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MOSCO STABILITY OF PROXIMAL MAPPINGS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper we establish criteria for the stability of the proximal mapping $\text{Prox}\,_{\varphi}^{f} = (\partial\varphi + \partial f)^{-1}$ associated to the proper lower semicontinuous convex functions φ and f on a reflexive Banach space X. We prove that, under certain conditions, if the convex functions φ_n converge in the sense of Mosco to φ and if ξ_n converges strongly to ξ, then $\text{Prox}\,_{\varphi_n}^{f}(\xi_n)$ converges weakly and, if f is also totally convex, then it converges strongly to $\text{Prox}\,_{\varphi}^{f}(\xi)$.

1. Preliminaries

Let X be a real reflexive Banach space with the norm $\|\cdot\|$ and let X^* be its dual with the norm denoted $\|\cdot\|_*$. Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous, convex function. Then the function f^*, the Fenchel conjugate of f, is also a proper, lower semicontinuous, convex function and $f^{**} = f$ (see, e.g., [12, pp. 78-79]). We assume that f is a Legendre function in the sense given to this term in [8, Definition 5.2]. This implies that both functions f and f^* have domains with nonempty interior, are differentiable on the interiors of their respective domains,

$$\text{dom}\,\nabla f = \text{int}\,\text{dom}\, f = \text{dom}\,\partial f, \tag{1.1}$$

$$\text{ran}\,\nabla f = \text{dom}\,\nabla f^* = \text{int}\,\text{dom}\, f^*, \tag{1.2}$$

and

$$\text{ran}\,\nabla f^* = \text{dom}\,\nabla f = \text{int}\,\text{dom}\, f, \tag{1.3}$$

and also

$$\nabla f = (\nabla f^*)^{-1}. \tag{1.4}$$

With the function f we associate the function $W^f : X^* \times X \to [0, +\infty]$ defined by

$$W^f(x^*, x) = f(x) - \langle x^*, x\rangle + f^*(x^*). \tag{1.5}$$

The function $D^f : X \times X \to [0, +\infty]$ defined by

$$D^f(y, x) = \begin{cases} W^f(\nabla f(x), y) & \text{if } x \in \text{int}\,\text{dom}\, f, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

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is the well known *Bregman distance associated to* f whose importance to convex optimization was first emphasized in [11]. The function $\nu_f : \operatorname{int}\operatorname{dom} f \times \mathbb{R}_+ \to [0,\infty]$ given by

$$\nu_f(x,t) = \inf\{D^f(y,x) : \|y-x\| = t\} \tag{1.7}$$

is the so called *modulus of total convexity of* f studied in [13]. The function f is called *totally convex* if $\nu_f(x,t) > 0$ for any $x \in \operatorname{int}\operatorname{dom} f$ and $t > 0$.

We denote by $\mathcal{F}_f$ the set of proper, lower semicontinuous and convex functions $\varphi : X \to (-\infty,+\infty]$ which satisfy the following conditions:

(i) $\operatorname{dom}\,\varphi \cap \operatorname{int}\operatorname{dom} f \neq \varnothing$;

(ii) $\varphi_f := \inf\{\varphi(x) : x \in \operatorname{dom}\,\varphi \cap \operatorname{dom}\, f\} > -\infty$.

With any $\varphi \in \mathcal{F}_f$ we associate the function $\operatorname{Env}_\varphi^f : X^* \to [-\infty,+\infty]$ given by

$$\operatorname{Env}_\varphi^f(\xi) = \inf\{\varphi(x) + W^f(\xi,x) : x \in X\}.$$

The function $\operatorname{env}_\varphi^f := \operatorname{Env}_\varphi^f \circ \nabla f$ was introduced and studied in [10]. The following result is a direct consequence of (1.4) and of [9, Propositions 3.22 and 3.23].

Proposition 1.1. *Suppose that* $\varphi \in \mathcal{F}_f$. *For any* $\xi \in \operatorname{int}\operatorname{dom} f^*$ *there exists a unique vector in* X, *denoted* $\operatorname{Prox}_\varphi^f(\xi)$, *such that*

$$\varphi(\operatorname{Prox}_\varphi^f(\xi)) + W^f(\xi, \operatorname{Prox}_\varphi^f(\xi)) = \operatorname{Env}_\varphi^f(\xi). \tag{1.8}$$

Moreover, $\operatorname{Prox}_\varphi^f(\xi) \in \operatorname{dom}\,\partial\varphi \cap \operatorname{int}\,\operatorname{dom}\, f$ *and we have*

$$\operatorname{Prox}_\varphi^f(\xi) = [\partial(\varphi+f)]^{-1}(\xi) = (\partial\varphi + \nabla f)^{-1}(\xi). \tag{1.9}$$

The function

$$\xi \to \operatorname{Prox}_\varphi^f(\xi) : \operatorname{int}\operatorname{dom}\, f^* \to \operatorname{dom}\partial\varphi \cap \operatorname{int}\operatorname{dom}\, f$$

is called *the proximal mapping relative to* f *associated to* φ. It is a natural generalization of a notion introduced and studied by J.-J. Moreau and R.T. Rockafellar since the early sixties (see [22, pp. 34-37] for more information on this notion). The function $\operatorname{prox}_\varphi^f := \operatorname{Prox}_\varphi^f \circ \nabla f$ was termed in [9, Definition 3.16] D^f*-proximal operator associated to* φ.

Note that if C is a nonempty, closed and convex subset of X such that $C \cap \operatorname{int}\operatorname{dom} f \neq \varnothing$, then its indicator function ι_C belongs to $\mathcal{F}_f$. Therefore, the functions

$$\operatorname{Proj}_C^f := \operatorname{Prox}_{\iota_C}^f : \operatorname{int}\operatorname{dom} f^* \to C \cap \operatorname{int}\operatorname{dom}\, f$$

and

$$\operatorname{proj}_C^f := \operatorname{prox}_{\iota_C}^f : \operatorname{int}\operatorname{dom} f^* \to C \cap \operatorname{int}\operatorname{dom} f$$

are well-defined. The function proj_C^f is the so called *Bregman projection with respect to* f *onto the set* C, a concept originating in [11]. The function Proj_C^f was studied in [1], [2], [3], [4] and in [15]. In [15] it was termed *projection relative to* f *onto* C.

In this paper we are concerned with the following question: Given the functions $\varphi_n, \varphi : X \to [-\infty,+\infty]$, $(n \in \mathbb{N})$, contained in $\mathcal{F}_f$, and such that the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ converges in the sense of Mosco to φ, and given a sequence of vectors $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \operatorname{int}\operatorname{dom} f^*$ which converges to some vector $\xi \in \operatorname{int}\operatorname{dom} f^*$, does the sequence $\left\{\mathrm{Prox}^f_{\varphi_n}(\xi_n)\right\}_{n\in\mathbb{N}}$ converge (weakly or strongly) to $\mathrm{Prox}^f_{\varphi}(\xi)$? Recall (see [19, Definition 1.1 and Lemma 1.10]) that the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to be *convergent in the sense of Mosco to* φ (and we write $\text{M-lim}_{n\to\infty}\varphi_n = \varphi$) if the following conditions are satisfied:

(M1) If $\{x_n\}_{n\in\mathbb{N}}$ is a weakly convergent sequence in X such that $\text{w-lim}_{k\to\infty} x_n = x$, and if $\{\varphi_{i_n}\}_{n\in\mathbb{N}}$ is a subsequence of $\{\varphi_n\}_{n\in\mathbb{N}}$, then $\liminf_{n\to\infty}\varphi_{i_n}(x_n) \geq \varphi(x)$;

(M2) For every $u \in X$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty}\varphi_n(u_n) = \varphi(u)$.

Also recall (cf. [19, Remark 1.4 and Lemma 1.10]) that a sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed and convex subsets of X is called *Mosco convergent to the subset* K *of* X (and we write $\text{M-lim}_{n\to\infty} K_n = K$) if $\text{M-lim}\,\iota_{K_n} = \iota_K$.

The question posed above is of interest in connection with the convergence and stability analysis of algorithms for solving variational inequalities emerging from convex optimization problems (see [5], [6], [16], [19], and the references therein). It is known (cf. [17, Theorems 3.1 and 4.1]) that it has positive answer when the space X is smooth and strictly convex, $f = \frac{1}{2}\|\cdot\|^2$ and the functions φ_n and φ are indicator functions of closed convex sets K_n and K, respectively, such that $\text{M-lim}_{n\to\infty} K_n = K$. In these circumstances, $\mathrm{Proj}^f_{K_n}(\xi)$ converges weakly to $\mathrm{Proj}^f_K(\xi)$ and the convergence is strong provided that the norm of the space is Fréchet differentiable and has the Kadeč-Klee property. It was shown in [21, Theorems 4.1 and 4.2] that, in the case of the Bregman projections, these results can be improved in order to guarantee weak, and even strong, convergence of $\mathrm{proj}^f_{K_n}(x_n)$ to $\mathrm{proj}^f_K(x)$ when X is an arbitrary reflexive Banach space, $\lim_{n\to\infty} x_n = x$ and f is a lower semicontinuous convex function satisfying certain additional conditions. An extension of some of these results occurs in [15, Section 4.6]. It shows that $\mathrm{Proj}^f_{K_n}(\xi_n)$ converges weakly to $\mathrm{Proj}^f_K(\xi)$ whenever the function f is either coercive or totally convex on bounded sets, $\text{M-lim}_{n\to\infty} K_n = K$ and $\lim_{k\to\infty}\xi_n = \xi$.

Our purpose in that follows is to prove that convergence results similar to those mentioned above still hold when, instead of projections relative to f onto closed convex subsets of X, one considers proximal mappings $\mathrm{Prox}^f_{\varphi_n}(\xi_n)$ and $\mathrm{Prox}^f_{\varphi}(\xi)$, while $\text{M-lim}_{n\to\infty}\varphi_n = \varphi$. When this happens we say that the proximal mapping $\mathrm{Prox}^f_{\cdot}$ is *stable with respect to the Mosco convergence*. The main result we have in this respect is Theorem 2.1 presented in Section 2. In Section 3 we show how Theorem 2.1 can be used in order to obtain improved Mosco stability criteria in various settings.

2. Mosco stability of the proximal mappings

The main result of this paper is the theorem presented below. It should be noted that the strong coercivity condition and the uniform convexity condition occurring at points (b) and (c) of the theorem are independent in the sense that each of these conditions can hold without the other being satisfied (see [15, Section 4.3]).

Theorem 2.1. *Let* φ *and* φ_n, $(n \in \mathbb{N})$, *be functions in* $\mathcal{F}_f$ *such that* $M\text{-lim}_{n\to\infty}\varphi_n = \varphi$ *and the sequence* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is uniformly bounded from below. Let* $\{\xi_n\}_{n\in\mathbb{N}}$ *be a* *sequence contained in* int dom f^* *and such that* $\lim_{n\to\infty} \xi_n = \xi \in$ int dom f^*. *If at least one of the next three conditions is satisfied:*

(a) *the sequence* $\{\xi_n\}_{n\in\mathbb{N}}$ *is constant (i.e., for any* $n \in \mathbb{N}$, $\xi_n = \xi$*);*
(b) *the function* f *is strongly coercive (i.e.,* $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$*);*
(c) *the function* f *is uniformly convex on bounded sets;*

then the following limits exist and we have

$$w\text{-}\lim_{n\to\infty} \mathrm{Prox}^f_{\varphi_n}(\xi_n) = \mathrm{Prox}^f_{\varphi}(\xi) \tag{2.1}$$

and

$$\lim_{n\to\infty} \mathrm{Env}^f_{\varphi_n}(\xi_n) = \mathrm{Env}^f_{\varphi}(\xi). \tag{2.2}$$

Moreover, if the function f *is totally convex, and if (2.1) and (2.2) are true, then the convergence in (2.1) is strong, that is,*

$$\lim_{n\to\infty} \mathrm{Prox}^f_{\varphi_n}(\xi_n) = \mathrm{Prox}^f_{\varphi}(\xi). \tag{2.3}$$

We present our proof of this theorem as a sequence of lemmas. In all the lemmas we assume that the functions φ and φ_n and the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ satisfy the conditions in the hypothesis of Theorem 2.1. We start with the following result.

Lemma 2.2. *If the sequence* $\{\xi_n\}_{n\in\mathbb{N}}$ *is constant, then (2.1) and (2.2) are true.*

Proof of Lemma 2.2. Let $u \in \operatorname{int}\operatorname{dom} f \cap \operatorname{dom}\varphi$. From (M2) we deduce that there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} \varphi_n(u_n) = \varphi(u). \tag{2.4}$$

Denote $\hat{x}_n = \mathrm{Prox}^f_{\varphi_n}(\xi)$ and $\hat{x} = \mathrm{Prox}^f_{\varphi}(\xi)$. From (1.8) we obtain

$$\varphi_n(\hat{x}_n) + W^f(\xi, \hat{x}_n) \le \varphi_n(u_n) + W^f(\xi, u_n), \quad \forall n \in \mathbb{N}. \tag{2.5}$$

We claim that the sequence on the right hand side in (2.5) is bounded. In order to show that, note that the sequence $\{\varphi_n(u_n)\}_{n\in\mathbb{N}}$ is bounded because it is convergent (see (2.4)). Since $\lim_{n\to\infty} u_n = u \in$ int dom f, it results that there exists a natural number n_0 such that, for all $n \ge n_0$, we have $u_n \in$ int dom f. The function f is lower semicontinuous and, therefore, it is continuous on int dom f (see, for instance, [20, Proposition 3.3]). Thus, the function $W^f(\xi, \cdot)$ is continuous on int dom f and, hence,

$$\lim_{n\to\infty} W^f(\xi, u_n) = W^f(\xi, u). \tag{2.6}$$

Consequently, the sequence $\left\{W^f(\xi, u_n)\right\}_{n\in\mathbb{N}}$ is bounded too and this proves our claim.

Let $c_1 \in \mathbb{R}$ be some upper bound of the sequence $\left\{\varphi_n(u_n) + W^f(\xi, u_n)\right\}_{n\in\mathbb{N}}$. By hypothesis, there exists a real number c_2 such that $\varphi_n(x) \ge c_2$, for all $x \in X$ and $n \in \mathbb{N}$. Consequently,

$$W^f(\xi, \hat{x}_n) \le c_2 - c_1, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

By the Moreau-Rockafellar Theorem (see, for instance, [7, Fact 3.1]), since $\xi \in$ int dom f^*, it results that $f - \langle \xi, \cdot \rangle$ is coercive and, therefore, its sublevel sets

$\mathrm{lev}_{\leq\alpha}(f - \langle\xi,\cdot\rangle)$ are bounded. So, the sublevel sets $\mathrm{lev}_{\leq\alpha}W^f(\xi,\cdot)$ are bounded too. Consequently, (2.7) implies that the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is bounded as being contained in $\mathrm{lev}_{\leq\alpha}W^f(\xi,\cdot)$ with $\alpha = c_2 - c_1$. Thus, there exists a subsequence $\{\hat{x}_{i_n}\}_{n\in\mathbb{N}}$ of $\{\hat{x}_n\}_{n\in\mathbb{N}}$ which converges weakly to some $v \in X$. Since the function $W^f(\xi,\cdot)$ is weakly lower semicontinuous we have $W^f(\xi,v) \leq \liminf_{n\to\infty} W^f(\xi,\hat{x}_{i_n}) \leq \alpha < +\infty$ showing that $v \in \mathrm{dom}\, W^f(\xi,\cdot)$. By (M1), (2.5), the weak lower semicontinuity of $W^f(\xi,\cdot)$, (2.4) and (2.6), we have

$$\begin{aligned}\varphi(v) + W^f(\xi,v) &\leq \liminf_{n\to\infty}\varphi_{i_n}(\hat{x}_{i_n}) + \liminf_{n\to\infty} W^f(\xi,\hat{x}_{i_n}) \\ &\leq \liminf_{n\to\infty}\mathrm{Env}^f_{\varphi_{i_n}}(\xi) \leq \limsup_{n\to\infty}\mathrm{Env}^f_{\varphi_{i_n}}(\xi) \\ &= \varphi(u) + W^f(\xi,u).\end{aligned} \tag{2.8}$$

Since u was arbitrarily chosen in $\mathrm{int}\,\mathrm{dom}\, f$, we conclude that $v = \hat{x}$. Therefore, the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ has a unique weak cluster point, i.e., $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $\hat{x}$. This proves (2.1).

In order to prove (2.2) observe that, since $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $v = \hat{x}$ and (M1) holds, the relations in (2.8) remain true when we write $\hat{x}_n$ and u_n instead of $\hat{x}_{i_n}$ and u_{i_n}, respectively. In particular, (2.8) is true when $\{u_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty} u_n = \hat{x}$ and $\lim_{n\to\infty}\varphi_n(u_n) = \varphi(\hat{x})$ (such sequences exist by (M2)). By consequence, we deduce

$$\mathrm{Env}^f_{\varphi}(\xi) \leq \liminf_{n\to\infty}\mathrm{Env}^f_{\varphi_n}(\xi) \leq \limsup_{n\to\infty}\mathrm{Env}^f_{\varphi_n}(\xi) \leq \mathrm{Env}^f_{\varphi}(\xi),$$

and this proves (2.2). □

Now we show that (2.1) and (2.2) still hold when the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is not necessarily constant, but condition (b) or condition (c) is satisfied.

Lemma 2.3. *If one of the conditions (b) or (c) is satisfied, then (2.1) and (2.2) are true.*

Proof of Lemma 2.3. Let $u \in \mathrm{dom}\,\varphi \cap \mathrm{int}\,\mathrm{dom}\, f$. From (M2), there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that (2.4) holds. Denote

$$\hat{x}_n = \mathrm{Prox}^f_{\varphi_n}(\xi_n) \quad \text{and} \quad \hat{x} = \mathrm{Prox}^f_{\varphi}(\xi). \tag{2.9}$$

From the definition of $\mathrm{Prox}^f_{\varphi_n}$ one has

$$\varphi_n(\hat{x}_n) + W^f(\xi_n,\hat{x}_n) \leq \varphi_n(u_n) + W^f(\xi_n,u_n), \quad \forall n \in \mathbb{N}. \tag{2.10}$$

The sequence $\{\varphi_n(u_n)\}_{n\in\mathbb{N}}$ is bounded by (2.4). Since $\lim_{n\to\infty} u_n = u \in \mathrm{int}\,\mathrm{dom}\, f$, it results that for some $n_0 \in \mathbb{N}$ we have $u_n \in \mathrm{int}\,\mathrm{dom}\, f$, for all $n \geq n_0$. The functions f and f^* are continuous on the interior of their respective domains. Thus, by (1.5), we have

$$\lim_{n\to\infty} W^f(\xi_n,u_n) = W^f(\xi,u), \tag{2.11}$$

showing that $\left\{W^f(\xi_n,u_n)\right\}_{n\in\mathbb{N}}$ is bounded. Hence, the right hand side in (2.10) is bounded by some real number c_3. Consequently,

$$W^f(\xi_n,\hat{x}_n) \leq c_2 - c_3,\ \forall n \in \mathbb{N}. \tag{2.12}$$

Assume that f is strongly coercive. Observe that

$$W^f(\xi_n, \hat{x}_n) - f^*(\xi_n) = f(\hat{x}_n) - \langle \xi_n, \hat{x}_n \rangle, \ \forall n \in \mathbb{N}. \tag{2.13}$$

The sequence $\{f^*(\xi_n)\}_{n\in\mathbb{N}}$ is convergent and, then, bounded. So, the sequence on the left hand side of (2.13) is bounded. Let M be a finite upper bound of it. We claim that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is also bounded. To prove that, suppose by contradiction that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is not bounded. Then, there exists a subsequence $\{\hat{x}_{k_n}\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|\hat{x}_{k_n}\| = \infty$. By (2.13), we have

$$M \geq \|\hat{x}_{k_n}\| \left(\frac{f(\hat{x}_{k_n})}{\|\hat{x}_{k_n}\|} - \|\xi_{k_n}\|_* \right), \ \forall n \in \mathbb{N},$$

where $\lim_{n\to\infty} f(\hat{x}_{k_n}) / \|\hat{x}_{k_n}\| = \infty$ because f is strongly coercive. Since $\{\|\xi_{k_n}\|_*\}_{n\in\mathbb{N}}$ is bounded, this is a contradiction. These prove that if f is strongly coercive, then $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is bounded.

Assume that the function f is uniformly convex on bounded sets. According to (1.2), $\xi_n \in \operatorname{int}\operatorname{ran} \nabla f$ for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $v_n \in \operatorname{int}\operatorname{dom} f$ such that $\nabla f(v_n) = \xi_n$. By (1.4), the sequence $\{v_n\}_{n\in\mathbb{N}}$ is contained in the set $\bigcup_{n\in\mathbb{N}} \partial f^*(\xi_n)$. Since the operator ∂f^* is monotone, it is also locally bounded on $\operatorname{int}\operatorname{dom} f$. So, there exists a neighborhood V of ξ such that $\bigcup_{\zeta\in V} \partial f^*(\zeta)$ is bounded. The sequence $\{\xi_n\}_{n\in\mathbb{N}}$ being convergent to ξ, there exists a positive integer n_0 such that $\xi_n \in V$ for all $n \geq n_0$. Hence, for any $n \geq n_0$, one has $v_n \in \partial f^*(\xi_n) \subseteq \bigcup_{\zeta\in V} \partial f^*(\zeta)$, and this implies that the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded. Convex functions which are lower semicontinuous and uniformly convex on bounded sets are totally convex on bounded sets (cf. [14, Section 4] or [15, Section 2]). Thus, the modulus of total convexity of f over the bounded set $E = \{v_n\}_{n\in\mathbb{N}}$, that is, the function

$$\nu_f(E, t) := \inf \left\{ D^f(y, v_n) : n \in \mathbb{N}, \ \|y - v_n\| = t \right\},$$

is positive for $t > 0$ and has the property that

$$\nu_f(E, ct) \geq c\nu_f(E, t), \tag{2.14}$$

whenever $c \geq 1$ and $t \geq 0$ (cf. [15, Section 2.10]). Observe that

$$\nu_f(E, \|\hat{x}_n - v_n\|) \leq D^f(\hat{x}_n, v_n) = W^f(\xi_n, \hat{x}_n), \quad \forall n \in \mathbb{N}. \tag{2.15}$$

Suppose that the sequence $\{\|\hat{x}_n - v_n\|\}_{n\in\mathbb{N}}$ is unbounded. Then it has a subsequence, denoted $\{\|\hat{x}_{j_n} - v_{j_n}\|\}_{n\in\mathbb{N}}$, such that, for all $n \in \mathbb{N}$, we have $\|\hat{x}_{j_n} - v_{j_n}\| \geq 1$ and $\lim_{n\to\infty} \|\hat{x}_{j_n} - v_{j_n}\| = \infty$. By (2.14), we have

$$\nu_f(E, \|\hat{x}_{j_n} - v_{j_n}\|) \geq \|\hat{x}_{j_n} - v_{j_n}\| \, \nu_f(E, 1), \quad \forall n \in \mathbb{N},$$

where $\nu_f(E, 1) > 0$. This and (2.15) imply

$$W^f(\xi_{j_n}, \hat{x}_{j_n}) \geq \|\hat{x}_{j_n} - v_{j_n}\| \, \nu_f(E, 1), \quad \forall n \in \mathbb{N},$$

which contradicts the boundedness of the sequence $\left\{W^f(\xi_n, \hat{x}_n)\right\}_{n\in\mathbb{N}}$ established above. Hence, the sequence $\{\|\hat{x}_n - v_n\|\}_{n\in\mathbb{N}}$ is bounded. Since $\{v_n\}_{n\in\mathbb{N}}$ is also bounded, it results that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is bounded too.

We prove next that if the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is bounded, then it converges weakly to $\hat{x}$. Since $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is bounded, it has a subsequence, denoted $\{\hat{x}_{k_n}\}_{n\in\mathbb{N}}$, which converges weakly to some $v \in X$. Observe that $\lim_{n\to\infty} (\langle \xi_{k_n}, \hat{x}_{k_n} \rangle - \langle \xi, v \rangle) = 0$.

Therefore, since f is convex and lower semicontinuous (and, hence, weakly lower semicontinuous) and since f^* is continuous on the interior of its domain, we deduce

$$\begin{aligned}\liminf_{n\to\infty} W^f(\xi_{k_n}, \hat{x}_{k_n}) &= f^*(\xi) - \langle \xi, v\rangle + \liminf_{n\to\infty} f(\hat{x}_{k_n}) \\ &\geq f^*(\xi) - \langle \xi, v\rangle + f(v) = W^f(\xi, v).\end{aligned}$$

Consequently, by (M1), (1.5), (2.4), (2.10) and (2.11), we have

$$\begin{aligned}\varphi(v) + W^f(\xi, v) &\leq \liminf_{n\to\infty} \varphi_{k_n}(\hat{x}_{k_n}) + \liminf_{n\to\infty} W^f(\xi_{k_n}, \hat{x}_{k_n}) \\ &\leq \liminf_{n\to\infty} \operatorname{Env}^f_{\varphi_{k_n}}(\xi_{k_n}) \leq \limsup_{n\to\infty} \operatorname{Env}^f_{\varphi_{k_n}}(\xi_{k_n}) \\ &\leq \lim_{n\to\infty} \left[\varphi_{k_n}(u_{k_n}) + W^f(\xi_{k_n}, u_{k_n})\right] = \varphi(u) + W^f(\xi, u).\end{aligned} \tag{2.16}$$

Since u was arbitrarily chosen in $\operatorname{int}\operatorname{dom} f$, we conclude that $v = \hat{x}$. Therefore, the sequence $\{\hat{x}_n\}_{n\in\mathbb{N}}$ has a unique weak cluster point, i.e., it converges weakly to $\hat{x}$.

Now we prove that, if $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $\hat{x}$, then (2.2) also holds. To this end observe that, since $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges weakly to $\hat{x} = v$, the relations in (2.16) are still satisfied when we replace k_n by n. In particular, this is true when $\{u_n\}_{n\in\mathbb{N}}$ is a sequence such that $\lim_{n\to\infty} u_n = \hat{x}$ and $\lim_{n\to\infty} \varphi_n(u_n) = \varphi(\hat{x})$ (such sequences exist by (M2)). In this case, by (2.16), we obtain

$$\begin{aligned}\operatorname{Env}^f_{\varphi}(\xi) &\leq \liminf_{n\to\infty} \operatorname{Env}^f_{\varphi_n}(\xi_n) \leq \limsup_{n\to\infty} \operatorname{Env}^f_{\varphi_n}(\xi_n) \\ &\leq \limsup_{n\to\infty} \left[\varphi_n(u_n) + W^f(\xi_n, u_n)\right] = \operatorname{Env}^f_{\varphi}(\xi),\end{aligned}$$

and this proves (2.2). □

The next result completes the proof of Theorem 2.1.

Lemma 2.4. *If the function f is totally convex, and if the equalities (2.1) and (2.2) are satisfied, then the sequence $\left\{\operatorname{Prox}^f_{\varphi_n}(\xi_n)\right\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{Prox}^f_{\varphi}(\xi)$, i.e., (2.3) holds.*

Proof of Lemma 2.4. Let $\hat{x}$ and $\hat{x}_n$ be given by (2.9). According to Proposition 1.1, we have that $\hat{x} = (\nabla f + \partial\varphi)^{-1}(\xi)$ and this implies that there exists a vector $\eta \in \partial\varphi(\hat{x})$ such that

$$\xi = \nabla f(\hat{x}) + \eta. \tag{2.17}$$

By (1.8) and (1.5), for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\operatorname{Env}^f_{\varphi_n}(\xi_n) - \operatorname{Env}^f_{\varphi}(\xi) &= \varphi_n(\hat{x}_n) - \varphi(\hat{x}) + f(\hat{x}_n) - f(\hat{x}) - \langle \xi, \hat{x}_n - \hat{x}\rangle \\ &\quad - \langle \xi_n - \xi, \hat{x}_n - \hat{x}\rangle - \langle \xi_n, \hat{x}\rangle + f^*(\xi_n) \\ &\quad - f^*(\xi) - \langle \xi, \hat{x}_n - \hat{x}\rangle + \langle \xi, \hat{x}_n\rangle.\end{aligned}$$

Taking (2.17) into account we deduce

$$\begin{aligned}\operatorname{Env}^f_{\varphi_n}(\xi_n) - \operatorname{Env}^f_{\varphi}(\xi) =& \varphi_n(\hat{x}_n) - \varphi(\hat{x}) + D^f(\hat{x}_n, \hat{x}) - \left[\langle \eta, \hat{x}_n - \hat{x}\rangle - \langle \xi_n - \xi, \hat{x}_n - \hat{x}\rangle\right] \\ &+ \left[f^*(\xi_n) - f^*(\xi) - \langle \xi, \hat{x}_n - \hat{x}\rangle\right] - \left[\langle \xi_n, \hat{x}\rangle - \langle \xi, \hat{x}_n\rangle\right],\end{aligned} \tag{2.18}$$

where the quantities between square brackets converge to zero as $n \to \infty$ because of (2.1), the continuity of f^* on the interior of its domain and the assumption that $\lim_{n\to\infty} \xi_n = \xi$. By (2.2), letting $n \to \infty$ in (2.18), we obtain

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left[\varphi_n(\hat{x}_n) - \varphi(\hat{x}) + D^f(\hat{x}_n, \hat{x})\right] = \limsup_{n\to\infty} \left[\varphi_n(\hat{x}_n) - \varphi(\hat{x}) + D^f(\hat{x}_n, \hat{x})\right] \\ &\geq \limsup_{n\to\infty} \left[\varphi_n(\hat{x}_n) - \varphi(\hat{x})\right] \geq \liminf_{n\to\infty} \left[\varphi_n(\hat{x}_n) - \varphi(\hat{x})\right] \geq 0, \end{aligned}$$

where the last inequality results from (M1). This and (2.1) imply $\lim_{n\to\infty} \varphi_n(\hat{x}_n) = \varphi(\hat{x})$. Consequently,

$$0 \leq \liminf_{n\to\infty} \nu_f(\hat{x}, \|\hat{x}_n - \hat{x}\|) \leq \limsup_{n\to\infty} \nu_f(\hat{x}, \|\hat{x}_n - \hat{x}\|) \leq \lim_{n\to\infty} D^f(\hat{x}_n, \hat{x}) = 0.$$

and, hence, $\lim_{n\to\infty} \nu_f(\hat{x}, \|\hat{x}_n - \hat{x}\|) = 0$. This can not happen unless $\lim_{n\to\infty} \|\hat{x}_n - \hat{x}\| = 0$ because, f being totally convex, the function $\nu_f(\hat{x}, \cdot)$ is strictly increasing on $[0, +\infty)$ and has $\nu_f(\hat{x}, 0) = 0$ (cf. [13, Proposition 1.2.2]). □

3. Corollaries of Theorem 2.1

Theorem 2.1 (b) applies when the space X is strictly convex and smooth and $f = \frac{1}{p} \|\cdot\|^p$ for some $p \in (1, +\infty)$. In this situation we obtain the following improved version of Theorem 3.1 of [17].

Corollary 3.1. *If the space X is strictly convex and smooth, if $f := \frac{1}{p} \|\cdot\|^p$ for some $p \in (1, +\infty)$, and if φ, $\varphi_n : X \to (-\infty, +\infty]$, $(n \in \mathbb{N})$, are proper, lower semicontinuous, convex functions such that M-$\lim_{n\to\infty} \varphi_n = \varphi$ and such that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is uniformly bounded from below, then the equalities (2.1) and (2.2) hold for any convergent sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{k\to\infty} \xi_n = \xi$.*

Recall that the Banach space X is called *E-space* if it is reflexive, strictly convex and has the Kadeč-Klee property. It was shown in [21] (see also [15, Section 3.2]) that X is an E-space if and only if the function $\|\cdot\|^p$ is totally convex for some $p \in (1, +\infty)$. These facts and Theorem 2.1(b) lead us to the following result which improves Theorem 4.1 of [17]:

Corollary 3.2. *If X is a smooth E-space and if $f := \frac{1}{p} \|\cdot\|^p$ for some $p \in (1, +\infty)$, and if φ, $\varphi_n : X \to (-\infty, +\infty]$, $(n \in \mathbb{N})$, are proper, lower semicontinuous, convex functions such that M-$\lim_{n\to\infty} \varphi_n = \varphi$ and such that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is uniformly bounded from below, then the sequence $\left\{\mathrm{Prox}^f_{\varphi_n}(\xi_n)\right\}_{n\in\mathbb{N}}$ converges strongly to $\mathrm{Prox}^f_{\varphi}(\xi)$ for any convergent sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset X$ such that $\lim_{k\to\infty} \xi_n = \xi$.*

Theorem 2.1 also allows us to improve upon a result concerning the stability of relative projections presented in [15, Theorem 4.5].

Corollary 3.3. *If f is a lower semicontinuous, convex, Legendre function in the reflexive Banach space X and if K and K_n, $(n \in \mathbb{N})$, are closed convex subsets of X which intersect int dom f and such that M-$\lim_{n\to\infty} K_n = K$, then for any sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset$ int dom f^* such that $\lim_{k\to\infty} \xi_n = \xi \in$ int dom f^* we have*

$$w\text{-}\lim_{n\to\infty} \mathrm{Proj}^f_{K_n}(\xi_n) = \mathrm{Proj}^f_K(\xi) \tag{3.1}$$

and

$$\lim_{n\to\infty} \mathrm{Env}^f_{K_n}(\xi_n) = \mathrm{Env}^f_K(\xi),$$

whenever one of the conditions (a), (b) or (c) of Theorem 2.1 is satisfied. Moreover, if (c) is satisfied, or one of the conditions (a) or (b) in Theorem 2.1 is satisfied and f is also totally convex, then the convergence in (3.1) is strong.

If the lower semicontinuous, convex, Legendre function f is Fréchet differentiable, then its gradient ∇f is continuous on $\mathrm{int}\,\mathrm{dom}\, f$ (cf. [20, p. 20]). Therefore, applying Theorem 2.1 to the sequence $\xi_n = \nabla f(x_n)$ when $\lim_{n\to\infty} x_n = x$ one can deduce the following improvement of Theorems 4.1 and 4.2 of [21].

Corollary 3.4. *If f is a lower semicontinuous, convex, Fréchet differentiable, Legendre function and if K and K_n, ($n \in \mathbb{N}$), are closed convex subsets of X which intersect int dom f and such that M-$\lim_{n\to\infty} K_n = K$, then for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset$ int dom f such that $\lim_{n\to\infty} x_n = x \in$ int dom f we have*

$$w\text{-}\lim_{n\to\infty} \mathrm{proj}^f_{K_n}(x_n) = \mathrm{proj}^f_K(x) \tag{3.2}$$

and

$$\lim_{n\to\infty} \mathrm{env}^f_{K_n}(x_n) = \mathrm{env}^f_K(x),$$

whenever one of the conditions (a) or (b) of Theorem 2.1 is satisfied. Moreover, if (c) is satisfied, or one of the conditions (a) or (b) of Theorem 2.1 is satisfied and f is also totally convex, then the convergence in (3.2) is strong.

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