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REGULARITY RESULTS FOR SOME LINEAR AND NONLINEAR EQUATIONS

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Dedicated to the memory of Filippo Chiarenza

Abstract. The main results concern local Hölder continuity of the solutions of classes of subelliptic equations with rough coefficients, as well as an application to regularity results for degenerate Monge-Ampère equations. A connection between Monge-Ampère equations and those of subelliptic type is furnished by an n-dimensional partial Legendre transformation, which reduces the question of regularity for Monge-Ampère equations to a similar one for quasilinear equations.

We begin by discussing some results obtained jointly with Eric Sawyer in [7] concerning local Hölder continuity for weak solutions of linear equations of divergence form with rough coefficients in a bounded domain $\Omega \subset R^n$. The general form of the equation that we consider is

$$Lu + \text{lower order terms} = f + T'\vec{g}, \tag{1}$$

where

$$Lu = \text{div } B(x)\nabla u$$

for an $n \times n$ nonnegative semidefinite matrix $B(x)$, $x \in \Omega$, where T' denotes the adjoint of a collection T of vector fields which are subunit relative to the matrix B, i.e.,

$$T'\vec{g} = \sum_i T'_i g_i = \sum_i \sum_{j=1}^n \partial_j \left(\alpha_{ij} g_i\right)$$

with

$$\left(\sum_{j=1}^n \alpha_{ij}(x)\xi_j\right)^2 \leq \xi' B(x)\xi$$

for each i and all $\xi \in R^n, x \in \Omega$. Inclusion of the term $T'\vec{g}$ among the nonhomogeneous terms is important for applications to equations of Monge-Ampère type. The exact form of the lower order terms is carefully discussed in [7], but here we will not give the details for the sake of simplicity.

The matrix $B(x)$ is always assumed to be nonnegative semidefinite. In fact, we will allow B (and so also L) to vary relative to another fixed quadratic form $\xi' Q(x)\xi$ satisfying

(1) there exist positive constants C_1 and C_2 such that for all $x \in \Omega, \xi \in R^n$,

$$C_1 \xi' Q(x)\xi \leq \xi' B(x)\xi \leq C_2 \xi' Q(x)\xi,$$

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(2) $Q(x)$ is bounded and measurable in Ω,
(3) $Q(x)$ is nonnegative semidefinite in Ω.

We seek conditions on Q which guarantee that all weak solutions u of *any* such equation (1) are Hölder continuous of order α with

$$||u||_{C^\alpha(K)} \leq C, \quad K \text{ compact}, K \subset \Omega,$$

where for appropriately large q,

$\alpha = \alpha(K; L^q$ data of the lower order terms; constants C_1, C_2 of equivalence above),

$C = C(K : L^q$ data of $f, \vec{g}$ and the lower order terms; constants C_1, C_2 above).

Here we use the notation

$$||u||_{C^\alpha(K)} = \sup_{x \in K} |u(x)| + \sup_{x,y \in K} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

to describe ordinary Hölder continuity with respect to Euclidean distance; variants will be mentioned below.

We will say that such a quadratic form $\mathcal{Q}(x,\xi) = \xi' Q(x) \xi$ is *L^q-subelliptic in* Ω. Note that if $q < \infty$ and $\mathcal{Q}(x,\xi)$ is L^q-subelliptic, then it is also L^∞-subelliptic since Ω is bounded.

Our first result is one of axiomatic type in the sense that it furnishes a set of conditions that are sufficient for L^q-subellipticity. We state the theorem first in a rough version; more details are given after the rough statement. We say that a function $d(x,y)$ defined for $x, y \in \Omega$ is a symmetric quasimetic if there is a positive constant κ such that for all $x, y, z \in \Omega$,

(1) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$,
(3) $d(x,y) \leq \kappa[d(x,z) + d(z,y)]$.

Theorem 1. *Let $Q(x)$ be a bounded, measurable matrix which is nonnegative semidefinite in a bounded domain $\Omega \subset R^n$. Then the corresponding quadratic form $\mathcal{Q}(x,\xi)$ is L^q-subelliptic in Ω for all sufficiently large q provided there is a symmetric quasimetric $d(x,y)$ which is measurable in each variable separately and*

(1) *there are positive constants ϵ, c, C such that*

$$c|x-y| \leq d(x,y) \leq C|x-y|^\epsilon \quad \textit{for all } x, y \in \Omega,$$

(2) *Lebesgue measure is a doubling measure for the d-balls $B(x,r)$ defined by*

$$B(x,r) = \{y \in \Omega : d(x,y) < r,\} \quad x \in \Omega, r > 0,$$

i.e., there is a constant C independent of x and r such that $|B(x,2r)| \leq C|B(x,r)|$,

(3) *there are appropriate Sobolev and Poincaré inequalities for d-balls (see below for details),*

(4) *there are appropriate approximating sequences of Lipschitz cut-off functions for d-balls (see below for details).*

The value of q is related to the doubling assumed in condition (2) and to the Sobolev estimate in condition (3); see below. The only d-balls for which the hypotheses above (as well as the more careful statements below) are required to hold are those balls $B(x,r)$ with $x \in \Omega$ and $0 < r < \delta\, dist(x, \partial\Omega)$ for a small, fixed choice of δ. Although this convention will always be part of our assumptions, we will not generally repeat it below.

In order to be more precise, we first explain what conditions (3) and (4) mean. Following this, a description of allowable q values is given.

We will use the notation

$$|\nabla w(x)|_Q^2 = \nabla w(x) \cdot Q(x) \nabla w(x) = \mathcal{Q}(x, \nabla w(x)),$$

and we write $r(B)$ for the radius of a d-ball B.

The Sobolev Assumption. Suppose that there are constants $\sigma > 1$ and $C > 0$ such that for all d-balls B and all $w \in W_0^{1,2}(B)$,

$$\left(\frac{1}{|B|} \int_B |w|^{2\sigma} dx \right)^{1/(2\sigma)} \leq C\, r(B) \left(\frac{1}{|B|} \int_B |\nabla w|_Q^2 dx \right)^{1/2} + C \left(\frac{1}{|B|} \int_B w^2 dx \right)^{1/2}.$$

The Poincaré Assumption. Suppose that there is a positive constant C such that for every d-ball B and every $w \in W^{1,2}(CB)$ (where CB denotes the ball with radius $Cr(B)$ which is concentric with B),

$$\left(\frac{1}{|B|} \int_B |w - \frac{1}{|B|} \int_B w|^2 dx \right)^{1/2} \leq C\, r(B) \left(\frac{1}{|B|} \int_{CB} |\nabla w|_Q^2 dx \right)^{1/2}.$$

The Cutoff Assumption. Assume there exist positive constants c, N, p, C_p such that for each d-ball $B(y,r)$, there is a sequence $\{\psi_j\}_{j=1}^{\infty}$ of Lipschitz functions satisfying

(1) $\operatorname{supp} \psi_1 \subset B(y,r)$,
(2) $\psi_j(x) = 1$ if $x \in B(y, cr)$ for each j,
(3) $\operatorname{supp} \psi_{j+1} \subset\subset \{x : \psi_j(x) = 1\}$ for each j,
(4) $\left(\frac{1}{|B(y,r)|} \int_{B(y,r)} |\nabla \psi_j(x)|_Q^p dx \right)^{1/p} \leq C_p \frac{j^N}{r}$.

We will need a more quantitative version of the doubling assumption in hypothesis (2) of Theorem 1 in order to describe the range of possible q values. The doubling hypothesis leads in a standard way to the existence of a *doubling exponent* D, i.e., to the existence of a positive number D such that

$$|B(x,r)| \leq C \left(\frac{r}{t} \right)^D |B(y,t)| \quad \text{whenever } B(y,t) \subset B(x,r),$$

with C independent of x, y, r, t.

With these more precise versions of the hypotheses in Theorem 1, it is possible to give a more exact description of the values of q allowed in the conclusion. If σ' denotes the dual index of the value σ in the Sobolev Assumption (i.e., $\sigma' = \sigma/(\sigma - 1)$) and p is the exponent in the Cutoff Assumption, then the conclusion of Theorem 1 holds for any q with

$$q > \max\{2\sigma', D\} \quad \text{provided } p > \max\{2\sigma', 4\}.$$

In fact, the range of q values can be replaced by the generally larger range

$$q > \max\{2\sigma', Q^*\}, \quad \text{where } Q^* = \limsup_{r\to 0} \max_{x\in\Omega} \frac{\log|B(x,r)|}{\log r},$$

with the same restrictions on p. The number Q^* is called the *upper dimension* of the quasimetric space; of course Q^* varies with the quasimetric d.

Remarks. (i) The Hölder constant C and exponent α in the conclusion of Theorem 1 depend also on the constants in all the assumptions above.

(ii) The restriction that $d(x,y) \le C|x-y|^\epsilon$ can be dropped in Theorem 1 if in the conclusion we replace the notion of Hölder continuity by d-Hölder continuity, i.e., if in the definition of Hölder continuity we replace Euclidean distance $|x-y|$ by $d(x,y)$. Note that the inequality $d(x,y) \le C|x-y|^\epsilon$ for all x,y is the same as assuming the Fefferman-Phong condition that there exist positive constants C and ϵ such that

$$D(x,r) \subset B(x, Cr^\epsilon), \quad x \in \Omega, 0 < r < \delta\, dist(x, \partial\Omega),$$

where $D(x,r)$ denotes the ordinary Euclidean ball with center x and radius r.

(iii) In some cases, several of the assumptions required for Theorem 1 are automatically true. This happens most notably in case the matrix $Q(x)$ is continuous and we choose $d(x,y)$ to be the Q*-subunit metric* (or *control metric*) $\delta(x,y)$ defined by letting $\delta(x,y)$ be the minimum time $t \ge 0$ required to connect x to y by Lipschitz continuous curves $\gamma(t)$ in Ω with

$$\left(\gamma'(t)\cdot\xi\right)^2 \le \xi\cdot Q(\gamma(t))\xi$$

for all $\xi \in R^n$ and all t. For such Q and $d(x,y) = \delta(x,y)$, it turns out that the Cutoff Assumption holds automatically with $p = \infty$, and moreover, the Poincaré Assumption implies the Sobolev Assumption for some value of $\sigma > 1$. Moreover, the inequality $\delta(x,y) \ge c|x-y|$ holds for some $c > 0$.

We state this important special case separately in the next theorem.

Theorem 2. *Suppose that $\mathcal{Q}(x,\xi)$ is a nonnegative semidefinite continuous quadratic form in Ω, and suppose that the subunit metric $\delta(x,y)$ is finite on $\Omega\times\Omega$. Denote by $K(x,r)$ the corresponding δ-subunit balls, and let Q^* be the upper dimension of these balls. If $\sigma > 1$, then $\mathcal{Q}(x,\xi)$ is L^q-subelliptic in Ω for $q > \max\{Q^*, 2\sigma'\}$ provided:*

1. *the doubling condition $|K(x,2r)| \le C|K(x,r)|$ holds,*
2. *there exists $\epsilon > 0$ such that the containment condition $D(x,r) \subset K(x, Cr^\epsilon)$ holds,*
3. *the Poincaré Assumption holds with $B(x,r) = K(x,r)$, and σ is the value for which the Sobolev estimate holds.*

We now pass from these axiomatic results to some specific examples in case the matrix $Q(x)$ is a *diagonal* matrix with the following entries on the main diagonal:

$$a_1(x)^2, \ldots, a_n(x)^2, \quad a_i(x) \ge 0, x \in \Omega.$$

Thus the basic quadratic form becomes

$$\mathcal{Q}(x,\xi) = \sum_{j=1}^{n} a_j(x)^2 \xi_j^2,$$

and we will be considering subelliptic equations (1) whose coefficient matrix $B(x)$ satisfies

$$C_1 \sum_{j=1}^{n} a_j(x)^2 \xi_j^2 \leq \xi' B(x) \xi \leq C_2 \sum_{j=1}^{n} a_j(x)^2 \xi_j^2, \quad x \in \Omega, \xi \in R^n.$$

In order to verify the assumptions required in Theorem 1, we will impose restrictions on the functions $a_j(x)$.

Definition 3. A collection of continuous vector fields $X_j = a_j(x)\frac{\partial}{\partial x_j}, 1 \leq j \leq n$, satisfies the flag condition at a point $x \in \Omega$ if for each index set $\mathcal{I}$ with $\emptyset \subset \mathcal{I} \subset\neq \{1, 2, \ldots, n\}$, there is $j \notin \mathcal{I}$ such that for any neighborhood $\mathcal{N}$ of x in Ω, a_j does not vanish identically on $(x + \mathcal{V}_{\mathcal{I}}) \cap \mathcal{N}$, where $\mathcal{V}_{\emptyset} = \{0\}$ and $\mathcal{V}_{\mathcal{I}} =$ span $\{e_i : i \in \mathcal{I}\}, e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i^{th} position.

The vector fields X_j are said to satisfy the flag condition in Ω if they satisfy the flag condition at every point of Ω.

An equivalent formulation of the flag condition at x is that there is an increasing sequence of vector spaces

$$\{0\} = \mathcal{V}_0 \subset\neq \mathcal{V}_1 \subset\neq \ldots \mathcal{V}_j \subset\neq \mathcal{V}_{j+1} \subset\neq \ldots \mathcal{V}_m = R^n,$$

and an increasing sequence of index sets

$$\emptyset \neq \mathcal{I}_1 \subset\neq \ldots \mathcal{I}_j \subset\neq \mathcal{I}_{j+1} \subset\neq \ldots \mathcal{I}_m = \{1, 2, \ldots, n\},$$

such that $\mathcal{V}_j = \text{span}\{e_i : i \in \mathcal{I}_j\}$ for $1 \leq j \leq m$, and a_j does not vanish identically on $(x + \mathcal{V}_j) \cap \mathcal{N}$ for any neighborhood $\mathcal{N}$ of x in Ω if $i \in \mathcal{I}_{j+1}, j \geq 0$.

An increasing sequence $\{\mathcal{V}_j\}_{j=1}^{m}$ as above is called a *flag* at x. Note that flags may vary from point to point.

While we are primarily interested in rough vector fields, it is proved in [7] that if $\{a_j\}_{j=1}^{n}$ are real analytic functions, then the flag condition and the Hörmander commutator condition are equivalent for the vector fields $\{a_j(x)\partial_j\}_{j=1}^{n}$:

Theorem 4. *A collection $\{a_j(x)\partial_j\}_{j=1}^{n}$ of real analytic (diagonal) vector fields in a domain $\Omega \subset R^n$ satisfies the flag condition in Ω if and only if it satisfies the Hörmander commutator condition in Ω, i.e., if and only if at each point $x \in \Omega$, there is a positive integer p such that the linear span of the vector fields $X_j = a_j\partial_j$ and their commutators up to order p,*

$$span\{X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \ldots, [X_{j_1}, [X_{j_2}, [\ldots[X_{j_{p-1}}, X_{j_p}]\ldots]]] : 1 \leq j_i \leq n\},$$

is equal to R^n.

Our main application of Theorem 1 is that for rough diagonal vector fields, the flag condition is equivalent to L^∞-subellipticity of the corresponding quadratic form, provided the vector fields satisfy appropriate reverse Hölder conditions. In order to describe these conditions, let $a(x)$ be a real-valued function with $a(x) \geq 0$, and let

$i \in \{1, ...n\}$. We say that $a(x) \in RH_\infty$ in the variable x_i, uniformly in the other variables, if there is a constant C such that for all one-dimensional intervals J and all $x = (x_1, ..., x_n)$,

$$\sup_{t \in J} a(x_1, ..., x_{i-1}, t, x_{i+1}, ...x_n) \leq C \frac{1}{|J|} \int_J a(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n)\, dt.$$

Theorem 5. *Suppose $\{a_j(x)\}_{j=1}^n, x \in \Omega \subset R^n$, satisfies $a_j \geq 0, a_j \in Lip(\Omega)$, and $a_j \in RH_\infty$ in x_i for all $i \neq j$, uniformly in the other variables. Then the quadratic form $\sum_{j=1}^n a_j(x)^2 \xi_j^2$ is L^∞-subelliptic in Ω if and only if $\{a_j(x)\partial_j\}_{j=1}^n$ satisfies the flag condition in Ω. Moreover, if the flag condition holds in Ω, then there exists $N \geq n$ depending only on the Lipschitz and RH_∞ constants of the a_j such that $\sum_{j=1}^n a_j(x)^2 \xi_j^2$ is L^q-subelliptic in Ω for all $q > N$.*

The value of N in Theorem 5 turns out to be the doubling exponent for Lebesgue measure of the *flag balls* (to be defined below) associated with the vector fields $\{a_j\}$. The flag balls turn out to determine a quasimetric and a space of homogenous type to which Theorem 1 applies. Complete details can be found in [7]. Let us now define the flag balls associated with the a_j.

We assume the vector fields $\{X_j\}_{j=1}^n = \left\{\frac{\partial}{\partial x_1}, a_2\,(x)\,\frac{\partial}{\partial x_2}, ..., a_n\,(x)\,\frac{\partial}{\partial x_n}\right\}$ are continuous in Ω, and that $a_j\,(x)$ is Lipschitz continuous in $x_2, ..., x_n$ uniformly in x_1, and reverse Hölder of infinite order in each variable x_i with $i \neq j$, uniformly in the remaining variables. Note that we have assumed $a_1(x) = 1$ for all x. This assumption causes no loss of generality since our results are local and at least one of the vector fields must be different from 0 at every point. We want to use the flag condition to construct a family of open rectangles

$$B\,(x, r) = \prod_{j=1}^n (x_j - B_j\,(x, r)\,, x_j + B_j\,(x, r))$$

for $x \in \Omega$ and $0 < r < \delta\; dist\,(x, \partial\Omega)$, that are related to the vector fields $\{X_j\}_{j=1}^n$ in the sense that there are positive constants c, C such that

$$(2) \qquad cB_j\,(x, r) \leq \sup_{z \in B(x,r)} r a_j\,(z) \leq CB_j\,(x, r)\,, \qquad x \in \Omega,\, 0 < r < \delta\; dist\,(x, \partial\Omega)\,,$$

for $1 \leq j \leq n$. Note that (2) says that the j^{th} sidelength $2B_j\,(x, r)$ of the rectangle $B\,(x, r)$ is comparable to r times the supremum of a_j over the rectangle $B\,(x, r)$. If the a_j were essentially constant, this would be the maximum distance a subunit curve could travel in the j^{th} direction in time r, and the rectangle $B\,(x, r)$ would be equivalent to the Fefferman-Phong control balls $K\,(x, r)$. The importance of (2) is that it provides a key link in proving that the rectangles $B\,(x, r)$ lead to a homogeneous space and a suitable subrepresentation formula, allowing verification of the Poincaré and Sobolev estimates required by Theorem 1.

The algorithm we employ below actually achieves the following stronger form of (2): there are positive constants c, C such that for every x, r with $x \in \Omega$ and $0 < r < \delta\; dist\,(x, \partial\Omega)$, there is a permutation $\{j_1, j_2, ..., j_n\}$ of $\{1, 2, ..., n\}$ with

$j_1 = 1$ satisfying

$$cB_{j_i}(x,r) \le \sup_{z: z_{j_\ell} = x_{j_\ell}, \ell \ge i \text{ and } \left|z_{j_\ell} - x_{j_\ell}\right| \le B_{j_\ell}(x,r), \ell < i} ra_{j_i}(z) \tag{3}$$
$$\le \sup_{z \in B(x,r)} ra_{j_i}(z) \le CB_{j_i}(x,r),$$

$x \in \Omega$, $0 < r < \delta\, dist\,(x, \partial\Omega)$ and $1 \le i \le n$.

Fix x, r with $x \in \Omega$ and $0 < r < \delta\, dist\,(x, \partial\Omega)$. Define

$$A_j(x,r) = \int_0^r a_j(x_1 + t, x_2, ..., x_n)\, dt, \tag{4}$$

$1 \le j \le n$, so that $A_1(x,r) = r$ and $A_j(x,r) \ge 0$ for all j. Since we are assuming that the a_j are reverse Hölder in x_1 uniformly in $x_2, ..., x_n$, it follows that

$$\sup_{z \in B(x,r)} ra_i(z) \text{ is essentially } \sup_{z \in B(x,r)} A_i(z,r),$$

and this motivates the use of A_i to implement our algorithm, which we now describe. We inductively define a rearrangement $\{j_2, ..., j_n\}$ of $\{2, ..., n\}$ and nonnegative numbers $B_{j_2}(x,r), ..., B_{j_n}(x,r)$ as follows: First define

$$A_{j_2}(x,r) = \max_{2 \le j \le n} A_j(x,r),$$
$$B_{j_2}(x,r) = A_{j_2}(x,r).$$

Then for $j \ne j_2$ set

$$\Phi_j^2(x,r) = \max\left\{A_j(z,r) : |z_i - x_i| \le \chi_{\{j_2\}}(i)\, B_i(x,r), 1 \le i \le n\right\},$$

and define

$$\Phi_{j_3}^2(x,r) = \max_{j \ne j_2} \Phi_j^2(x,r),$$
$$B_{j_3}(x,r) = \Phi_{j_3}^2(x,r).$$

Assuming $B_{j_2}(x,r), ..., B_{j_m}(x,r)$ have already been defined, then for $j \notin \{j_2, ..., j_m\}$, set

$$\Phi_j^m(x,r) = \max\left\{A_j(z,r) : |z_i - x_i| \le \chi_{\{j_2,...,j_m\}}(i)\, B_i(x,r), 1 \le i \le n\right\},$$

and define

$$\Phi_{j_{m+1}}^m(x,r) = \max_{j \notin \{j_2,...,j_m\}} \Phi_j^m(x,r),$$
$$B_{j_{m+1}}(x,r) = \Phi_{j_{m+1}}^m(x,r).$$

This inductively defines $B_{j_2}(x,r), ..., B_{j_n}(x,r)$.

Note. If we assume that the vector fields $\{X_i\}_{i=1}^n$ satisfy the flag condition in Definition 3, then we have the important property that $B_j(x,r) > 0$ for $2 \le j \le n$ and $r > 0$.

We now define open rectangles

$$B(x,r) = (x_1 - r, x_1 + r) \times \prod_{j=2}^n (x_j - B_j(x,r), x_j + B_j(x,r)), \tag{5}$$

for $x \in \Omega$, $0 < r < \delta\, dist\,(x, \partial\Omega)$, which we refer to as "flag balls". Note again that if δ is sufficiently small, then the rectangles $B\,(x,r)$ are well-defined and contained in Ω. Finally, we emphasize that the permutation $\{j_2, ..., j_n\}$ of $\{2, ..., n\}$ used to define the flag ball $B\,(x,r)$ depends on both x and r, and is analogous in spirit to the choice of N-tuple used to compute a corresponding quasimetric in Chapter 1, section 4 of [6].

In passing, we mention that in the special case when none of the $A_j(x,r)$ with $r > 0$ vanish, it is possible to derive an analogue of Theorem 5 in which the role played by the flag balls $B(x,r)$ is instead played by a much simpler collection of rectangles $A(x,r)$ defined by

$$A(x,r) = (x_1 - r, x_1 + r) \times \prod_{j=2}^{n} (x_j - A_j(x,r), x_j + A_j(x,r))\,, \tag{6}$$

provided the following condition holds: there are positive constants C and δ such that for $j = 2, ..., n$,

$$C^{-1} A_j(x,r) \leq A_j(z,r) \leq C A_j(x,r), \quad z \in A(x,r), x \in \Omega, 0 < r < \delta dist(x, \partial\Omega). \tag{7}$$

We refer to condition (7) as the *noninterference condition*, and we call the rectangles $A(x,r)$ *noninterference balls*. The noninterference condition (7) can be shown to be a corollary of the following *strong noninterference condition*:

$$r \left\{ \sup_{z \in A(x,r)} \left| \frac{\partial a_j}{\partial x_i}(z) \right| \right\} A_i(x,r) \leq C A_j(x,r), \quad x \in \Omega, 0 < r < \delta dist(x, \partial\Omega),$$

for $2 \leq i, j \leq n$. See [7] for details.

An application to Monge-Ampère equations.

We consider regularity of the generalized Monge-Ampère equation

$$\det D^2 u = k(x, u, Du), \quad x \in \Omega, \tag{8}$$

where $D^2 u$ is the Hessian matrix of u, k is smooth (infinitely differentiable) and nonnegative in $\Omega \times R^1 \times R^n$, and Ω is a convex domain in R^n with smooth boundary. For example, if k depends only on x, we have the classical Monge-Ampère equation, while the choice

$$k = k_n(x) \left(1 + |\nabla u|^2\right)^{\frac{n+2}{2}}$$

corresponds to the equation of prescribed Gaussian curvature $k_n(x)$.

When k is strictly positive, the Dirichlet problem

$$\det D^2 u = k \quad \text{in } \Omega, \quad u = \phi \quad \text{on } \partial\Omega$$

is elliptic and the solutions u are smooth up to the boundary when k and ϕ are smooth (Caffarelli, Nirenberg and Spruck [2]). However, if k is permitted to vanish, such regularity may fail spectacularly. For example, in two dimensions the function

$$u(x,y) = \frac{1}{18}(x^2 + y^2)^{3/2}$$

fails to be C^3 at the origin while $\det D^2u(x,y) = x^2+y^2$ is analytic and u has analytic trace on $x^2+y^2=1$.

In [4] and [5], it is shown that when k, ϕ and $\partial\Omega$ are smooth and k is nonnegative, then solutions of the Dirichlet problem are in $C^{1,1}(\bar{\Omega})$. Constructions in [1] and a specific example due to Sibony reported in [3] and [5] show that this result is best possible. On the other hand, Guan [3] showed in two dimensions that if k vanishes of finite type in a certain way, then every $C^{1,1}$ convex solution $u(x,y)$ is smooth provided $u_{yy} \geq c > 0$ in Ω. Note that u_{yy} vanishes at the origin in the example above.

In order to extend Guan's two dimensional result to higher dimensions, [8] employs an n-dimensional analogue of the partial Legendre transform. Given $u \in C^{1,1}$, we consider the mapping from

$$x = (x_1, x_2, ..., x_n) \quad \longrightarrow \quad (s, t_2, ..., t_n) = (s,t),$$

$t \in R^{n-1}$, given by

$$s = x_1, \quad t_2 = u_{x_2}(x), \ldots, t_n = u_{x_n}(x).$$

Assuming that the $(n-1)\times(n-1)$ determinant $\det(u_{x_i x_j})_{i,j\geq 2} \neq 0$, we may invert the transformation to obtain functions $v_l(s,t)$, $l = 2, \ldots, n$:

$$x_1 = s, \quad x_2 = v_2(s,t), \ldots, x_n = v_n(s,t).$$

Following [3], the basic strategy of [8] is to determine regularity of u in Ω by studying regularity of the functions v_l in the transformed domain. It is shown in [8] that the vector-valued function $v = (v_l)_{l=2}^n = (x_l(s,t))_{l=2}^n$ is a weak solution of the divergence form quasilinear system

$$\mathcal{L}v_l \equiv \left\{ \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial t'} k \left(co \left[\frac{\partial v}{\partial t'} \right] \right)' \frac{\partial}{\partial t} \right\} v_l = 0, \quad 2 \leq l \leq n,$$

where $\left(co\left[\frac{\partial v}{\partial t'}\right]\right)'$ denotes the transposed cofactor matrix of $\frac{\partial v}{\partial t'}$. If k is positive, the system is elliptic. A significant feature of the system in case k is allowed to vanish is that the degeneracy of the system is incorporated solely in the function k which appears in the coefficient matrix, assuming that $\det \frac{\partial t}{\partial x'} = \det \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{i,j=2}^n > 0$. In principle, this allows the use of the subelliptic regularity results discussed earlier.

Unfortunately, this system is not diagonal in the principal terms. On the other hand, the matrix

$$M = \left(co \left[\frac{\partial v}{\partial t'} \right] \right)'$$

satisfies the divergence-free property $\partial_{t'} M = \vec{0}'$, and consequently, if we differentiate the system above, we obtain that the vector-valued function $p = Dv = \left(\frac{\partial v_i}{\partial t_j}\right)_{2\leq i\leq n, 1\leq j\leq n}$, with $s = t_1$, satisfies the divergence form quasilinear system

$$\mathcal{L}p \equiv \left\{ \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial t'} k M(p) \frac{\partial}{\partial t} \right\} p = f((s,t), v, p, Dp),$$

that is diagonal in the principal terms, and has nonhomogeneous vector term f that is quadratic in Dp. In [8], we use the system for p to derive the following generalization of Guan's result to higher dimensions. The result gives subelliptic conditions on k which imply smoothness of convex solutions $u \in C^{2,1}(\Omega)$ to the generalized Monge-Ampère equation (8), provided u has $n-1$ nonzero principal curvatures.

Theorem 6. *Let $u \in C^{2,1}(\Omega)$ be a convex solution to (8) with k smooth on $\Omega \times R^1 \times R^n$ satisfying*

$$k(x,u,Du) \approx \left(|x_1|^{2m} + \psi(x)\right) K(x,u,Du), \quad x \in \Omega,$$

where K is smooth and positive on $\Omega \times R^1 \times R^n$, ψ is smooth and nonnegative on Ω, m is a positive integer, and $\psi^{1/(2m)}$ is Lipschitz continuous. If

$$d = det \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{i,j=2}^{n} > 0 \tag{9}$$

at all points of Ω, then $u \in C^\infty(\Omega)$.

It is of interest to note that the conclusion of the theorem fails if (9) is replaced by the assumption that a minor of the Hessian of size $(n-2)\times(n-2)$ is nonvanishing. For example, the function

$$u(x) = (x_1^2 + x_2^2)^{3/2} + \sum_{j=3}^{n} \frac{x_j^2}{2}$$

is in $C^{2,1}(\Omega)$ but fails to be in $C^3(\Omega)$ even though (8) holds with $k = 18(x_1^2 + x_2^2)$ and

$$\det \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{i,j=3}^{n} = 1.$$

Note that for this function u, the determinant d in (9) vanishes when $k=0$.

Theorem 6 applies to the equation of prescribed Gaussian curvature $k_n(x)$, namely,

$$\det D^2 u = k_n(x) \left(1 + |Du|^2\right)^{\frac{n+2}{2}}, \tag{10}$$

with the following geometric consequence. If u is a $C^{2,1}$ convex function whose graph has smooth Gaussian curvature k_n, and consequently u satisfies (10), and if $k_n(x) \approx |x|^{2m}$ for some $m = 1, 2, \ldots$, then u is smooth provided $k_{n-1}(0) > 0$. Here k_{n-1} denotes the elementary symmetric function of order $n-1$ of the principal curvatures of u. In fact, in order to apply Theorem 6, it is enough to rotate coordinates so that $d(0) = k_{n-1}(0)$ in (9).

Some generalizations of Theorem 6 in which the assumption $u \in C^{2,1}$ is replaced by a weaker assumption will appear in [9]. These results still rely on the subellipticity facts from [7] discussed earlier.

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