3.8, where $x_n^* \in J(T_{\mu_m}(T_{\mu_n}z_n) - T_{\mu_n}z_n)$ for all $n \in \mathbf{N}$. Therefore, (II) holds. □

We know the following results for nonexpansive mappings from Lemma 3.9; see [9].

Lemma 3.10. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{T^i : i = 0, 1, 2, \dots\}$ satisfy the condition (I) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(T)$;*

(ii) *$\{T_n\}$ with $T_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{T^i : i = 0, 1, 2, \dots\}$ satisfy the conditions (I) and (II) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(T)$.*

Proof. Let $S = \{0, 1, 2, \dots\}$, $\mathcal{S} = \{T^i : i \in S\}$, $D = B(S)$ and $\mu_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$ for all $n \in \mathbf{N}$ and $f \in D$. We have $F(\mathcal{S}) = F(T) \neq \emptyset$ and know that $\{\mu_n\}$ is a sequence of monotone convergent means on D with $\lim_{n\to\infty}\|\mu_n - l_k^*\mu_n\| = 0$ for all $k \in S$ and $T_{\mu_n}x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x$ for every $x \in C$. By Lemma 3.9, we get Lemma 3.10. □

Lemma 3.11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let S_1 and S_2 be nonexpansive mappings of C into itself with $S_1S_2 = S_2S_1$ and $F(S_1) \cap F(S_2) \neq \emptyset$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S_1^i S_2^j$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{S_1^i S_2^j : i, j = 0, 1, 2, \dots\}$ satisfy the condition (I) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(S_1) \cap F(S_2)$;*

(ii) *$\{T_n\}$ with $T_n = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S_1^i S_2^j$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \{S_1^i S_2^j : i, j = 0, 1, 2, \dots\}$ satisfy the conditions (I) and (II) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(S_1) \cap F(S_2)$.*

Proof. Let $S = \{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\}$, $\mathcal{S} = \{S_1^i S_2^j : (i, j) \in S\}$, $D = B(S)$ and $\mu_n(f) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} f(i, j)$ for every $n \in \mathbf{N}$ and $f \in D$. We have $F(\mathcal{S}) = F(S_1) \cap F(S_2) \neq \emptyset$ and know that $\{\mu_n\}$ is a sequence of monotone convergent means on D with $\lim_{n\to\infty}\|\mu_n - l_{(k,m)}^*\mu_n\| = 0$ for each $(k, m) \in S$ and $T_{\mu_n}x = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S_1^i S_2^j x$ for every $x \in C$. By Lemma 3.9, we get Lemma 3.11. □

Let C be a nonempty closed convex subset of E. A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s)T(t)$ for every $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for each $s \geq 0$ and $x, y \in C$;

(iv) for all $x \in C$, $s \longmapsto T(s)x$ is continuous.

We have the following results for one-parameter nonexpansive semigroups by Lemma 3.9; see [9].

Lemma 3.12. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Let $\{t_n\} \subset (0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$. Then, the following hold:*

(i) *$\{T_n\}$ with $T_n\cdot = \frac{1}{t_n}\int_0^{t_n} T(s)\cdot ds$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \mathcal{S}$ satisfy the condition (I) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(\mathcal{S})$;*

(ii) *$\{T_n\}$ with $T_n\cdot = \frac{1}{t_n}\int_0^{t_n} T(s)\cdot ds$ ($\forall n \in \boldsymbol{N}$) and $\mathcal{T} = \mathcal{S}$ satisfy the conditions (I) and (II) with $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) = F(\mathcal{S})$.*

Proof. (i). Let $S = (0, \infty)$ and let D be the Banach space $C(S)$ of all bounded continuous real valued functions on S. Let $\lambda_s(f) = \frac{1}{s}\int_0^s f(t)\,dt$ for every $s > 0$ and $f \in D$. We know that $\{\lambda_s\}$ is a net of means on D with $\lim_{s\to\infty} \|\lambda_s - l_k^*\lambda_s\| = 0$ for each $k \in (0, \infty)$ and $T_{\lambda_s}x = \frac{1}{s}\int_0^s T(t)x\,dt$ for every $x \in C$. By Lemma 3.9 (i), we get Lemma 3.12 (i).
(ii). Let $S = (0, \infty)$ and let D be a set of all bounded Lebesque measurable real valued functions on S. Let $\lambda_s(f) = \frac{1}{s}\int_0^s f(t)\,dt$ for every $s > 0$ and $f \in D$. We have the mappings $t \mapsto \sup_n f_n(t)$ and $t \mapsto \inf_n f_n(t)$ are in D for every bounded sequence $\{f_n : n \in \mathbf{N}\}$ of D. We also know that $\{\lambda_s\}$ is a net of monotone convergent means on D with $\lim_{s\to\infty} \|\lambda_s - l_k^*\lambda_s\| = 0$ for each $k \in (0, \infty)$ and $T_{\lambda_s}x = \frac{1}{s}\int_0^s T(t)x\,dt$ for every $x \in C$. From Lemma 3.9 (ii), we get Lemma 3.12 (ii). □

4. Strong convergence theorem of Browder's type

Using the method of [26] (see also [28, 29]), we get the following.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself which satisfy $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^{\infty} F(T_n)$ and the condition (I). Define a sequence $\{x_n\}$ in C as follows: $x \in C$ and*

$$x_n = \alpha_n x + (1 - \alpha_n)T_n x_n \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset (0, 1)$ If $\lim_{n\to\infty} \alpha_n = 0$, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x$, where $P_{F(\mathcal{T})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{T})$.

Proof. Let $U_n = \alpha_n x + (1 - \alpha_n)T_n$ for every $n \in \mathbf{N}$. We have $U_n : C \longrightarrow C$ and U_n is a contraction for all $n \in \mathbf{N}$ since T_n is nonexpansive and $0 < \alpha_n < 1$. So, for each $n \in \mathbf{N}$, there exists a unique element $x_n \in C$ such that $x_n = \alpha_n x + (1 - \alpha_n)T_n x_n$. By (I), we get $F(\mathcal{T}) = \cap_{n=1}^{\infty} F(T_n)$. Let $z \in \cap_{n=1}^{\infty} F(T_n)$. We obtain

$$\begin{aligned} \|x_n - z\| &= \|\alpha_n(x - z) + (1 - \alpha_n)(T_n x_n - z)\| \\ &\leq \alpha_n\|x - z\| + (1 - \alpha_n)\|T_n x_n - z\| \\ &\leq \alpha_n\|x - z\| + (1 - \alpha_n)\|x_n - z\| \end{aligned}$$

for every $n \in \mathbf{N}$. So, we have $\|x_n - z\| \leq \|x - z\|$ for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is bounded. Further, we have that

$$\|x_n - T_n x_n\| = \alpha_n\|x - T_n x_n\| \leq \alpha_n(\|x - z\| + \|T_n x_n - z\|) \leq 2\alpha_n\|x - z\|$$

for each $n \in \mathbf{N}$. From $\lim_{n\to\infty} \alpha_n = 0$, we get

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

So, from (I), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \tag{4.1}$$

for all $T \in \mathcal{T}$. We get

$$\begin{aligned}
&\alpha_n(x - z, J(x_n - z)) \\
&= \alpha_n(x_n - z, J(x_n - z)) + (1 - \alpha_n)((x_n - T_n x_n) - (z - T_n z), J(x_n - z)) \\
&= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\{(x_n - z, J(x_n - z)) - (T_n x_n - T_n z, J(x_n - z))\} \\
&\geq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\{\|x_n - z\|^2 - \|T_n x_n - T_n z\| \cdot \|x_n - z\|\} \geq \alpha_n\|x_n - z\|^2
\end{aligned}$$

for every $n \in \mathbf{N}$. So, we obtain

$$\|x_n - z\|^2 \leq (x - z, J(x_n - z)) \tag{4.2}$$

for all $n \in \mathbf{N}$ and $z \in \cap_{n=1}^\infty F(T_n)$. We also have

$$\begin{aligned}
(x_n - x, J(x_n - z)) &= \frac{1 - \alpha_n}{\alpha_n}(T_n x_n - x_n, J(x_n - z)) \\
&= \frac{1 - \alpha_n}{\alpha_n}\{(T_n x_n - z, J(x_n - z)) - (x_n - z, J(x_n - z))\} \\
&= \frac{1 - \alpha_n}{\alpha_n}\{(T_n x_n - z, J(x_n - z)) - \|x_n - z\|^2\} \leq 0 \qquad (4.3)
\end{aligned}$$

for each $n \in \mathbf{N}$ and $z \in \cap_{n=1}^\infty F(T_n)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ and let μ be a Banach limit. Let g be a real valued function on C defined by $g(y) = \mu_i\|x_{n_i} - y\|^2$ for every $y \in C$. By [33] , we know that g is continuous and convex and g satisfies $\lim_{\|y\|\to\infty} g(y) = \infty$. So, there exists $x_0 \in C$ such that $g(x_0) = \inf_{y\in C} g(y)$. Let $y_1, y_2 \in C$ such that $g(y_1) = g(y_2) = \inf_{y\in C} g(y)$ and suppose that $y_1 \neq y_2$. Let B be a bounded subset of E containing sequences $\{x_{n_i} - y_1\}$ and $\{x_{n_i} - y_2\}$. By (2.1), there exists $g_B \in G$ such that

$$\begin{aligned}
\left\|x_{n_i} - \frac{y_1 + y_2}{2}\right\|^2 &= \left\|\frac{1}{2}(x_{n_i} - y_1) + \frac{1}{2}(x_{n_i} - y_2)\right\|^2 \\
&\leq \frac{1}{2}\|x_{n_i} - y_1\|^2 + \frac{1}{2}\|x_{n_i} - y_2\|^2 - \frac{1}{4}g_B(\|y_1 - y_2\|)
\end{aligned}$$

for every $i \in \mathbf{N}$ which implies

$$g\Big(\frac{y_1 + y_2}{2}\Big) \leq \frac{1}{2}g(y_1) + \frac{1}{2}g(y_2) - \frac{1}{4}g_B(\|y_1 - y_2\|) < \inf_{y\in C} g(y).$$

This is a contradiction. So, we obtain $y_1 = y_2$. Therefore, there exists a unique element y_0 of C such that $g(y_0) = \inf_{y\in C} g(y)$. Suppose $y_0 \notin F(T)$ for some $T \in \mathcal{T}$. Let B be a bounded subset of E containing sequences $\{x_{n_i} - y_0\}$ and $\{x_{n_i} - Ty_0\}$.

We have

$$\begin{aligned}\Big\|x_{n_i}-\frac{Ty_0+y_0}{2}\Big\|^2 &\le \frac{1}{2}\|x_{n_i}-y_0\|^2+\frac{1}{2}\|x_{n_i}-Ty_0\|^2-\frac{1}{4}g_B(\|y_0-Ty_0\|)\\ &\le \frac{1}{2}\|x_{n_i}-y_0\|^2+\frac{1}{2}\{\|x_{n_i}-Tx_{n_i}\|+\|Tx_{n_i}-Ty_0\|\}^2\\ &\quad -\frac{1}{4}g_B(\|y_0-Ty_0\|)\\ &\le \frac{1}{2}\|x_{n_i}-y_0\|^2+\frac{1}{2}\{\|x_{n_i}-Tx_{n_i}\|+\|x_{n_i}-y_0\|\}^2\\ &\quad -\frac{1}{4}g_B(\|y_0-Ty_0\|)\\ &=\frac{1}{2}\|x_{n_i}-y_0\|^2\\ &\quad +\frac{1}{2}\{\|x_{n_i}-Tx_{n_i}\|^2 \ +2\|x_{n_i}-Tx_{n_i}\|\cdot\|x_{n_i}-y_0\|+\|x_{n_i}-y_0\|^2\}\\ &\quad -\frac{1}{4}g_B(\|y_0-Ty_0\|)\end{aligned}$$

for some $g_B \in G$. This implies

$$g\Big(\frac{Ty_0+y_0}{2}\Big)\le\frac{1}{2}g(y_0)+\frac{1}{2}g(y_0)-\frac{1}{4}g_B(\|y_0-Ty_0\|)<\inf_{y\in C}g(y)$$

by (4.1). This is a contradiction. So, we get $y_0 \in F(\mathcal{T})$. It follows from (4.2) and Lemma 2.3 that $\mu_i\|x_{n_i}-y_0\|^2 \le \mu_i(x-y_0, J(x_{n_i}-y_0)) \le 0$. There exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty}\|x_{n_{i_j}}-y_0\|=0$$

because

$$\lim_{j\to\infty}\|x_{n_{i_j}}-y_0\|=\liminf_{i\to\infty}\|x_{n_i}-y_0\|\le\mu_i\|x_{n_i}-y_0\|^2\le 0.$$

On the other hand, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i}\to z_1$ and $x_{n_j}\to z_2$. Then, from (4.1) we have that for any $T\in\mathcal{T}$,

$$\|z_1-Tz_1\|\le\|z_1-x_{n_i}\|+\|x_{n_i}-Tx_{n_i}\|+\|Tx_{n_i}-Tz_1\|\to 0$$

as $i\to\infty$. So, we get $z_1\in\cap_{n=1}^{\infty}F(T_n)$. Similarly, $z_2\in\cap_{n=1}^{\infty}F(T_n)$. By (4.3), we obtain $(x_{n_i}-x, J(x_{n_i}-z_2))\le 0$ for all $i\in\mathbf{N}$ and $(x_{n_j}-x, J(x_{n_j}-z_1))\le 0$ for each $j\in\mathbf{N}$. Since

$$\begin{aligned}|(x_{n_i}-x, J(x_{n_i}-z_2))&-(z_1-x, J(z_1-z_2))|\\ &\le|(x_{n_i}-x, J(x_{n_i}-z_2))-(z_1-x, J(x_{n_i}-z_2))|\\ &\quad+|(z_1-x, J(x_{n_i}-z_2))-(z_1-x, J(z_1-z_2))|\\ &\le\|x_{n_i}-z_1\|\cdot\|x_{n_i}-z_2\|\\ &\quad+|(z_1-x, J(x_{n_i}-z_2))-(z_1-x, J(z_1-z_2))|\end{aligned}$$

for every $i\in\mathbf{N}$ and J is norm to weak* uniformly continuous on bounded subsets of E, we have $(z_1-x, J(z_1-z_2))\le 0$. Similarly, $(z_2-x, J(z_2-z_1))\le 0$. So, we get $\|z_1-z_2\|^2=(z_1-z_2, J(z_1-z_2))\le 0$, that is, $z_1=z_2$. Therefore, $\{x_n\}$ converges strongly to some element of $\cap_{n=1}^{\infty} F(T_n) = F(\mathcal{T})$. Hence, we can define a mapping P of C onto $F(\mathcal{T})$ by $Px = \lim_{n\to\infty} x_n$ because x is an arbitrary point of C. By (4.3), we obtain $(Px - x, J(Px - z_0)) \leq 0$ for all $x \in C$ and $z_0 \in F(\mathcal{T})$. So, P is a sunny nonexpansive retraction of C onto $F(\mathcal{T})$ from Lemma 2.4. □

We have the following result for nonexpansive mappings by Lemma 3.1 (i) and Theorem 4.1.

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)Tx_n$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.*

We get the following result for convex combination of nonexpansive mappings by Lemma 3.2 (i) and Theorem 4.1.

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)(\gamma_n Sx_n + (1 - \gamma_n)Tx_n)$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ with $a \leq b$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}x$, where $P_{F(S)\cap F(T)}$ is a sunny nonexpansive retraction of C onto $F(S) \cap F(T)$.*

We have the following result [17] for accretive operators from Lemma 3.5 (i) and Theorem 4.1.

Theorem 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator with $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$ and $A^{-1}0 \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)J_{\lambda_n} x_n$ ($\forall n \in \boldsymbol{N}$), where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$. If $\liminf_{n\to\infty} \lambda_n > 0$, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$, where $P_{A^{-1}0}$ is a sunny nonexpansive retraction of C onto $A^{-1}0$.*

We get the following result for the W-mappings from Lemma 3.6 (i) and Theorem 4.1.

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable. Let $S_1, S_2, \ldots$ be infinite nonexpansive mappings of C into itself with $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\beta_1, \beta_2, \ldots$ be real numbers with $0 < \beta_i \leq b < 1$ for every $i \in \boldsymbol{N}$ for some $b \in (0, 1)$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_n, \beta_{n-1}, \ldots, \beta_1$ for every $n \in \boldsymbol{N}$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1 - \alpha_n)W_n x_n$ ($\forall n \in \boldsymbol{N}$), where $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is a sunny nonexpansive retraction of C onto F.*

We have the following result for nonexpansive semigroups by Lemma 3.9 (i) and Theorem 4.1.

Theorem 4.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F := F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and being invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $x^* \in E^*$, the function $t \mapsto (T(t)x, x^*)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)T_{\mu_n}x_n$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is a sunny nonexpansive retraction of C onto F.*

We get the following results for nonexpansive mappings from Lemmas 3.10 (i) and 3.11 (i) and Theorem 4.1.

Theorem 4.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)\frac{1}{n}\sum_{i=0}^{n-1} T^i x_n$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.*

Theorem 4.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S_1 and S_2 be nonexpansive mappings of C into itself such that $S_1S_2 = S_2S_1$ and $F(S_1) \cap F(S_2) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)} \sum_{k=0}^{n} \sum_{i+j=k} S_1^i S_2^j x_n$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(S_1)\cap F(S_2)}x$, where $P_{F(S_1)\cap F(S_2)}$ is a sunny nonexpansive retraction of C onto $F(S_1) \cap F(S_2)$.*

We have the following result for one-parameter nonexpansive semigroups by Lemma 3.12 (i) and Theorem 4.1.

Theorem 4.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $x \in C$ and $\{x_n\}$ be a sequence by $x_n = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\, ds$ $(\forall n \in \boldsymbol{N})$, where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\{t_n\} \subset (0,\infty)$ with $\lim_{n\to\infty} t_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{S})}x$, where $P_{F(\mathcal{S})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{S})$.*

5. Strong convergence theorem of Halpern's type

Using the method employed in [24], we get the following.

Theorem 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $\{T_n\}$ and $\mathcal{T}$ be families of nonexpansive mappings of C into itself which satisfy $\emptyset \neq F(\mathcal{T}) \subset$*

$\cap_{n=1}^{\infty} F(T_n)$ *and the conditions (I) and (III). Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_n(\beta_n x + (1-\beta_n) x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\beta_n\} \subset [0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and* $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$*. If* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$*, then* $\{x_n\}$ *converges strongly to* $P_{F(\mathcal{T})}x$*, where* $P_{F(\mathcal{T})}$ *is a sunny nonexpansive retraction of* C *onto* $F(\mathcal{T})$*.*

Proof. We have $F(\mathcal{T}) = \cap_{n=1}^{\infty} F(T_n)$ by (I). Let $z \in \cap_{n=1}^{\infty} F(T_n)$. We have $\|x_n - z\| \le \|x - z\|$ for every $n \in \mathbf{N}$. In fact, suppose that $\|x_n - z\| \le \|x - z\|$ for some $n \in \mathbf{N}$. We get

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(x-z) + (1-\alpha_n)\{T_n(\beta_n x + (1-\beta_n)x_n) - z\}\| \\ &\le \alpha_n\|x-z\| + (1-\alpha_n)\{\beta_n\|x-z\| + (1-\beta_n)\|x_n - z\|\} \\ &\le \|x-z\|. \end{aligned}$$

So, $\{x_n\}$ is bounded. Next, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n x + (1-\alpha_n)T_n(\beta_n x + (1-\beta_n)x_n) \\ &\quad - \alpha_{n-1}x - (1-\alpha_{n-1})T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &= \|(\alpha_n - \alpha_{n-1})x + (1-\alpha_n)\{T_n(\beta_n x + (1-\beta_n)x_n) \\ &\qquad\qquad - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\} \\ &\quad + (1-\alpha_n)\{T_{n-1}(\beta_n x + (1-\beta_n)x_n) \\ &\qquad\qquad - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\} \\ &\quad + (\alpha_{n-1} - \alpha_n)T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &\le |\alpha_n - \alpha_{n-1}| \cdot \|x - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\| \\ &\quad + (1-\alpha_n)\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \\ &\quad + (1-\alpha_n)\|\{\beta_n x + (1-\beta_n)x_n\} - \{\beta_{n-1}x + (1-\beta_{n-1})x_{n-1}\}\| \\ &\le |\alpha_n - \alpha_{n-1}| \cdot M_1 \\ &\quad + (1-\alpha_n)\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \\ &\quad + (1-\alpha_n)\{|\beta_n - \beta_{n-1}| \cdot (\|x\| + \|x_{n-1}\|) + (1-\beta_n)\|x_n - x_{n-1}\|\} \end{aligned}$$

for each $n = 2, 3, \ldots$, where $M_1 = \sup_{n \in \mathbf{N}\setminus\{1\}} \|x - T_{n-1}(\beta_{n-1}x + (1-\beta_{n-1})x_{n-1})\|$. Since a sequence $\{\beta_n x + (1-\beta_n)x_n\}$ is bounded, there exists $M_2 > 0$ such that

$$\|T_n(\beta_n x + (1-\beta_n)x_n) - T_{n-1}(\beta_n x + (1-\beta_n)x_n)\| \le a_{n-1}M_2$$

for all $n = 2, 3, \ldots$ by (III). Therefore, we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + a_{n-1})M \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - x_{n-1}\| \end{aligned} \tag{5.1}$$

for every $n = 2, 3, \dots$, where $M = \max\{M_1, M_2, \sup_{n\in\mathbf{N}\setminus\{1\}}\{\|x\| + \|x_{n-1}\|\}\}$. Let $m, n \in \mathbf{N}$. By (5.1), we obtain

$$\begin{aligned}
&\|x_{n+m+1} - x_{n+m}\| \le (|\alpha_{n+m} - \alpha_{n+m-1}| + |\beta_{n+m} - \beta_{n+m-1}| + a_{n+m-1})M \\
&\quad + (1-\alpha_{n+m})(1-\beta_{n+m})\|x_{n+m} - x_{n+m-1}\| \\
&\le (|\alpha_{n+m} - \alpha_{n+m-1}| + |\beta_{n+m} - \beta_{n+m-1}| + a_{n+m-1})M \\
&\quad + (1-\alpha_{n+m})(1-\beta_{n+m})\{(|\alpha_{n+m-1} - \alpha_{n+m-2}| \\
&\qquad + |\beta_{n+m-1} - \beta_{n+m-2}| + a_{n+m-2})M \\
&\qquad\quad + (1-\alpha_{n+m-1})(1-\beta_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\|\} \\
&\le \{(|\alpha_{n+m} - \alpha_{n+m-1}| + |\alpha_{n+m-1} - \alpha_{n+m-2}|) \\
&\quad + (|\beta_{n+m} - \beta_{n+m-1}| + |\beta_{n+m-1} - \beta_{n+m-2}|) + (a_{n+m-1} + a_{n+m-2})\}M \\
&\qquad + (1-\alpha_{n+m})(1-\beta_{n+m})(1-\alpha_{n+m-1})(1-\beta_{n+m-1})\|x_{n+m-1} - x_{n+m-2}\| \\
&\le \cdots \\
&\le M \cdot \sum_{k=m}^{n+m-1} (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k| + a_k) \\
&\quad + \|x_{m+1} - x_m\| \cdot \prod_{k=m+1}^{n+m} (1-\alpha_k)(1-\beta_k).
\end{aligned}$$

So, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|x_{n+1} - x_n\| &= \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \\
&\le M \cdot \sum_{k=m}^{\infty} (|\alpha_{k+1} - \alpha_k| + |\beta_{k+1} - \beta_k| + a_k)
\end{aligned}$$

for each $m \in \mathbf{N}$. Therefore, we get $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - T_n(\beta_n x + (1-\beta_n)x_n)\| + \|T_n(\beta_n x + (1-\beta_n)x_n) - T_n x_n\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n\|x - T_n(\beta_n x + (1-\beta_n)x_n)\| + \beta_n\|x - x_n\|
\end{aligned}$$

for all $n \in \mathbf{N}$, we have $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Let $m \in \mathbf{N}$ and take $n \in \mathbf{N}$ with $n > m$. By (III), there exists $M_B > 0$ such that

$$\begin{aligned}
\|x_n - T_m x_n\| &\le \|x_n - T_n x_n\| + \|T_n x_n - T_{n-1}x_n\| + \cdots + \|T_{m+1}x_n - T_m x_n\| \\
&\le \|x_n - T_n x_n\| + M_B \cdot \sum_{k=m}^{n-1} a_k.
\end{aligned}$$

So, we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} \|x_n - T_m x_n\| = 0.$$

So, let $\{\gamma_m\} \subset (0,1)$ such that $\lim_{m\to\infty} \gamma_m = 0$ and $\limsup_{n\to\infty} \|x_n - T_m x_n\| \le b\gamma_m^2$ for each $m \in \mathbf{N}$, where $b \in (0,\infty)$ with $b > \sup_{m\in\mathbf{N}}\{\limsup_{n\to\infty} \|x_n - T_m x_n\|\}$ and let $\{y_m\}$ be a sequence of C such that $y_m = \gamma_m x + (1-\gamma_m)T_m y_m$ for every $m \in \mathbf{N}$.

By Theorem 4.1, $\lim_{m\to\infty} y_m = z \in F(\mathcal{T})$. Let μ be a Banach limit. Since

$$\|x_n - T_m y_m\|^2 \leq \|x_n - T_m x_n\|^2 + \|x_n - y_m\|^2 + 2\|x_n - T_m x_n\| \cdot \|x_n - y_m\| \tag{5.2}$$

for each $n, m \in \mathbf{N}$, we have

$$\begin{aligned} \mu_n\|x_n - T_m y_m\|^2 &\leq \mu_n\|x_n - y_m\|^2 \\ &\quad + \limsup_{n\to\infty}(\|x_n - T_m x_n\|^2 + 2\|x_n - T_m x_n\| \cdot \|x_n - y_m\|) \end{aligned} \tag{5.3}$$

for all $m \in \mathbf{N}$. From

$$(1-\gamma_m)(x_n - T_m y_m) = (x_n - y_m) - \gamma_m(x_n - x),$$

we obtain

$$\begin{aligned} (1-\gamma_m)^2\|x_n - T_m y_m\|^2 &\geq \|x_n - y_m\|^2 - 2\gamma_m(x_n - x, J(x_n - y_m)) \\ &= (1-2\gamma_m)\|x_n - y_m\|^2 + 2\gamma_m(x - y_m, J(x_n - y_m)) \end{aligned} \tag{5.4}$$

for every $m, n \in \mathbf{N}$. Hence, we have

$$\begin{aligned} &(1-\gamma_m)^2\mu_n\|x_n - T_m y_m\|^2 \\ &\qquad \geq (1-2\gamma_m)\mu_n\|x_n - y_m\|^2 + 2\gamma_m\mu_n(x - y_m, J(x_n - y_m)) \end{aligned}$$

for all $m \in \mathbf{N}$. By (5.3), we have

$$\begin{aligned} &(1-\gamma_m)^2\{\mu_n\|x_n - y_m\|^2 + \limsup_{n\to\infty}(\|x_n - T_m x_n\|^2 + 2\|x_n - T_m x_n\| \cdot \|x_n - y_m\|)\} \\ &\qquad \geq (1-2\gamma_m)\mu_n\|x_n - y_m\|^2 + 2\gamma_m\mu_n(x - y_m, J(x_n - y_m)) \end{aligned}$$

and hence

$$\begin{aligned} &\frac{\gamma_m}{2}\mu_n\|x_n - y_m\|^2 \\ &\quad + \frac{(1-\gamma_m)^2}{2\gamma_m}\limsup_{n\to\infty}(\|x_n - T_m x_n\|^2 + 2\|x_n - T_m x_n\| \cdot \|x_n - y_m\|) \\ &\qquad \geq \mu_n(x - y_m, J(x_n - y_m)) \end{aligned} \tag{5.5}$$

for each $m \in \mathbf{N}$. Let $\varepsilon > 0$. Since E is norm to weak* uniformly continuous on bounded subsets of E and $y_m \to z$, there exists $m_1 \in \mathbf{N}$ such that for every $m \geq m_1$,

$$|(x - z, J(x_n - z)) - (x - z, J(x_n - y_m))| < \frac{\varepsilon}{3} \tag{5.6}$$

$$|(x - z, J(x_n - y_m)) - (x - y_m, J(x_n - y_m))| < \frac{\varepsilon}{3} \tag{5.7}$$

for all $n \in \mathbf{N}$. Since $\gamma_m \to 0$ and $\limsup_{n\to\infty}\|x_n - T_m x_n\| \leq b\gamma_m^2$ $(\forall m \in \mathbf{N})$, from (5.5) there exists $m_2 \in \mathbf{N}$ such that

$$\mu_n(x - y_m, J(x_n - y_m)) < \frac{\varepsilon}{3}$$

for each $m \geq m_2$. Hence, there exists $m_0 \in \mathbf{N}$ such that for every $m \geq m_0$,

$$\begin{aligned}\mu_n(x-z, J(x_n-z)) &= \mu_n(x-z, J(x_n-z)) - \mu_n(x-z, J(x_n-y_m)) \\ &\quad + \mu_n(x-z, J(x_n-y_m)) - \mu_n(x-y_m, J(x_n-y_m)) \\ &\quad + \mu_n(x-y_m, J(x_n-y_m)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since ε is arbitrary, we have

$$\mu_n(x-z, J(x_n-z)) \leq 0.$$

Further, by $\|x_{n+1}-x_n\| \to 0$, we get

$$|(x-z, J(x_n-z)) - (x-z, J(x_{n+1}-z))| \to 0.$$

Therefore, we obtain

$$\limsup_{n\to\infty}(x-z, J(x_n-z)) \leq 0 \tag{5.8}$$

by [24, Proposition 2]. It follows from $\|\{\beta_n x + (1-\beta_n)x_n - z\} - (x_n - z)\| \to 0$ that

$$\limsup_{n\to\infty}(x-z, J(\beta_n x + (1-\beta_n)x_n - z)) \leq 0. \tag{5.9}$$

Since

$$(1-\alpha_n)\{T_n(\beta_n x + (1-\beta_n)x_n) - z\} = (x_{n+1}-z) - \alpha_n(x-z)$$

and

$$(1-\beta_n)(x_n-z) = \beta_n x + (1-\beta_n)x_n - z - \beta_n(x-z),$$

from Lemm 2.2 we have

$$(1-\alpha_n)^2\|T_n(\beta_n x + (1-\beta_n)x_n) - z\|^2 \geq \|x_{n+1}-z\|^2 - 2\alpha_n(x-z, J(x_{n+1}-z))$$

and

$$\begin{aligned}(1-\beta_n)^2\|x_n-z\|^2 \;\geq\; & \|\beta_n x + (1-\beta_n)x_n - z\|^2 \\ & -2\beta_n(x-z, J(\beta_n x + (1-\beta_n)x_n - z))\end{aligned}$$

for all $n \in \mathbf{N}$. Let $\varepsilon > 0$. By (5.8) and (5.9), there exists $n_0 \in \mathbf{N}$ such that

$$2(x-z, J(x_n-z)) < \varepsilon$$

and

$$2(x-z, J(\beta_n x + (1-\beta_n)x_n - z)) < \varepsilon$$

for every $n \geq n_0$. So, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\alpha_n)^2\|T_n(\beta_n x + (1-\beta_n)x_n) - z\|^2 + 2\alpha_n(x-z, J(x_{n+1}-z)) \\ &\leq (1-\alpha_n)^2\|\beta_n x + (1-\beta_n)x_n - z\|^2 + 2\alpha_n(x-z, J(x_{n+1}-z)) \\ &\leq (1-\alpha_n)^2\{(1-\beta_n)^2\|x_n-z\|^2 + 2\beta_n(x-z, J(\beta_n x + (1-\beta_n)x_n - z))\} \\ &\quad + 2\alpha_n(x-z, J(x_{n+1}-z)) \\ &\leq (1-\alpha_n)(1-\beta_n)\|x_n-z\|^2 + (1-\alpha_n)\beta_n\varepsilon + \alpha_n\varepsilon \\ &\leq (1-\alpha_n)(1-\beta_n)\|x_n-z\|^2 + \{1-(1-\alpha_n)(1-\beta_n)\}\varepsilon\end{aligned}$$

for every $n \geq n_0$. Hence, we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq (1-\alpha_n)(1-\beta_n)\{(1-\alpha_{n-1})(1-\beta_{n-1})\|x_{n-1} - z\|^2 \\
&\qquad + (1 - (1-\alpha_{n-1})(1-\beta_{n-1}))\varepsilon\} + \{1 - (1-\alpha_n)(1-\beta_n)\}\varepsilon \\
&= (1-\alpha_n)(1-\beta_n)(1-\alpha_{n-1})(1-\beta_{n-1})\|x_{n-1} - z\|^2 \\
&\qquad + \{1 - (1-\alpha_n)(1-\beta_n)(1-\alpha_{n-1})(1-\beta_{n-1})\}\varepsilon \\
&\leq \cdots \\
&\leq \|x_{n_0} - z\|^2 \cdot \prod_{k=n_0}^{n} (1-\alpha_k)(1-\beta_k) + \Big\{1 - \prod_{k=n_0}^{n} (1-\alpha_k)(1-\beta_k)\Big\}\varepsilon
\end{aligned}$$

for each $n \geq n_0$. Therefore, $\limsup_{n\to\infty} \|x_{n+1} - z\|^2 \leq \varepsilon$. Since ε is arbitrary, we get $x_n \to z \in F(\mathcal{T})$. Hence, we can define a mapping P of C onto $F(\mathcal{T})$ by $Px = \lim_{n\to\infty} x_n$. From Theorem 4.1, P is a sunny nonexpansive retraction of C onto $F(\mathcal{T})$. □

W remark that in Theorem 5.1, the condition (III) is replaced by the following condition: For every bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_n x - T_{n+1} x\| : x \in B\} < \infty.$$

We get the following result [24] for nonexpansive mappings by Lemma 3.1 (i) and (ii) and Theorem 5.1.

Theorem 5.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.

We have the following result [16] for nonexpansive mappings by Lemma 3.2 (ii) and Theorem 5.1.

Theorem 5.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S and T be nonexpansive mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)(\gamma_n S + (1-\gamma_n)T)(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ and $\{\gamma_n\} \subset [a, b]$

for some $a, b \in (0,1)$ *with* $a \leq b$ *satisfies* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)}x$, *where* $P_{F(S)\cap F(T)}$ *is a sunny nonexpansive retraction of* C *onto* $F(S) \cap F(T)$.

We have the following result [17] for accretive operators from Lemma 3.5 (ii) and Theorem 5.1.

Theorem 5.4. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable and let* $A \subset E \times E$ *be an accretive operator with* $\overline{D(A)} \subset C \subset \cap_{\lambda>0} R(I + \lambda A)$ *and* $A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{\lambda_n}(\beta_n x + (1 - \beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, $\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = 0$ *and* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ *and* $\{\lambda_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} \lambda_n > 0$ *and* $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x$, *where* $P_{A^{-1}0}$ *is a sunny nonexpansive retraction of* C *onto* $A^{-1}0$.

We get the following result [23] for the W-mappings by Lemma 3.6 (ii) and Theorem 5.1.

Theorem 5.5. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable. Let* $S_1, S_2, \ldots$ *be infinite nonexpansive mappings of* C *into itself with* $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ *and let* $\beta_1, \beta_2, \ldots$ *be real numbers with* $0 < \beta_i \leq b < 1$ *for every* $i \in \boldsymbol{N}$ *for some* $b \in (0, 1)$. *Let* W_n *be the* W*-mapping generated by* $S_n, S_{n-1}, \ldots, S_1$ *and* $\beta_n, \beta_{n-1}, \ldots, \beta_1$ *for every* $n \in \boldsymbol{N}$. *Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) W_n(\gamma_n x + (1 - \gamma_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ *and* $\{\gamma_n\} \subset [0, 1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0$, $\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \gamma_n) = 0$ *and* $\sum_{n=1}^{\infty}(|\alpha_n - \alpha_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_F x$, *where* P_F *is a sunny nonexpansive retraction of* C *onto* F.

We also have the following result.

Theorem 5.6. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* E *whose norm is uniformly Gâteaux differentiable and let* $\{T_n\}$ *and* $\mathcal{T}$ *be families of nonexpansive mappings of* C *into itself which satisfy* $\emptyset \neq F(\mathcal{T}) \subset \cap_{n=1}^{\infty} F(T_n)$ *and the conditions (I) and (II). Let* $\{x_n\}$ *be a sequence generated as follows:* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n(\beta_n x + (1 - \beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0, 1)$ *and* $\{\beta_n\} \subset [0, 1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and* $\prod_{n=1}^{\infty}(1 - \alpha_n)(1 - \beta_n) = 0$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(\mathcal{T})}x$, *where* $P_{F(\mathcal{T})}$ *is a sunny nonexpansive retraction of* C *onto* $F(\mathcal{T})$.

Proof. As in the proof of Theorem 5.1, we have $F(\mathcal{T}) = \cap_{n=1}^{\infty} F(T_n)$ and $\{x_n\}$ is bounded. Since

$$\|x_{n+1} - T_n x_n\| \leq \|x_{n+1} - T_n(\beta_n x + (1 - \beta_n)x_n)\|$$

$$+\|T_n(\beta_n x + (1-\beta_n)x_n) - T_n x_n\|$$
$$\leq \alpha_n\|x - T_n(\beta_n x + (1-\beta_n)x_n)\| + \beta_n\|x - x_n\|$$

for every $n \in \mathbf{N}$, we get $\lim_{n\to\infty}\|x_{n+1} - T_n x_n\| = 0$. From (II), $\lim_{n\to\infty}\|x_n - T_m x_n\| = 0$ for every $m \in \mathbf{N}$. As in the proof of Theorem 5.1, $x_n \to P_{F(\mathcal{T})}x$, where $P_{F(\mathcal{T})}$ is a sunny nonexpansive retraction of C onto $F(\mathcal{T})$. In fact, let $\{\gamma_m\} \subset (0,1)$ such that $\lim_{m\to\infty}\gamma_m = 0$ and let $\{y_m\}$ be a sequence of C generated by $y_m = \gamma_m x + (1-\gamma_m)T_m y_m$ for every $m \in \mathbf{N}$. By Theorem 4.1, $\lim_{m\to\infty} y_m = z \in F(\mathcal{T})$. From (5.2) and (5.4), we get

$$\frac{\gamma_m}{2}\|x_n - y_m\|^2 + \frac{(1-\gamma_m)^2}{2\gamma_m}(\|x_n - T_m x_n\|^2 + 2\|x_n - T_m x_n\| \cdot \|x_n - y_m\|) \geq (x - y_m, J(x_n - y_m))$$

for each $m, n \in \mathbf{N}$ which implies

$$\limsup_{n\to\infty}(x - y_m, J(x_n - y_m)) \leq \frac{\gamma_m}{2}\limsup_{n\to\infty}\|x_n - y_m\|^2$$

for all $m \in \mathbf{N}$. Let $\varepsilon > 0$. Since $\lim_{m\to\infty}\gamma_m = 0$, there exists $m_3 \in \mathbf{N}$ such that for every $m \geq m_3$,

$$\limsup_{n\to\infty}(x - y_m, J(x_n - y_m)) < \frac{\varepsilon}{3}.$$

Hence, there exists $m_4 \in \mathbf{N}$ such that

$$\begin{aligned}\limsup_{n\to\infty}(x - z, J(x_n - z)) &\leq \limsup_{n\to\infty}|(x - z, J(x_n - z)) - (x - z, J(x_n - y_m))| \\ &\quad + \limsup_{n\to\infty}|(x - z, J(x_n - y_m)) - (x - y_m, J(x_n - y_m))| \\ &\quad + \limsup_{n\to\infty}(x - y_m, J(x_n - y_m)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for each $m \geq m_4$ by (5.6) and (5.7). So, we obtain (5.8) and (5.9). Therefore, $x_n \to P_{F(\mathcal{T})}x$. □

We get the following result [14] for accretive operators by Lemma 3.5 (iii) and Theorem 5.6.

Theorem 5.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $A \subset E \times E$ be an accretive operator with $\overline{D(A)} \subset C \subset \cap_{\lambda>0}R(I + \lambda A)$ and $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)J_{\lambda_n}(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ and $\{\lambda_n\} \subset (0,\infty)$ satisfies $\lim_{n\to\infty}\lambda_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{A^{-1}0}x$, where $P_{A^{-1}0}$ is a sunny nonexpansive retraction of C onto $A^{-1}0$.

We have the following result for nonexpansive semigroups from Lemma 3.9 (ii) and Theorem 5.6.

Theorem 5.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S be a semigroup. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C such that $F := F(\mathcal{S}) \neq \emptyset$ and let D be a subspace of $B(S)$ containing constants and being invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $x^* \in E^*$, the function $t \mapsto (T(t)x, x^*)$ is in D and the mappings $t \mapsto \sup_n f_n(t)$ and $t \mapsto \inf_n f_n(t)$ are in D for each bounded sequence $\{f_n : n \in \boldsymbol{N}\}$ of D. Let $\{\mu_n\}$ be a sequence of monotone convergent means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T_{\mu_n}(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is a sunny nonexpansive retraction of C onto F.

We get the following results for nonexpansive mappings by Lemmas 3.10 (ii) and 3.11 (ii) and Theorem 5.6.

Theorem 5.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n}\sum_{i=0}^{n-1} T^i(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is a sunny nonexpansive retraction of C onto $F(T)$.

Theorem 5.10. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let S_1 and S_2 be nonexpansive mappings of C into itself such that $S_1S_2 = S_2S_1$ and $F := F(S_1) \cap F(S_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}\sum_{i+j=k} S_1^i S_2^j(\beta_n x + (1-\beta_n)x_n) \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\beta_n\} \subset [0,1)$ satisfy $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$. Then, $\{x_n\}$ converges strongly to $P_F x$, where P_F is a sunny nonexpansive retraction of C onto F.

We have the following result for one-parameter nonexpansive semigroups from Lemma 3.12 (ii) and Theorem 5.6.

Theorem 5.11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is uniformly Gâteaux differentiable and let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F := F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)(\beta_n x + (1-\beta_n)x_n)\, ds \ (\forall n \in \boldsymbol{N}),$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\{\beta_n\} \subset [0,1)$ *satisfy* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ *and* $\prod_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n) = 0$ *and* $\{t_n\} \subset (0,\infty)$ *with* $\lim_{n\to\infty} t_n = \infty$*. Then,* $\{x_n\}$ *converges strongly to* $P_F x$*, where* P_F *is a sunny nonexpansive retraction of* C *onto* F*.*

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