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SOLVABILITY OF A DIRICHLET PROBLEM FOR DIVERGENCE FORM PARABOLIC EQUATION

MARIA ALESSANDRA RAGUSA

Abstract. We consider parabolic equations with discontinuous coefficients and prove that if the known term belongs to the Morrey space $L^{p,\lambda}$ then the first order derivatives of the solution of an associate Dirichlet problem belong to the same space. We also obtain local Hölder continuity for solution.

1. Introduction

In this note we are concerned with existence, uniqueness and global regularity in $H_0^{1,p,\lambda}(Q_T)$ of the weak solution of the second order differential equation

$$\mathcal{L}u \equiv u_t - \sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} = \operatorname{div} f \tag{1.1}$$

in the cylinder $Q_T = \Omega \times (-T, 0)$ where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded open set with sufficiently smooth boundary and $T > 0$.

In our treatment we assume $x = (x_1, \dots, x_n, t) = (x', t) \in \mathbb{R}^{n+1}$, $u_t = \frac{\partial u}{\partial t}$, $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ the spatial gradient of u.

For reader's convenience we recall the definition

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \big(x'_n \geq 0\big),$$
$$\mathbb{R}^{n+1}_- = \mathbb{R}^{n+1} \cap \big(x'_n \leq 0\big).$$

We also assume $\mathcal{L}$ to be a linear parabolic operator in divergence form whose possibly discontinuous coefficients are taken in the space VMO at first defined by Sarason in [13] (see Section 2 below for precise definition). This hypotheses implies a number of good properties, for example bounded uniformly continuous functions are in the vanishing mean oscillation class as well as functions of the Sobolev spaces $W^{\tau,n/\tau}$ for $\tau \in]0,1]$.

Our main result is the well-posedness in $H_0^{1,p,\lambda}(Q_T)$ of the Cauchy-Dirichlet problem

$$\begin{cases} \mathcal{L}u = \operatorname{div} f & Q_T, \\ u = 0 & \text{on } \partial\Omega \times (-T, 0) \\ u(x', -T) = 0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

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The result has been studied in Lebesgue spaces (see [12]), now we improve it showing that fine regularity of the right-hand side f increase the regularity of the first derivatives of the solution.

We recall that if the coefficients a_{ij} are real bounded measurable functions existence of weak solutions of linear parabolic equations and some kind of regularity has been studied by Kaplan in [5] and [6].

The present note generalizes, for linear second order elliptic equations, the local regularity result obtained by Marino and Maugeri in [7], as showed in detail in Appendix, because they obtain

$$u \in L^p_{loc}(-T, 0, H^{1,p}(\Omega))$$

while here is proved that

$$u \in L^{p,\lambda}(-T, 0, H^{1,p}(\Omega)).$$

We point out that in this note we answer to the question, arised when I present during a conference the note [12] , if the space H and the Lebesgue space L^p can be exchanged. It is true comparing our main theorem with Theorem 2.10 in [12]. In fact in the above mentioned paper it is proved that if u satisfy the parabolic equation (1.1) for $f \in L^p(Q_T)$, then

$$u \in H^q_p(-T, 0, L^p(\Omega)), \quad q = 1 - \frac{1}{p}$$

and our main theorem showed that if $f \in L^{p,\lambda}(Q_T)$ then

$$u \in L^{p,\lambda}(-T, 0, H^{1,p}(\Omega)).$$

In realizing the program it is not used Nash's techniques ([8]) because is easier for the author to obtain interior and boundary estimates as consequence of combined Morrey regularity and representation formula for the solution of (1.1) and its derivatives expressed, similarly to [2] and [3], in terms of singular integral operators and commutators with parabolic Calderón - Zygmund kernel (see [9]) and some less singular operators, that are in some sense taking after Hardy's operators.

A combination of these estimates with solvability in L^p spaces proved in [12] leads to the well-posedness of the Cauchy - Dirichlet problem (3.4) in Morrey class.

Finally we observe that global a priori estimate and the known properties of Morrey spaces for suitable p and λ (see [1]) allows us to derive global Hölder regularity result for the solution u.

2. Preliminary Tools and Representation Formula

We assume throughout the paper that Ω is an open bounded subset in $\mathbb{R}^n$, $n \geq 3$.

Definition 2.1. We set

$$H^{1,p,\lambda}_0(Q_T) = \Big\{ u \in H^{1,p}_0(Q_T) : Du = (\partial_{x_1} u, \dots, \partial_{x_n} u) \text{ is such that } \partial_{x_i} u \text{ and } \partial_t u \in L^{p,\lambda}(Q_T), \forall i = 1, \dots, n \Big\}.$$

Let us suppose the linear parabolic operator Ł having principal part

$$a_{ij}(x) = a_{ji}(x), \quad \forall x \in Q_T, \quad \forall i, j = 1, .., n$$

such that

$$\exists \mu > 0 : \mu^{-1}|\xi^2| \leq a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ a. e. } x \in Q_T.$$

Before the definition of parabolic Calderón-Zygmund kernel we set parabolic metric the quantity

$$d(x, y) = \rho(x - y)$$

where

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}$$

and the balls of radius σ and center 0 respect to metric are the ellipsoids

$$E_\sigma(0) = \{x = (x', t) \in \mathbb{R}^{n+1} : \frac{|x'|^2}{\sigma^2} + \frac{t^2}{\sigma^4} < 1\}.$$

In the sequel we denote E_σ as an ellipsoid in $\mathbb{R}^{n+1}$ of radius σ.

Definition 2.2. (see [9]). A function k is a Parabolic Calderón-Zygmund kernel on $\mathbb{R}^{n+1}$ with respect to the parabolic metric ρ if

(1) k is smooth on $\mathbb{R}^{n+1}\backslash\{0\}$;
(2) $k(rx', r^2t) = r^{-(n+2)}k(x', t)$, $\forall r > 0$ (homogeneity condition);
(3) $\displaystyle\int_{\rho(x)=r} k(x)d\sigma(x) = 0$, $\forall r > 0$ (cancellation property on ellipsoids).

In the following it will be useful to consider the fundamental solution of the constant coefficient operator $\mathcal{L}_0$ obtained by $\mathcal{L}$ freezing the coefficients at a fixed point $x_0 \in Q_T$.

$$\Gamma(x_0, \xi) = \begin{cases} \frac{1}{(4\pi(t+T)^{(\frac{n}{2})}\sqrt{\det\{a_{ij}(x_0)\}}} \exp\left(-\frac{\sum_{i,j=1}^n A_{ij}(x_0)\xi_i\xi_j}{4(t+T)}\right) & t + T > 0 \\ 0 & t + T < 0 \end{cases} \tag{2.1}$$

where A_{ij} are the entries of the inverse matrix of $\{a_{ij}\}_{i,j=1,\dots,n}$.

For our purposes it is fruitful to give the definition of John - Nirenberg class of Bounded Mean Oscillation functions (see [4]) and, as subclass, the Sarason class VMO of Vanishing Mean Oscillation functions (see [13]).

Definition 2.3. A locally integrable function f belongs to the John-Nirenberg space BMO of functions with bounded mean oscillation if the following quantity is finite

$$\|f\|_* \equiv \sup_{E_\sigma \subset \mathbb{R}^{n+1}} \frac{1}{|E_\sigma|} \int_{E_\sigma} |f(x) - f_{E_\sigma}|dx,$$

where

$$f_{E_\sigma} = \frac{1}{|E_\sigma|} \int_{E_\sigma} f(x)dx$$

and E_σ is any ellipsoid in $\mathbb{R}^{n+1}$ of radius σ.

Let us now introduce the space whom the coefficients a_{ij} belong.

Definition 2.4. Let us set $f \in BMO$ and E_σ in the class of the ellipsoids of $\mathbb{R}^{n+1}$ having radius $\sigma > 0$ and

$$\eta(R) = \sup_{E_\sigma \subset \mathbb{R}^{n+1}} \sup_{\sigma \leq R} \frac{1}{|E_\sigma|} \int_{E_\sigma} |f(x) - f_{E_\sigma}| dx$$

the VMO modulus of the function f.

We say that $f \in VMO$ if

$$\lim_{R \to 0} \eta(R) = 0.$$

Let us note that replacing $\mathbb{R}^{n+1}$ by Q_T we obtain the definitions of $BMO(Q_T)$ and $VMO(Q_T)$ preserving its character.

We now describe the Morrey spaces $L^{p,\lambda}(Q_T)$ whom the known term belongs.

Definition 2.5. Let $1 < p < \infty, 0 \leq \lambda < n+2$. We say that a locally integrable function f belongs to the Morrey class $L^{p,\lambda}(Q_T)$ if

$$\|f\|^p_{L^{p,\lambda}(Q_T)} = \sup_{\substack{\rho > 0 \\ x \in Q_T}} \frac{1}{\rho^\lambda} \int_{Q_T \cap E_\rho(x)} |f(y)|^p dy < +\infty.$$

Let us set $T(x) \equiv T(x', t; x', t)$ and $T(x', t; x'', t) = x' - \frac{2x'_n}{a_n(x'', t)} a_{nn}(x'', t)$, $x', x'' \in \mathbb{R}^n$ and any fixed, $t \in \mathbb{R}$ and $a_n(x', t) = (a_{in}(x', t)_{i=1,\dots,n})$ is the last column (row) of the matrix $\{a_{ij}\}_{i,j=1,\dots,n}$. Let us also consider $k(x, \cdot)$ a variable PCZ kernel and the following singular integral operators and commutators

$$Kf(x) = P.V. \int_{\mathbb{R}^{n+1}} k(x, x-y) f(y) dy$$

and

$$C[a, f] = P.V. \int_{\mathbb{R}^{n+1}} k(x, x-y)[a(y) - a(x)] f(y) dy.$$

In addition we use in the sequel the following notation for integral operators having nonsingular variable kernel

$$\tilde{K} f(x) = \int_{\mathbb{R}^{n+1}_+} k(x, T(x) - y) f(y) dy$$

and

$$\tilde{C}[a, f] = \int_{\mathbb{R}^{n+1}_+} k(x, T(x) - y)[a(y) - a(x)] f(y) dy.$$

Let us now recall $L^{p,\lambda}$ estimates (see [9] Theorem 3.3) based on technique consisting in eigenfunctions expansion of the kernel.

Theorem 2.6. *Let $1 < p < \infty$, $0 < \lambda < n+2$, $f \in L^p(\mathbb{R}^{n+1}_+)$ and $a \in BMO(\mathbb{R}^{n+1}_+)$. There exists a constant c independent of f such that*

$$||\tilde{K} f||_{L^{p,\lambda}(\mathbb{R}^{n+1}_+)} \leq c \cdot ||f||_{L^{p,\lambda}(\mathbb{R}^{n+1}_+)}. \tag{2.2}$$

and

$$||\tilde{C}[a, f]||_{L^{p,\lambda}(\mathbb{R}^{n+1}_+)} \leq c \cdot ||a||_* \cdot ||f||_{L^{p,\lambda}(\mathbb{R}^{n+1}_+)}. \tag{2.3}$$

The same results concerning $L^{p,\lambda}$ estimates for the operators $K(f)$ and $C[a,f]$ are considered in [14] and [10], precisely if $f \in L^{p,\lambda}(\mathbb{R}^{n+1})$ then

$$\|Kf\|_{L^{p,\lambda}(\mathbb{R}^{n+1})} \le c\|f\|_{L^{p,\lambda}(\mathbb{R}^{n+1})},$$

and

$$\|C(a,f)\|_{L^{p,\lambda}(\mathbb{R}^{n+1})} \le c\|a\|_*\|f\|_{L^{p,\lambda}(\mathbb{R}^{n+1})}.$$

The above estimates can be repeated substituting $\mathbb{R}^{n+1}$ with a subset Q of $\mathbb{R}^{n+1}$.

In addition we shall use the following result proved in [11] concerning with singular integrals having as kernel an homogeneous function.

Theorem 2.7. *Let Q be an open subset of $\mathbb{R}^{n+1}$, $1 < p < \infty$, $0 \le \nu < n+2$ such that $\nu + \alpha p < n+2$ and $k \in C(\mathbb{R}^{n+1} \setminus \{0\})$ be an homogeneous function of degree $-\alpha$, $\alpha \in]0, n+2[$, Then, for every $f \in L^{p,\nu}(Q)$ the operator*

$$Tf(x) = \int_Q k(x-y)f(y)dy$$

is defined, belongs to $L^{q,\mu}(Q)$ where $\frac{1}{p} + \frac{\alpha}{n+2} = \frac{1}{q} + 1$, $\mu = \frac{\nu q}{p}$ and exists a constant $c > 0$ independent on f such that

$$\|Tf\|_{L^{q,\mu}(Q)} \le c\|f\|_{L^{p,\nu}(Q)}.$$

3. Interior and Boundary Estimates and Main Results

Let us consider $\mathcal{E}$ an arbitrary set, $\mathcal{E} \subset\subset Q_T$, and

$$\mathcal{E}^+ = \{(x_1, .., x_n, t) \in \mathcal{E} : x_n > 0, t < 0\}.$$

Let us also introduce $\Theta \in C_0^\infty(Q_T)$ a cut-off function, $0 \le \Theta \le 1$, $\Theta(x) = 1$ in $E_{\frac{\sigma}{2}}$, $\Theta(x) = 0$ outside E_σ, $|\nabla\Theta| \le \frac{c}{\sigma}$, for some $E_\sigma \subset\subset Q_T$.

If u is a solution of $\mathcal{L}u = \operatorname{div} f$ on Q_T with zero boundary data, we can consider u as a solution of the equation

$$\mathcal{L}(\Theta u) = \operatorname{div} G + g$$

for $G = -a_{ij}\Theta_{x_i}u + \Theta f$ and $g = -a_{ij}\Theta_{x_j}u_{x_i} - \Theta_{x_j}f + \Theta_t u$.

Let us state the following interior and boundary representation formula as considered in [12].

Theorem 3.1. *(Interior representation formula). Let the hypotheses of symmetry and ellipticity for $a_{ij} \in VMO \cap L^\infty(\mathbb{R}^{n+1})$ hold and let E_σ be an arbitrary ellipsoid contained in Q_T, $G \in [C_0^\infty(E_\sigma)]^n$ and $g \in C_0^\infty(E_\sigma)$. Consider $v = \Theta u \in C_0^\infty(E_\sigma)$ as a solution of*

$$\mathcal{L}(\Theta u) = \operatorname{div} G + g.$$

Then,

$$\begin{aligned} (3.1)\quad v_{xi}(x) &= (\Theta u)_{xi}(x) \\ &= P.V. \int_{E_\sigma} \Gamma_{ij}(x, x-y)\{[a_{hj}(x) - a_{hj}(y)](\Theta u)_{x_h}(y) - G_j(y)\}dy \end{aligned}$$

$$+\int_{E_\sigma} g(y)\Gamma_i(x,x-y)dy + G_j(x)\int_{\sum_{n+1}} \Gamma_i(x,y)\eta_j d\sigma_y$$

where η_j stands for the j-th component of the outer normal to the following surface $\Sigma_{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

Using the representation formula, it is possible to prove the following interior estimate.

Theorem 3.2 (Interior estimate)**.** *Assume $a_{ij} \in VMO \cap L^\infty(\mathbb{R}^{n+1})$ symmetric and uniformly elliptic, u a weak solution of (1.1) such that $\frac{\partial u}{\partial x_i} u \in L^p(Q_T)$ $i = 1, \dots, n$, $f \in [L^{p,\lambda}(Q_T)]^n$, $2 < p < \infty$, $0 < \lambda < n+2$. Then $\exists \sigma_0 > 0$ and $\exists c > 0$ independent on f and u such that $\forall \sigma < \sigma_0$ we have*

$$\|\nabla u\|_{L^{p,\lambda}(E_{\frac{\sigma}{2}})} \leq c\left(\|u\|_{L^{p,\lambda}(E_\sigma)} + \|f\|_{L^{p,\lambda}(E_\sigma)}\right). \tag{3.2}$$

Theorem 3.3 (Boundary representation formula)**.** *Let us assume the above hypotheses about the coefficients a_{ij}, for some $E_\sigma \subset\subset Q_T$, $G \in [C^\infty(\overline{E}_\sigma^+)]^n$ and $g \in C^\infty(\overline{E}_\sigma^+)$ vanish in a neighborhood of $\mathbb{R}_+^{n+1} \cap \partial E_\sigma$.*

If v is a restriction to E_σ^+ of some function in $C_0^\infty(E_\sigma)$ vanishing in $\{\{x_n = 0\} \times]-T, 0[\} \cap \overline{E}_\sigma^+$ and satisfies the equation $\mathcal{L}u = \operatorname{div} G + g$ in $\overline{E}_\sigma^+$, then

$$v_{x_i} = P.V.\int_{E_\sigma^+} \Gamma_{ij}(x,x-y)\left\{[a_{hj}(x) - a_{hj}(y)]v_{x_h}(y) - G_j(y)\right\}dy$$
$$+ c_{ij}(x)G_j(x) + \int_{E_\sigma^+} \Gamma_i(x,x-y)g(y)dy + I_i(x), \quad \forall x \in E_\sigma^+,$$

where $c_{ij}(x) = \int_{\sum_{n+1}} \Gamma_i(x,y)\eta_j d\sigma_y$ are bounded functions arising from the interior representation formula

$$I_i(x) = \int_{E_\sigma^+} \Gamma_{ij}(x,T(x)-y)\left\{[a_{hj}(x) - a_{hj}(y)]\, v_{x_h}(y) - G_j(y)\right\}dy$$
$$-\int_{E_\sigma^+} \Gamma_i(x,T(x)-y)g(y)dy, \quad i = 1, \dots, n-1$$

and

$$I_n(x) = \int_{E_\sigma^+} B_h(y)\Gamma_{hj}(x,T(x)-y)\{[a_{hj}(x) - a_{hj}(y)]v_{x_h}(y) - G_j(y)\}dy$$
$$-\int_{E_\sigma^+} B_h(y)\Gamma_i(x,T(x)-y)g(y)dy,$$

where B_h are bounded functions that have L^∞ norm expressed in term of the ellipticity constant μ.

Theorem 3.4 (Boundary estimate)**.** *Let $a_{ij} \in VMO \cap L^\infty(\mathbb{R}^{n+1})$ and in addition symmetry and ellipticity conditions be true. Let us assume $f \in [L^{p,\lambda}(Q_T)]^n$ where $2 < p < +\infty$, $0 < \lambda < n+2$. Then there exists $\sigma_0 > 0$ such that $\forall \sigma < \sigma_0$ and for* *every u weak solution of* (1.1) *which vanishes on* $\{\{x_n = 0\}\times]-T, 0[\} \cap \overline{E}_\sigma^+$ *such that* $\frac{\partial u}{\partial x_i} \in L^p(Q_T)$ $i = 1, \dots, n$, *we have*

$$\|\nabla u\|_{L^{p,\lambda}(E^+_{\sigma/2})} \leq c\left(\|u\|_{L^{p,\lambda}(E^+_\sigma)} + \|f\|_{L^{p,\lambda}(Q_T)}\right) \tag{3.3}$$

for a suitable constant c independent on u and f.

Before we prove the above boundary estimate let us state the main result.

Theorem 3.5. *Let* $a_{ij} \in VMO \cap L^\infty(\mathbb{R}^{n+1})$ *be symmetric and uniformly elliptic. Suppose that* $f \in [L^{p,\lambda}(Q_T)]^n$ *where* $2 < p < +\infty$, $0 < \lambda < n + 2$.

Then the Cauchy-Dirichlet problem

$$\begin{cases} \mathcal{L}u = \operatorname{div} f & Q_T, \\ u = 0 & \text{on } \partial\Omega \times (-T, 0) \\ u(x', -T) = 0 & \text{in } \Omega. \end{cases} \tag{3.4}$$

has a unique solution $u \in H_0^{1,p,\lambda}(Q_T)$.

Proof of Theorem 3.4. Let us consider the above boundary representation formula

$$v_{x_i}(x) = C_{ij}[a_{hj}, v_{x_h}](x) - K_{ij}(G_j)(x) + c_{ij}(x)G_j(x) + T_i g(x) + I_i(x).$$

The first two integrals are singular and of the kind considered in [10], [14] as in [9], the term $T_i(g)$ is a bounded nonsingular integral. Then

$$\|\nabla v\|_{L^{p,\lambda}(E^+_\sigma)} \leq c\Big(\|a\|_* \cdot \|\nabla v\|_{L^{p,\lambda}(E^+_\sigma)} + \|G\|_{L^{p,\lambda}(E^+_\sigma)} + \|Tg\|_{L^{p,\lambda}(E^+_\sigma)} + \|I_i\|_{L^{p,\lambda}(E^+_\sigma)}\Big) \tag{3.5}$$

Let us study a majorization for $\|Tg\|_{L^{p,\lambda}(E^+_\sigma)}$. For it we are inspired by Theorem 2.6 in [11] and obtain

$$\|Tg\|_{L^{p,\lambda}(E^+_\sigma)} \leq c\|g\|_{L^{\tilde{p},\nu}(E^+_\sigma)}, \tag{3.6}$$

for $\frac{1}{\tilde{p}} + \frac{\alpha}{n+1} = \frac{1}{p} + 1; \lambda = \nu\frac{p}{\tilde{p}}, \nu + \alpha\tilde{p} < n + 2.$

Then from the definition of g

$$\|g\|_{L^{\tilde{p},\nu}(E^+_\sigma)} \leq \|g\|_{L^{p,\nu}(E^+_\sigma)} \leq c \cdot \left(\|\nabla u\|_{L^{p,\nu}(E^+_\sigma)} + \|f\|_{L^{p,\nu}(E^+_\sigma)} + \|u\|_{L^{p,\nu}(E^+_\sigma)}\right). \tag{3.7}$$

We also have

$$\|G\|_{L^{p,\lambda}(E^+_\sigma)} \leq c \cdot \left(\|u\|_{L^{p,\lambda}(E^+_\sigma)} + \|f\|_{L^{p,\lambda}(E^+_\sigma)}\right). \tag{3.8}$$

To majorize the term I_i using $L^{p,\lambda}$ estimates for nonsingular integral operators $\tilde{K}$ and $\tilde{C}$ and Theorem 2.7, we have

$$\|I\|_{L^{p,\lambda}(E^+_\sigma)} \leq c \cdot \left(\|a\|_* \cdot \|\nabla v\|_{L^{p,\lambda}(E^+_\sigma)} + \|G\|_{L^{p,\lambda}(E^+_\sigma)} + \|g\|_{L^{\tilde{p},\nu}(E^+_\sigma)}\right). \tag{3.9}$$

From the VMO hypothesis on the coefficients a_{ij}, (3.5), (3.6), (3.9) and let $\mu = \min(\lambda, \nu + p)$ then $\mu \le \lambda$, we obtain

$$\|\nabla v\|_{L^{p,\mu}(E_\sigma^+)} \le c \cdot \Big(\|G\|_{L^{p,\lambda}(E_\sigma^+)} + \|g\|_{L^{p,\nu}(E_\sigma^+)} \Big) .$$

Consider (3.7) and (3.8), we have

$$\|\nabla v\|_{L^{p,\mu}(E_\sigma^+)} \le c \cdot \Big(\|u\|_{L^{p,\lambda}(E_\sigma^+)} + \|f\|_{L^{p,\lambda}(Q_T)} + \|\nabla u\|_{L^{p,\nu}(Q_T)} \Big) . \tag{3.10}$$

It follows from the L^p result ([12]) that $\forall p > 2$ we have

$$\|\nabla u\|_{L^p(Q_T)} \le c \cdot \|f\|_{L^p(Q_T)}$$

and for $\nu = 0$ (3.10) implies

$$\|\nabla u\|_{L^{p,\mu}(E^+_{\frac{\sigma}{2}})} \le \|\nabla v\|_{L^{p,\mu}(E_\sigma^+)} \le c \cdot \Big(\|u\|_{L^{p,\lambda}(Q_T)} + \|f\|_{L^{p,\lambda}(Q_T)} \Big) .$$

If $\lambda \le p$ we get the conclusion, if $\lambda > p$ we obtain

$$\|\nabla u\|_{L^{p,p}(E^+_{\frac{\sigma}{2}})} \le c \cdot \Big(\|u\|_{L^{p,\lambda}(Q_T)} + \|f\|_{L^{p,\lambda}(Q_T)} \Big) .$$

Using again (3.10) for $\nu = p$ we have

$$\|\nabla u\|_{L^{p,\mu_1}(E^+_{\frac{\sigma}{2}})} \le c \cdot \Big(\|u\|_{L^{p,\lambda}(Q_T)} + \|f\|_{L^{p,\lambda}(Q_T)} \Big) .$$

where $\mu_1 = \min(\lambda, 2p)$. The improvement p is constant then in a finite number of steps we obtain (3.3). □

Proof of Theorem 3.5. Existence and uniqueness of the solution u of the Cauchy-Dirichlet problem (3.4) are true because are proved above interior and boundary estimates, the estimates for u_t are obtained writing $u_t = \mathcal{L}u + (a_{ij}u_{x_i})_{x_j}$ and applying Jensen's inequality (see [9]), because of $L^{p,\lambda} \subset L^p$ and for the study made in the case L^p in [12]. □

As a consequence of Theorem 3.5 and well known properties of Morrey spaces contained in [1] it is easy to have the following result.

Corollary 3.6. *Let u be a solution of* (3.4), $f \in [L^{p,\lambda}(Q_T)]^n$. *If $n+2-p < \lambda < n+2$ then, the solution u is a Hölder continuous function with exponent $\alpha = 1 - \frac{n+2-\lambda}{p}$.*

4. Appendix

We wish to point out that the main theorem prove that $\nabla u \in L^{p,\lambda}(Q_T)$, or equivalently

$$u \in L^{p,\lambda}(-T, 0, H^{1,p}(\Omega)), \quad \forall 2 < p < +\infty, \quad 0 < \lambda < n+2$$

introducing the following norm

$$\|u\|_{L^{p,\lambda}(-T,0,H^{1,p}(\Omega))} = \sup_{0<\sigma<\operatorname{diam} Q_T, X^0 \in Q_T} \left(\frac{1}{\sigma^\lambda} \int_{(-T,0)\cap(t^0-\sigma^2,t^0)} \Big[\int_{\Omega \cap B(x^0,\sigma)} |\nabla u|^p dx \Big] dt \right)^{\frac{1}{p}} .$$

This result can be view, as suggested by A. Maugeri, as a generalization, only for linear second order parabolic equations, of the study made by Marino and Maugeri in [7] because of the authors showed that if Ω is a bounded open set of $\mathbb{R}^n$ and $N \in \mathbb{N}$,

$$u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap C^0([-T, 0]; L^\infty(\Omega, \mathbb{R}^N))$$

is a solution of the second order nonlinear variational system

$$-\sum_{i=1}^{n} D_i a^i(x, u, Du) + \frac{\partial u}{\partial t} = B^i(x, u, Du)$$

where $a^i(x, u, Du)$ and $B^i(x, u, Du)$, $i = 1, \ldots, n$, $Du = (D_1 u, \ldots, D_n u)$, are vectors of $\mathbb{R}^N$ defined in $\Lambda = Q_T \times \mathbb{R}^N \times \mathbb{R}^{nN}$, measurable in x, continuous in (u, p), $\frac{\partial a^i(x,u,p)}{\partial p_k^j}$, $i, j = 1, \ldots, n$, $k = 1, \ldots, N$,are bounded in Ω such that $a^i(x, u, p)$ satisfy the strong ellipticity condition and the vectors a^i, B^i have a quadratic growth, then there exists $\overline{p} > 2$ such that $\forall p \in (2, \overline{p})$

$$D_i u \in L^p_{loc}(\Omega, \mathbb{R}^N), \quad i = 1, \ldots, n$$

and holds

$$u \in L^p_{loc}(-T, 0, H^{1,p}(\Omega, \mathbb{R}^N)).$$

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Maria Alessandra Ragusa
Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
E-mail address: maragusa@dmi.unict.it