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# RECURRENT DIMENSIONS AND DIOPHANTINE CONDITIONS OF DISCRETE DYNAMICAL SYSTEMS GIVEN BY CIRCLE MAPPINGS

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Abstract. In this paper we study recurrent dimensions of discrete dynamical systems given by circle diffeomorphisms, using a renormalization method. We estimate the upper and the lower recurrent dimensions according to some algebraic properties of irrational rotation numbers of the circle mappings and we show that the gap values between the upper and the lower dimensions, which measure unpredictability levels of orbits, take positive values if the rotation numbers have good approximation properties by rational numbers.

## 1. Introduction

In this paper we study recurrent dimensions of discrete dynamical systems given by a circle diffeomorphism $f : S^1 \to S^1$. The rotation number of $f$ is defined by

$$\rho(f) = \lim_{n\to\infty} \frac{\hat{f}^n(x) - x}{n}$$

where $\hat{f} : \mathbf{R} \to \mathbf{R}$ is a lift of $f$ such that $\pi \circ \hat{f} = f \circ \pi, \quad \pi : \mathbf{R} \to \mathbf{R}/\mathbf{Z}(= S^1)$ is a covering map. Our purpose of this paper is to estimate the recurrent dimensions of the discrete orbits $\Sigma_x = \{f^n(x) : n \in \mathbf{N}_0\}$ according to the algebraic properties of $\rho(f)$.

The following theorem by Poincaré is well known.

**Theorem 1.1** (Poincaré, 1885). *If $f : S^1 \to S^1$ is a homeomorphism without periodic points, then there exist a rotation $R_\alpha(x) := x + \alpha (mod\ 1)$ and a continuous surjective monotone map $h : S^1 \to S^1$, which satisfies*

$$h \circ f = R_\alpha \circ h$$

*and $\alpha$ is an irrational number and equal to the rotation number of $f$. Consequently, $\rho(f)$ is independent of $x$.*

In the case of Theorem 1.1 we say that $f$ is semi-conjugate to the rotation $R_\alpha$ or $h$ is a semi-conjugacy between $f$ and $R_\alpha$. Furthermore, if $h$ is strictly monotone (one-to-one), we say that $f$ is conjugate to the rotation $R_\alpha$ or $h$ is a conjugacy between $f$ and $R_\alpha$.

If $f$ is sufficiently smooth, $f$ is conjugate to a rotation. The following theorem was given by Denjoy.

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**Theorem 1.2** (Denjoy, 1932). *If $f : S^1 \to S^1$ is $C^2$-diffeomorphism without periodic points, then $f$ is topologically conjugate to a rotation. That is, the conjugacy $h$ between $f$ and the rotation is a homeomorphism.*

The regularity of the conjugacy was studied by many authors. Here we introduce the estimate by Katznelson and Ornstein [1].

We say that $g$ is $C^{m+\delta}$-class where $m \geq 1$ is an iteger and $0 \leq \delta < 1$, if $g$ is $C^m$ and its $m$-th derivative is Hölder continuous with its exponent $\delta$.

**Theorem 1.3** (Katznelson and Ornstein, 1989). *Let $f : S^1 \to S^1$ be a $C^k$-diffeomorphism, $k > 0$, without periodic points and its rotation number $\alpha$ satisfies the Diophantine condition for $\beta \geq 0$:*

$$|\alpha - \frac{p}{q}| > \frac{C}{q^{2+\beta}} \qquad (*)$$

*for all $p/q \in \mathbf{Q}$. Then, if $\beta + 2 < k$, the conjugacy $h$ between $f$ and the rotation $R_\alpha$ is of class $C^{k-1-\beta-\varepsilon}$ for all $\varepsilon > 0$.*

In our previous papers ([7], [8], [9]) we introduce the gaps of recurrent dimensions, which are differences between the upper and the lower recurrent dimensions, as the index parameters, which measure unpredictability levels of the orbits.

In view of Theorem 1.2 and 1.3 we estimate the gaps of recurrent dimensions of the discrete orbit $\Sigma_x$, given by a $C^k$-class function $f$, in the following cases.

(I) The rotation number satisfies the assumption $\beta + 2 < k$ and the conjugacy $h$ is smooth : $C^\gamma$-class, $\gamma \geq 1$.
(II) The rotation number satisfies $2 \leq k \leq \beta + 2$ and $h$ is a homeomorphism.

Our plan of this paper is as follows. In section 2 we introduce the classifications of irrational numbers to parametrize the Diophantine condition $(*)$ and give defintions of recurrent dimensions. In section 3 we estimate the gaps of recurrent dimensions in the case (I) and in section 4 we treat the case (II). In section 5, introducing a renormalization technique and showing some fractal structures of the intervals given by the circle mapping, we prove some Lemmas, which are used to estimate the recurrent dimensions in section 4. In section 6 (Appendix) we give some numerical results on recurrent dimensions of quasi-periodic orbits given by the rotations according to the classifications of irrational rotation numbers.

## 2. Classification of irrational numbers

Let $\tau$ be an irrational number. In our previous papers ([6], [7], [8]) we introduce the following classifications according to (good or bad) levels of approximation by rational numbers.

We say that $\tau$ is an $\alpha$-order Roth number if there exists $\alpha \geq 0$ such that, for every $\beta : \beta > \alpha$, there exists a constant $c_\beta > 0$, which satisfies

$$|\tau - \frac{q}{p}| \geq \frac{c_\beta}{p^{2+\beta}}$$

for all rational numbers $q/p \in \mathbf{Q}$.

Let $\{n_i/m_i\}$ be the Diophantine approximation of $\tau$. Then we call $\tau$ an $\alpha$-order weak Liouville number if there exists a subsequence $\{m_{k_j}\} \subset \{m_j\}$, which satisfies

$$|\tau - \frac{n_{k_j}}{m_{k_j}}| < \frac{c}{m_{k_j}^{2+\alpha}}, \quad \forall j$$

for some constants $c, \alpha > 0$.

Furthermore, we can parametrize the Diophantine condition $(*)$ as follows:

Let $R(\alpha)$ be the set of $\alpha$-order Roth numbers and $wL(\beta)$ the set of $\beta$-order weak Liouville numbers. In [8] we have shown that

$$\begin{aligned} &R(\alpha) \subset R(\alpha'),\ \alpha \leq \alpha',\ wL(\beta) \subset wL(\beta'),\ \beta \geq \beta', \\ &R(\alpha)^c \subset \bigcap_{\beta<\alpha} wL(\beta), \quad wL(\beta) \subset \bigcap_{\beta>\alpha} R(\alpha)^c, \\ &R(0)^c = \bigcup_{\beta>0} wL(\beta) \end{aligned}$$

where the complements are considered in the set of all irrational numbers. Thus, for each irrational number $\tau$, there exists a constant $d_0$, which specifies the levels of (bad or good) approximable properties by rational numbers:

$$\begin{aligned} &\inf\{\alpha : \tau \text{ is an } \alpha\text{-order Roth number}\} \\ &\quad = \sup\{\beta : \tau \text{ is a } \beta\text{-order weak Liouville number}\} := d_0. \end{aligned} \tag{2.1}$$

In our previous paper ([7]) we introduced a $d_0$-(D) condition for a pair of irrational numbers. For a single irrational case, let us say that $\tau$ satisfies a $d_0$-(D) condition if (2.1) holds.

**Definitions of recurrent dimensions:**

Define the first $\varepsilon$-recurrent time by

$$M_\varepsilon(x) = \min\{m \in \mathbf{N} : |f^m(x) - x| < \varepsilon\}.$$

and the upper and lower recurrent dimensions by

$$\overline{D}_x = \limsup_{\varepsilon\to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}, \qquad \underline{D}_x = \liminf_{\varepsilon\to 0} \frac{\log M_\varepsilon(x)}{-\log \varepsilon}.$$

Then we can define the gaps of recurrent dimensions by $G_x = \overline{D}_x - \underline{D}_x$. (See [6] or [7] for further details.)

If the gap values $G_x$ take positive values, we cannot exactly determine or predict the $\varepsilon$-recurrent time of the orbits. Thus we propose the value $G_x$ as the parameter, which measures the unpredictability level of the orbit.

## 3. Smooth conjugacy case

In this section we consider the case where the conjugacy $h$ between the circle map $f$ and the rotaion is $C^\gamma$-class, $\gamma \geq 1$. First we note that the metric in $S^1$ is induced by the covering (quotient) map $\pi : \mathbf{R} \to S^1$ such that

$$|x - y| := \inf_{m\in\mathbf{Z}} |x - y - m|, \quad x, y \in S^1$$

where we use the same notation as that of usual absolute values as far as not being confused.

**Theorem 3.1.** *Let $f : S^1 \to S^1$ be a $C^3$-diffeomorphism without periodic points and its rotation number $\alpha$ satisfies the $d_0$-(D) condition for $0 \le d_0 < 1$. Then, for each $x \in S^1$, we have*

$$\underline{D}_x \le \frac{1}{1+d_0}, \quad \overline{D}_x \ge 1.$$

*Consequently, we have*

$$G_x \ge 1 - \frac{1}{1+d_0}.$$

*Proof.* Since the Diophantine condition $(*)$ in Theorem 1.3 is satisfied with $\beta = 1 - \varepsilon_0 > d_0$ for some sufficiently small $\varepsilon_0 > 0$, the conjugacy $h$ is $C^{1+\varepsilon_0-\varepsilon}$-class for every $\varepsilon > 0$. Thus we can admit $C^1$-conjugacy $h : h \circ f = R_\alpha \circ h$. Since $f^n(x) = h^{-1} \circ R_\alpha^n \circ h$ and Lipschitz continuity conditions of $h$ and $h^{-1}$, which are given by the Mean Value Theorem, such that

$$C_1|x-y| \le |h(x) - h(y)| \le C_2|x-y|, \quad x, y \in S^1 : |x-y| \le \frac{1}{2}$$

for some $C_2 > C_1 > 0$, we can take an integer $m$ :

$$\begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\le C_1^{-1}|\alpha n - m|, \\ |\alpha n - m| &\le \frac{1}{2}, \end{aligned} \tag{3.1}$$

and also an integer $m'$ :

$$\begin{aligned} |f^n(x) - x| &= |h^{-1} \circ R_\alpha^n \circ h(x) - (h^{-1} \circ h)(x)| \\ &\ge C_2^{-1}|\alpha n - m'|, \\ |\alpha n - m'| &\le \frac{1}{2}. \end{aligned} \tag{3.2}$$

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number $\alpha$ of $f$. It follows from $d_0$-(D) condition that for every $\varepsilon > 0$ there exists a subsequence $\{p_{k_j}\}$ such that

$$|\alpha p_{k_j} - q_{k_j}| \le \frac{c}{p_{k_j}^{1+d_0-\varepsilon}}, \quad \forall j.$$

Thus we have

$$|f^{p_{k_j}}(x) - x| \le \frac{cC_1^{-1}}{p_{k_j}^{1+d_0-\varepsilon}} := \varepsilon_j.$$

It follows from the definition that we can estimate lower recurrent dimension

$$\begin{aligned} \underline{D}_x &= \liminf_{\varepsilon \to \infty} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\ &= \liminf_{j \to \infty} \inf_{\varepsilon_{j+1} \le \varepsilon \le \varepsilon_j} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\ &\le \liminf_{j \to \infty} \frac{\log M(\varepsilon_j)}{-\log \varepsilon_j} \end{aligned}$$

$$\begin{aligned} &\leq \lim_{j\to\infty} \frac{\log p_{k_j}}{-\log c + \log C_1 + (1+d_0-\varepsilon)\log p_{k_j}} \\ &= \frac{1}{1+d_0-\varepsilon} \end{aligned}$$

for every $\varepsilon > 0$.

Next we show the lower estimate. Here we use the following elementary property of the Dophantine sequence that

$$\frac{1}{p_k(p_{k+1}+p_k)} < |\alpha - \frac{q_k}{p_k}| < \frac{1}{p_k p_{k+1}} < \frac{1}{p_k^2} \tag{3.3}$$

and

$$\inf_{r\in\mathbf{N}} |\alpha n - r| \geq |\alpha p_k - q_k| \tag{3.4}$$

holds for every $n : 1 \leq n < p_{k+1}$. It follows from (3.2) that we have

$$|f^n(x) - x| \geq C_2^{-1}|\alpha p_k - q_k| \geq \frac{1}{2C_2 p_{k+1}} := \varepsilon_k$$

for every $n : 1 \leq n < p_{k+1}$. Thus we can estimate the upper recurrent dimension

$$\begin{aligned} \overline{D}_x &= \limsup_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &= \limsup_{k\to\infty} \sup_{\varepsilon_{k+1}\leq\varepsilon\leq\varepsilon_k} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\ &\geq \limsup_{k\to\infty} \frac{\log M(\varepsilon_k)}{-\log\varepsilon_k} \\ &\geq \lim_{k\to\infty} \frac{\log p_{k+1}}{\log 2C_2 + \log p_{k+1}} = 1 \end{aligned}$$

and from the definition of the gap values we obtain the conclusion. □

## 4. Topological conjugate case

Next we consider the case (II). $f$ has a unique invariant probability measure $\mu$, defined by $\mu(A) = \lambda(h(A))$ where $h$ is the conjugacy between $f$ and the rotation and $\lambda$ is a Lebesgue measure.

Let $\{q_k/p_k\}$ be the Diophantine sequence of the rotation number $\alpha$ of $f$ and denote

$$\begin{aligned} m_k(x) &= |f^{p_k}(x) - x|, \\ \alpha_k &= |p_k\alpha - q_k|, \end{aligned}$$

then we consider the subsets $A,\ B$ of $S^1$, defined by

$$\begin{aligned} A &= \{x \in S^1 : \limsup_{k\to\infty} \frac{m_k(x)}{\alpha_k} > 0\}, \\ B &= \{x \in S^1 : \limsup_{k\to\infty} \frac{\alpha_k}{m_k(x)} > 0\}. \end{aligned}$$

We note that

$$\alpha_k = \int_{S^1} m_k(x) d\mu(x) \tag{4.1}$$

(see [3]).

We can estimate the measure of these subsets:

**Lemma 4.1.** *Let $f : S^1 \to S^1$ be a $C^2$-diffeomorphism. Then we have*

$$\lambda(A) = \lambda(B) = 1. \tag{4.2}$$

*Proof.* Since $f$ and $f^{-1}$ is differentiable, it follows from the Mean Value Theorem that

$$K_1|x - y| \leq |f(x) - f(y)| \leq K_2|x - y|, \quad x, y \in S^1 : |x - y| \leq \frac{1}{2},$$

for some $K_2 > K_1 > 0$. Thus we can easily show that

$$x \in A \Longleftrightarrow f(x) \in A,$$
$$x \in B \Longleftrightarrow f(x) \in B.$$

Since $f$ is ergodic (cf.[3]), the invariant sets $A$ and $B$ have full measures or null measures.

First we show that $\lambda(A) > 0$. Define

$$A_0 = \{x \in S^1 : \limsup_{k\to\infty} \frac{m_k(x)}{\alpha_k} > c_0\}$$

for sufficiently small $c_0 > 0$ and assume that $\mu(A_0) = 0$, that is, $\mu(A_0^c) = 1$. Since

$$A_0^c = \{x \in S^1 : \limsup_{k\to\infty} \frac{m_k(x)}{\alpha_k} \leq c_0\},$$

for $x \in A_0^c$ there exists a small constant $\varepsilon_0 > 0$ and a large number $k_0$ such that, if $k \geq k_0$,

$$\frac{m_k(x)}{\alpha_k} \leq c_0 + \varepsilon_0.$$

It follows from (4.1) that we have a contradiction:

$$\alpha_{k_0} = \int_{A_0^c} m_{k_0}(x) d\mu(x) \leq (c_0 + \varepsilon_0)\alpha_{k_0}\mu(A_0^c) = (c_0 + \varepsilon_0)\alpha_{k_0}.$$

Thus we obtain $\mu(A) > \mu(A_0) > 0$. Since $h$ is a homeomorphism, we have $\lambda(A) > 0$. It follows from ergodicity of $f$ that we have $\lambda(A) = 1$.

For the set $B$ we can show the conclusion similary as follows. Denote

$$B_0 = \{x \in S^1 : \limsup_{k\to\infty} \frac{\alpha_k}{m_k(x)} > c_0'\}$$

for some small $c_0' > 0$ and assume that $\mu(B_0^c) = 1$. If $x \in B_0^c$, there exists a sufficiently small constant $\varepsilon_0' > 0$ and a large number $k_0'$ such that

$$\alpha_{k_0'} \leq (c_0' + \varepsilon_0')m_{k_0'}(x).$$

Thus we have a contradiction,

$$\alpha_{k_0'} = \int_{B_0^c} m_{k_0'}(x) d\mu(x) \geq \frac{\alpha_{k_0'}}{c_0' + \varepsilon_0'},$$

which gives $\mu(B) > \mu(B_0) > 0$. Applying the previous argument, we can conclude that $\lambda(B) = 1$. □

*Remark* 4.2. It is known that the circle mapping $f$ is conjugate to an irrational rotation if and only if its minimal invariant set (a non-empty compact invariant set which is minimal) is equal to $S^1$. Thus we can easily show that the invariant subsets $A, B$ are dense in $S^1$.

**Theorem 4.3.** *Let $f : S^1 \to S^1$ be a $C^2$-diffeomorphism without periodic points and its rotation number $\alpha$. Then we have*

$$\overline{D}_x \geq 1, \quad a.e. \ \ x \in S^1. \tag{4.3}$$

*Proof.* Let $x \in A$. Then there exists a constant $c_1 > 0$ and subsequences $\{\alpha_{k_j}\}$, $\{m_{k_j}(x)\}$, which satisfy

$$m_{k_j}(x) \geq c_1 \alpha_{k_j}, \quad \forall j \geq j_1$$

for a sufficiently large $j_1$. By Lemma 5.2 in the next section there exists a constant $b_0 : 0 < b_0 < 1$ such that

$$|f^n(x) - x| \geq b_0 m_{k_j}(x) \tag{4.4}$$

holds for every $n < p_{k_j+1}$. It follows from the property of Diophantine sequence that we have

$$\alpha_{k_j} \geq \frac{c_2}{p_{k_j+1}}$$

for some $c_2 > 0$. Thus, by putting

$$|f^n(x) - x| \geq \frac{b_0 c_1 c_2}{p_{k_j+1}} := \varepsilon_j,$$

we can estimate

$$\begin{aligned}
\overline{D}_x &= \limsup_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&= \limsup_{j\to\infty} \sup_{\varepsilon_{j+1}\leq\varepsilon\leq\varepsilon_j} \frac{\log M(\varepsilon)}{-\log\varepsilon} \\
&\geq \limsup_{j\to\infty} \frac{\log M(\varepsilon_j)}{-\log\varepsilon_j} \\
&\geq \lim_{j\to\infty} \frac{\log p_{k_j+1}}{-\log b_0 c_1 c_2 + \log p_{k_j+1}} = 1.
\end{aligned}$$

□

**Theorem 4.4.** *Let $f : S^1 \to S^1$ be a $C^2$-diffeomorphism without periodic points and its rotation number $\alpha$ satisfies the $d_0$-(D) condition for $d_0 > 0$. Then we have*

$$\underline{D}_x \leq \frac{1}{1 + d_0}, \quad a.e. \ \ x \in S^1. \tag{4.5}$$

*Consequently, we can estimate the gap values by*

$$G_x \geq \frac{d_0}{1 + d_0}, \quad a.e. \ \ x \in S^1.$$

*Proof.* Let $x \in B$, then there exists a constant $c_0 > 0$ which satisfies

$$m_k(x) \le c_0 \alpha_k, \quad \forall k \ge k_0$$

for a sufficiently large $k_0$. From the $d_0$-(D) condition, it follows that, for each $\beta : 0 < \beta < d_0$, we can admit a subsequence $\{p_{k_j}\}$ and a constant $c_\beta$ such that

$$|p_{k_j}\alpha - q_{k_j}| \le \frac{c_\beta}{p_{k_j}^{1+\beta}}.$$

Putting

$$\varepsilon_j := \frac{c_0 c_\beta}{p_{k_j}^{1+\beta}},$$

we note that

$$|f^{p_{k_j}}(x) - x| \le \varepsilon_j \quad \forall j \ge j_0$$

for a sufficiently large $j_0$. Thus we can estimate

$$\begin{aligned}
\underline{D}_x &= \liminf_{\varepsilon\to\infty} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&= \liminf_{j\to\infty} \inf_{\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j} \frac{\log M(\varepsilon)}{-\log \varepsilon} \\
&\le \liminf_{j\to\infty} \frac{\log M(\varepsilon_j)}{-\log \varepsilon_j} \\
&\le \lim_{j\to\infty} \frac{\log p_{k_j}}{-\log c_0 c_\beta + (1+\beta)\log p_{k_j}} \\
&= \frac{1}{1+\beta}
\end{aligned}$$

for every $\beta < d_0$.

Considering $x$ in the set $A \cap B$, which is also of full measure, we can estimate the gap value $G_x$. □

## 5. Renormalization method

In this section, following the notations and arugment in [3]and applying some renormalizatin techniques, we prove Lemma, which is used in the proof of Theorem 4.3.

Denote a subinterval $[a, b]$ in $[0, 1]$ by $J$. Define the space $\mathcal{S}(J)$ of maps $g : J \to J$, which satisfies the following conditions: $g(a) = g(b) \in (a, b)$, there exsits a discontinuity point $c \in (a, b)$ such that $g$ is continuous and monotone increasing on $[a, c)$ and $(c, b]$, and

$$\lim_{t\uparrow c} g(t) = b, \quad \lim_{t\downarrow c} g(t) = a.$$

For $J = [a, b]$, denote $J' := (a, c)$, $J'' := (c, b)$, then we can admit the following two cases:

$$\text{(i) } g(J') \subset J'', \quad J' \subset g(J''),$$
$$\text{(ii) } g(J'') \subset J', \quad J'' \subset g(J')$$

The case (i): For $g \in \mathcal{S}(J)$, since $g$ is monotone increasing on $J', J''$, we define

$$a(g) = \max\{k \in \mathbf{N} : \ g^i(J') \subset J'', \quad \forall i = 1, ..., k\}.$$

Then we have the ordered intervals

$$J', g(J'), ..., g^{a(g)}(J'),$$

and $g^{a(g)+1}(J') \cap J' \neq \emptyset$. Furthermore, the closure of $J' \cup g(J') \cup \cdots \cup g^{a(g)+1}(J')$ covers the closed interval $J$.

We define the first return map $\mathcal{R}(g) : K \to K$ of $g$ to $K$ for an interval $K \subset J$ by

$$\mathcal{R}(g)(x) = g^k(x)$$

where $k = k(x) = \min\{i \in \mathbf{N} : \ g^i(x) \in K\}$. Denote $J(g) = \overline{J' \cup g^{a(g)+1}(J')}$, then $\mathcal{R}(g)$ of $g$ to $J(g)$ is in $\mathcal{S}(J(g))$ and

$$\mathcal{R}(g)(J'' \cap J(g)) \subset J' = J' \cap J(g),$$

and

$$\mathcal{R}(g)|_{J'} = (g|_{J''})^{a(g)} \circ (g|_{J'}), \quad \mathcal{R}(g)|_{J'' \cap J(g)} = g|_{J''}.$$

For the case (ii) we can define the number $a(g)$ similarly.

Now, using the circle mapping $f : S^1 \to S^1$, which has no periodic points, we inductively define the renormalization sequences of intervals $\{J_n\}$ and of the return maps $\{\varphi_n\} : \varphi_n \in \mathcal{S}(J_n)$ and of the numbers $\{a_n\}$, which determine the continued fraction expantion for the rotation number of $f$.

Define $J_0 = [0, 1] := I$, $\varphi_0 : J_0 \to J_0$, $\varphi_0 = f$ and denote the interioir of the right component of $J_0 \backslash \{c\}$ by $J_0'$ and the interioir of the other component of $J_0 \backslash \{c\}$ by $J_0''$ and

$$a_1 = \begin{cases} a(f) + 1 & \text{if } \ f(J_0') \subset J_0'' \\ 1 & \text{if } \ J_0' \supset f(J_0''), \end{cases}$$

$$J_1 = \begin{cases} J(\varphi_0) & \text{if } \ f(J_0') \subset J_0'' \\ I & \text{if } \ J_0' \supset f(J_0''), \end{cases}$$

and

$$\varphi_1 = \begin{cases} \mathcal{R}(f) & \text{if } \ f(J_0') \subset J_0'' \\ f & \text{if } \ J_0' \supset f(J_0''). \end{cases}$$

Now suppose that $n \geq 2$ and $J_1, ..., J_{n-1}, \varphi_1, ..., \varphi_{n-1}$ are defined and that $\varphi_{n-1} : J_{n-1} \to J_{n-1}$ has no fixed points, then we inductively define the interval $J_n$, the return map $\varphi_n$ to $J_n$ and the integer $a_n$ by

$$J_n = J(\varphi_{n-1}), \quad \varphi_n = \mathcal{R}(\varphi_{n-1}) : J_n \to J_n,$$
$$a_n = a(\varphi_{n-1}).$$

On the other hand, if $\varphi_{n-1} : J_{n-1} \to J_{n-1}$ has fixed points, then we must let $a_n = +\infty$ and stop the inductive process, but, since we assume that $f$ has no periodic points, that is, $\varphi_n$ has no fixed points, we can define each sequence infinitely.

Thus, if $f(J_0') \subset J_0''$,

$$a_1 = a(f) + 1, \qquad \varphi_1 = \mathcal{R}(f),$$
$$a_n = a(\mathcal{R}^{n-1}(f)), \quad \varphi_n = \mathcal{R}^n(f), \quad n = 2, 3, ...$$

and, if $J_0' \supset f(J_0'')$,

$$a_1 = 1, \quad \varphi_1 = f,$$
$$a_n = a(\mathcal{R}^{n-2}(f)), \quad \varphi_n = \mathcal{R}^{n-1}(f), \quad n = 2, 3, ....$$

Let $J_n'$ be the interior of the left component of $J_n \backslash \{c\}$ if $n$ is odd and of the right component if $n$ is even. Let $J_n''$ be the interior of the other component of $J_n \backslash \{c\}$. Then we have

$$J_n' = J_{n-1}'' \cap J_n, \quad J_n'' = J_{n-1}' \cap J_n = J_{n-1}'$$

and $\varphi_n(J_n') \subset J_n''$ for all $n \geq 1$. Also we have $\varphi_1|_{J_1'} = f, \quad \varphi_1|_{J_1''} = f^{a_1}$ and

$$\varphi_n|_{J_n'} = \varphi_{n-1}|_{J_{n-1}''},$$
$$\varphi_n|_{J_n''} = (\varphi_{n-1}|_{J_{n-1}''})^{a(\varphi_{n-1})} \circ (\varphi_{n-1}|_{J_{n-1}'}).$$

Therefore by induction we have

$$\varphi_n|_{J_n'} = f^{p_{n-1}}, \quad \varphi_n|_{J_n''} = f^{p_n}$$

where $p_n$ is defined inductively by

$$p_0 = 1, \quad p_1 = a_1,$$
$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad \text{for} \quad n \geq 1.$$

If $n$ is even, we have

$$J_n = [f^{p_{n-1}}(c), f^{p_n}(c)], \quad J_n' = (c, f^{p_n}(c)), \quad J_n'' = (f^{p_{n-1}}(c), c)$$

and, if $n$ is odd,

$$J_n = [f^{p_n}, f^{p_{n-1}}(c)], \quad J_n' = (f^{p_n}(c), c), \quad J_n'' = (c, f^{p_{n-1}}(c)).$$

In [3] the fractal tiling structures of these intervals were introduced. Here we show the self-similar (fractal) structures of the tiling intervals by giving these intervals directly and successively.

**Lemma 5.1** ([3])**.** *Under the above setting it holds for* $0 \leq j \leq p_{n+1}$ *that*

$$f^j(c) \in J_n \Longleftrightarrow j = ip_n + p_{n-1}, \quad i \in \{0, ..., a_{n+1}\} \tag{5.1}$$

*and also the interval* $J$ *has the fractal tiling structures such that*

$$J = cl[\{\bigcup_{i=0}^{p_{n-1}-1} f^i(J_n')\} \bigcup \{\bigcup_{i=0}^{p_n-1} f^i(J_n'')\}] \tag{5.2}$$

*where all intervals in the union are disjoint.*

*Proof.* We consider the case $n$ is even, since we can show the odd case similarly. Let us start with $J_n'' = (f^{p_{n-1}}(c), c), \quad J_n' = (c, f^{p_n}(c))$. On the right side of $J_n'$ the intervals are successively preceded as follows:

$$(c, f^{p_n}(c)), (f^{p_n}(c), f^{(a_n-1)p_{n-1}+p_{n-2}}(c)) = f^{(a_n-1)p_{n-1}+p_{n-2}}(J_n''),$$
$$(f^{(a_n-1)p_{n-1}+p_{n-2}}(c), f^{(a_n-2)p_{n-1}+p_{n-2}}(c)) = f^{(a_n-2)p_{n-1}+p_{n-2}}(J_n''), ...,$$
$$..., (f^{2p_{n-1}+p_{n-2}}(c), f^{p_{n-1}+p_{n-2}})(c), (f^{p_{n-1}+p_{n-2}}(c), f^{p_{n-2}}(c)) = f^{p_{n-2}}(J_n''),$$

that is,

$$J_n'', c, J_n', f^{(a_n-1)p_{n-1}+p_{n-2}}(J_n''), f^{(a_n-2)p_{n-1}+p_{n-2}}(J_n''), ..., f^{p_{n-2}}(J_n'')$$

and we denote

$$\begin{aligned} J_n^{(3)} &= J'_n \cup \{f^{p_n}(c)\} \cup f^{(a_n-1)p_{n-1}+p_{n-2}}(J''_n) \cup \{f^{(a_n-1)p_{n-1}+p_{n-2}}(c)\} \\ &\quad \cup f^{(a_n-2)p_{n-1}+p_{n-2}}(J''_n) \cup \cdots \cup \{f^{p_{n-1}+p_{n-2}}(c)\} \cup f^{p_{n-2}}(J''_n), \end{aligned}$$

that is,

$$J_n^{(3)} = (c, f^{p_{n-2}}(c)).$$

Similarly, we can consider the following intervals on the left side of $J''_n$.

$$f^{p_{n-3}}(J_n^{(3)}), f^{p_{n-1}-(a_n-1)p_{n-2}}(J_n^{(3)}), ..., f^{p_{n-1}-p_{n-2}}(J_n^{(3)}), J''_n$$

and denote

$$\begin{aligned} J_n^{(4)} &= f^{p_{n-3}}(J_n^{(3)}) \cup \{f^{p_{n-1}-(a_n-1)p_{n-2}}(c)\} \cup f^{p_{n-1}-(a_n-1)p_{n-2}}(J_n^{(3)}) \\ &\quad \cup \cdots \cup \{f^{p_{n-1}-p_{n-2}}(c)\} \cup f^{p_{n-1}-p_{n-2}}(J_n^{(3)}) \cup \{f^{p_{n-1}}(c)\} \cup J''_n \\ &= (f^{p_{n-3}}(c), c). \end{aligned}$$

Successively, we can define the sequences $\{J_n^{(2j)}\}$, $\{J_n^{(2j-1)}\}, j = 2, 3, ..., n/2$. Note that $p_0 = 1, \lim_{s\to c-0} f(s) = f(c-) = 1$, then, we can enlarge the intervals successively and finally, we can cover the interval $(c, 1) = (c, f(c-))$

$$(c, 1) = (c, f(c-)) = f^{p_0}(J_n^{(n)}).$$

Put $J_n^{(0)} = f^{p_0}(J_n^{(n)})$, then we can put the intervals on the left side of $(f^{p_1}(c), c)$ successively:

$$f^{p_1-(a_1-1)p_0}(J_n^{(0)}), f^{p_1-(a_1-2)p_0}(J_n^{(0)}), ..., f^{p_1-p_0}(J_n^{(0)}), (f^{p_1}(c), c).$$

Since $p_1 - (a_1 - 1)p_0 = p_0 + p_{-1}$, $p_{-1} = 0$ and $\lim_{s\to c+0} f(s) = f(c+) = 0$, we have

$$f^{p_1-(a_1-1)p_0}(J_n^{(0)}) = f^1(J_n^{(0)}) = (f(c+), f^2(c-)) = (0, f^2(c-)).$$

Thus we can cover the interval $(0, c)$. □

Now we can show the following lemma.

**Lemma 5.2.** *Under the same Hypotheses as Theorem 4.3 there exists a constant* $b_0 : 0 < b_0 < 1$ *such that*

$$|f^j(x) - x| \geq b_0 m_n(x)$$

*holds for every* $j < p_{n+1}$.

*Proof.* For the circle mapping $f$ and each $x \in S_1$ we can construct the renormalization sequence with $c = x$. We use an improved version of Denjoy's inequality (see Lemma 3.4 in [3]):

For a sufficiently small $\varepsilon_0 > 0$ there exists a number $n_0$ such that, if $n \geq n_0$, then

$$\|Df^{p_n} - 1\| \leq \varepsilon_0 \quad (5.3)$$

holds where $\|\cdot\|$ denotes the usual supremum norm of the space of continuous functions.

Following the definitions of the first return maps by the renormalization method, we can consider the case

$$f^{p_n(a_{n+1}-1)+p_{n-1}}(c) < f^{p_{n+1}}(c) < c < f^{p_n}(c). \quad (5.4)$$

Fix a constant $b_0 : 0 < b_0 < 1 - 2\varepsilon_0$. In view of (5.1) and (5.2) it is sufficient to show that

$$|f^{p_n(a_{n+1}-1)+p_{n-1}}(c) - c| \geq b_0 |f^{p_n}(c) - c|.$$

Assume that

$$|f^{p_n(a_{n+1}-1)+p_{n-1}}(c) - c| < b_0 |f^{p_n}(c) - c|. \tag{5.5}$$

Using the Mean Value Theorem, we can take some $\xi \in (c, f^{p_n}(c))$, which satisfies

$$|f^{p_n(a_{n+1}-1)+p_{n-1}}(f^{p_n}(c)) - f^{p_n(a_{n+1}-1)+p_{n-1}}(c)| = |Df^{p_n(a_{n+1}-1)+p_{n-1}}(\xi)||f^{p_n}(c) - c|.$$

It follows from (5.3) that we can estimate

$$\begin{aligned} |f^{p_{n+1}}(c) - f^{p_n(a_{n+1}-1)+p_{n-1}}(c)| &= |Df^{p_n(a_{n+1}-1)+p_{n-1}}(\xi)||f^{p_n}(c) - c| \\ &\geq (1 - \varepsilon_0)|f^{p_n}(c) - c| \\ &> b_0 |f^{p_n}(c) - c|. \end{aligned}$$

By using (5.5) we can obtain $c < f^{p_{n+1}}(c)$, which contradicts (5.4). □

## 6. Appendix: numerical calculations

Since the definitions of recurrent dimensions are simple, it is easy to calculate these dimensions numerically. Here we give some numerical results on the recurrent dimensions of quasi-periodic orbits given by the rotation $R_\alpha(x) = x + \alpha \pmod 1$. (We treat the case of usual circle diffeomorphisms in our forthcoming paper.) By using "Mathematica" we investigate the following cases of the rotation number $\alpha$:

$$(1)\ \frac{\sqrt{5}+1}{2} \quad (2)\ \sqrt{2} \quad (3)\ [0, 2, 2, 2, 2^5, 2, 2] \quad (4)\ [0, 2, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, 2^{2^6}]$$

where the notations $[\cdot, \cdot, \cdots]$ are continued fraction expansions. The case (1) Fibonacci number is $[1, 1, 1, 1, \cdots]$, $\sqrt{2} = [1, 2, 2, 2, 2, \cdots]$ and these two numbers are in the "constant type" irrational class or called "badly approximable" such that $d_0 = 0$. We choose the numbers of (3) and (4) as examples of an $\alpha$-order weak Liouville number or an $\alpha$-order Roth number. Since it follows from the definitions that the order $\alpha$ is given by $m_{j+1} \simeq m_j^{1+\alpha}$ (see [8] for details), we can numerically calculate the values of $d_0$ by estimating $(\log m_{j+1}/\log m_j) - 1$ as follows: (3) $d_0 = 1.39992$ (4) $d_0 = 4.04439$.

Since the upper and lower recurrent dimensions are given by

$$\begin{aligned} \overline{D}_x &= \limsup_{k\to\infty} \sup_{\varepsilon_{k+1} \leq \varepsilon \leq \varepsilon_k} \frac{\log M(\varepsilon)}{-\log \varepsilon}, \\ \underline{D}_x &= \liminf_{j\to\infty} \inf_{\varepsilon_{j+1} \leq \varepsilon \leq \varepsilon_j} \frac{\log M(\varepsilon)}{-\log \varepsilon} \end{aligned}$$

(the proof of these relations was given in [7]), the asymptotic behavior of the sequence $\{D_k\}$ defined by

$$D_k = \frac{\log M(\varepsilon_k)}{-\log \varepsilon_k}$$

is most strongly related to the gap values of recurrent dimensions. We calculate the recurrent dimensions of the orbits given by the rotation $R_\alpha(x)$ as follows:

Let

$$x[n] := n\alpha \pmod 1, \quad E[n] := |x[n] - x[1]|, \quad n = 1, ..., M_1,$$

then, define

$$e[i] := c^{-i}, \quad c > 1,$$
$$m[i] := \min\{n : e[i+1] < E[n] < e[i], \quad n = 2, ..., M_1\}, \quad i = 1, ..., M_2$$

where we estimate the minimum values by using double loops such that "Do" and "If" for $n = 2, ..., M_1$ in the loop "Do" for $i = 1, ..., M_2$, not using "Min", but using "Break". Define

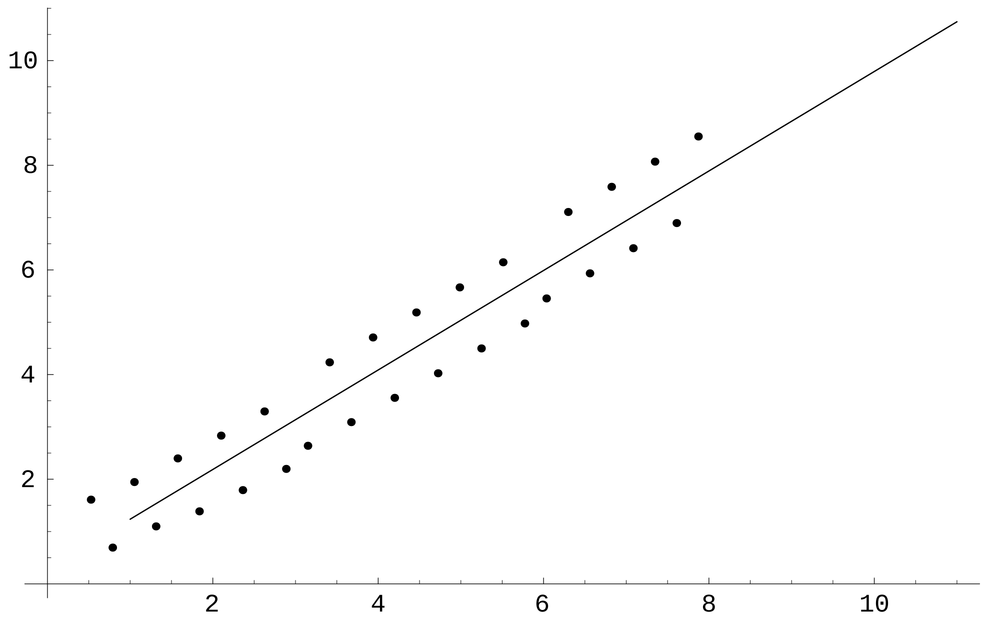
$$X[i] := -\log e[i], \quad Y[i] := \log m[i],$$

then we apply a linear regression command "Fit" to the data list $\{(X[i], Y[i]) : i = 1, ..., M_2\}$. Then we consider the slope of the line as one kind of mean values between upper and lower recurrent dimensions. We obtain the following results by taking the constants as

$$M_1 = 50000, \quad M_2 = 24 \sim 30, \quad c = 1.3$$

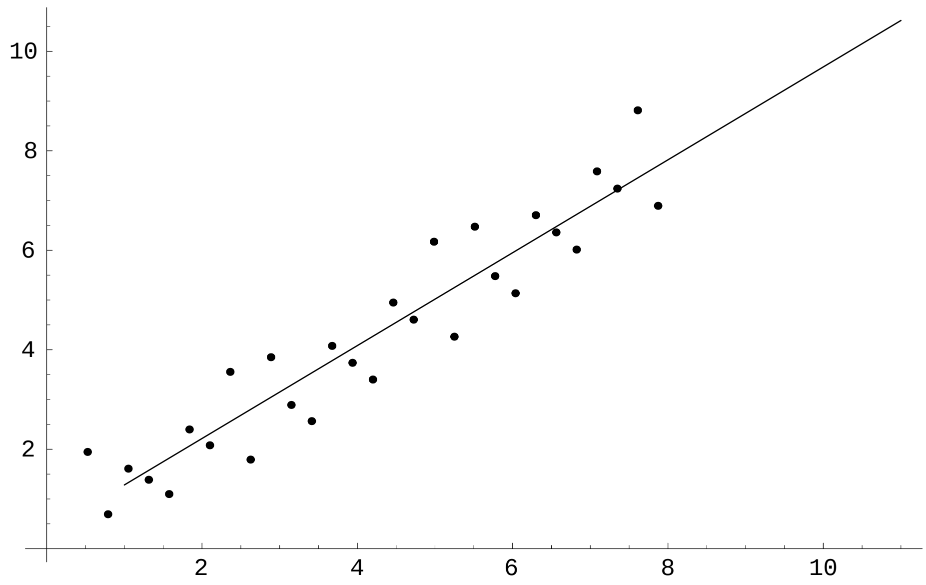
for each rotation number $\alpha$ from the case (1) to (4).

(1) $\alpha = \dfrac{\sqrt{5}+1}{2}$



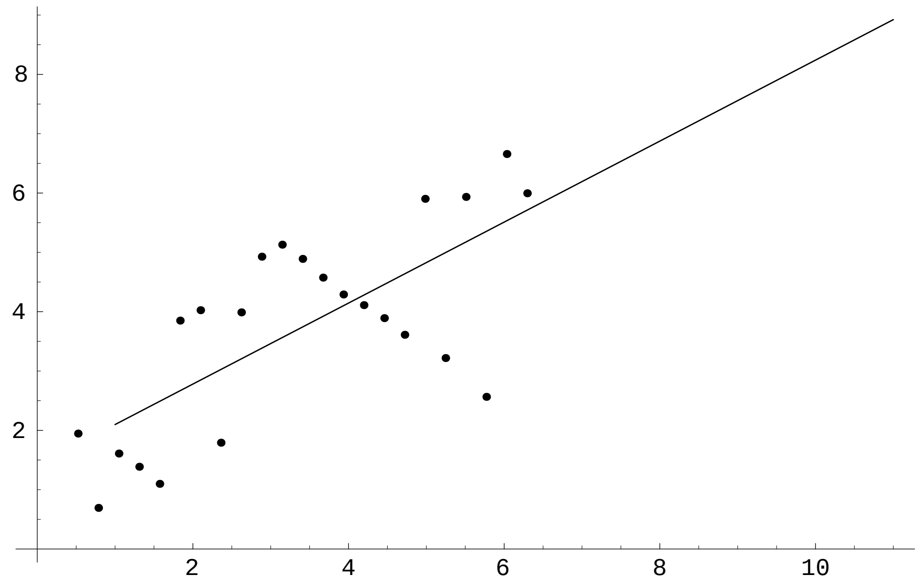

Fit: 0.284687 + 0.950747 x

(2) $\sqrt{2}$



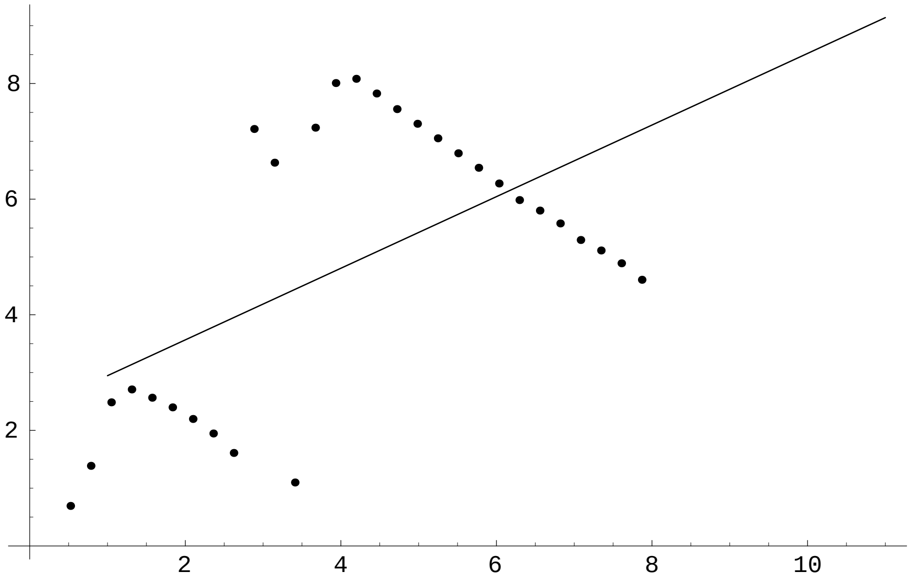

Fit: 0.348277 + 0.933742 x

(3) $[0, 2, 2, 2, 2^5, 2, 2]$




Fit: 1.41526 + 0.682509 x

(4) $[0, 2, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, 2^{2^6}]$




Fit: 2.32644 + 0.619417 x

According to the cases from (1) to (4) we can obtain the table of values $d_0$, $G_0 = d_0/(1+d_0)$, $D$: the slope of the line and $G_1$: the gap value of recurrent dimensions obtained by $D$ where we assume that $D = (\overline{D} + \underline{D})/2$ and $\overline{D} = 1$, then we have $G_1 = \overline{D} - \underline{D} = 2(1 - D)$.

**Table: Gaps of Recurrent Dimensions**

| Rot. num. | $d_0$ | $G_0$ | $D$ | $G_1$ |
|---|---|---|---|---|
| $(\sqrt{5}+1)/2$ | 0 | 0 | 0.950747 | 0.0985069 |
| $\sqrt{2}$ | 0 | 0 | 0.933742 | 0.132516 |
| $[0, 2, 2, 2, 2^5, 2, 2]$ | 1.39992 | 0.583319 | 0.682509 | 0.634982 |
| $[0, 2, 2^{2^2}, 2^{2^3}, 2^{2^4}, 2^{2^5}, 2^{2^6}]$ | 4.04439 | 0.80176 | 0.619417 | 0.761167 |

For the cases (1) and (2) the gap values $G_1$ are almost equal to 0.1, which almost matches the lower estimate $G_0$ of the gap values given in Theorem 3.1. On the other hand we can admit the positive gap values for the case (3) and (4). Applying

Theorem 3.1, we can show that $\overline{D} \geq 1$ holds. Under one another considerable assumption that $\underline{D} \sim D$ we can estimate the gap values as follows: $0.32 \sim$ for the case (3) and $0.38 \sim$ for the case (4).

We can expect that the gap values are deeply related to the variance of these numerical values of $D$. In the forthcoming paper we will try further numerical analysis on the gap values of recurrent dimensions by using Linear Regression Analysis for the case of the various diffeomorphisms.

## References


[1] Y. Katznelson and D. Ornstein, *The absolute continuity of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory Dynam. Systems 9 (1989), 681-690.

[2] Y. A. Khinchin, Continued Fractions, The University of Chicago Press 1964. 28 # 5037

[3] W. de Melo and S. van Strien, One-Dimensional Dynamics, Springer, Berlin, 1993.

[4] K. Naito, *Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation*, J. Differential Equations 141 (1997), 179-200.

[5] K. Naito, *Correlation dimensions of quasi-periodic orbits with frequencies given by quasi Roth numbers*, J. Korean Math. Soc. 37 (2000), 857-870.

[6] K. Naito, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations*, Trans. Amer. Math. Soc. 354 (2002), 1137-1151.

[7] K. Naito, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations II: Gaps of dimensions and Diophantine conditions*, Discrete Contin. Dyn. Syst. 11 (2004), 449-488.

[8] K. Naito, *Classifications of Irrational Numbers and Recurrent Dimensions of Quasi-Periodic Orbits*, J. Nonlinear Convex Anal. 5 (2004), 169-185.

[9] K. Naito, *Recurrent dimensions of quasi-periodic orbits with multiple frequencies: Extended common multiples and Diophantine conditions*, in Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis (Tokyo 2003), W. Takahashi, T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2004, pp. 367-380.

[10] D. Rand, S. Östlund, J. Sethna and E. Siggia, *Universal properties of the transition from quasiperiodicity to chaos in dissipative systems*, Phys. D 8 (1983), 303-342.

[11] W. M. Schmidt, Diophantine Approximation, Springer Lecture Notes in Math. 785, 1980.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