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$W^{2,p}$-A PRIORI ESTIMATES FOR THE NEUTRAL POINCARÉ PROBLEM*

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To the memory of Filippo Chiarenza

Abstract. A degenerate oblique derivative problem is studied for uniformly elliptic operators with low regular coefficients in the framework of Sobolev's classes $W^{2,p}(\Omega)$ for *arbitrary* $p>1$. The boundary operator is prescribed in terms of a directional derivative with respect to the vector field $\boldsymbol{\ell}$ that becomes tangential to $\partial\Omega$ at the points of some non-empty subset $\mathcal{E}\subset\partial\Omega$ and is directed outwards Ω on $\partial\Omega\setminus\mathcal{E}$. Under quite general assumptions of the behaviour of $\boldsymbol{\ell}$, we derive *a priori* estimates for the $W^{2,p}(\Omega)$-strong solutions for any $p\in(1,\infty)$.

Introduction

The lecture deals with regularity in Sobolev's spaces $W^{2,p}(\Omega)$, $\forall\ p\in(1,\infty)$, of the strong solutions to the oblique derivative problem

$$\begin{cases}\mathcal{L}u:=a^{ij}(x)D_{ij}u=f(x) & \text{a.e. } \Omega,\\ \mathcal{B}u:=\partial u/\partial\boldsymbol{\ell}=\varphi(x) & \text{on } \partial\Omega\end{cases}\tag{1}$$

where $\mathcal{L}$ is a uniformly elliptic operator with low regular coefficients and $\mathcal{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $\boldsymbol{\ell}(x)=(\ell^1(x),\dots,\ell^n(x))$ defined on $\partial\Omega$, $n\geq 3$. Precisely, we are interested in the Poincaré problem (1) (cf. [17, 20, 16]), that is, a situation when $\boldsymbol{\ell}(x)$ becomes *tangential* to $\partial\Omega$ at the points of a non-empty subset $\mathcal{E}$ of $\partial\Omega$.

From a mathematical point of view, (1) is *not* an elliptic boundary value problem. In fact, it follows from the general PDEs theory that (1) is a *regular (elliptic)* problem *if and only if* the Shapiro–Lopatinskij complementary condition is satisfied which means $\boldsymbol{\ell}$ must be transversal to $\partial\Omega$ when $n\geq 3$ and $|\boldsymbol{\ell}|\neq 0$ as $n=2$. If $\boldsymbol{\ell}$ is *tangent* to $\partial\Omega$ then (1) is a *degenerate* problem and new effects occur in contrast to the regular case. It turns out that the qualitative properties of (1) depend on the behaviour of $\boldsymbol{\ell}$ near the set of tangency $\mathcal{E}$ and especially on the way the normal component $\gamma\boldsymbol{\nu}$ of $\boldsymbol{\ell}$ (with respect to the outward normal $\boldsymbol{\nu}$ to $\partial\Omega$) changes or no its sign on the trajectories of $\boldsymbol{\ell}$ when these cross $\mathcal{E}$. The main results were obtained by Hörmander [6], Egorov and Kondrat'ev [2], Maz'ya [8], Maz'ya and Paneah [9], Melin and Sjöstrand [10], Paneah [15] and good surveys and details can be found in

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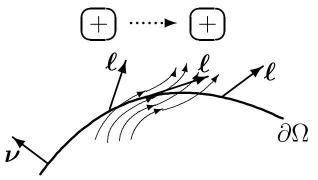
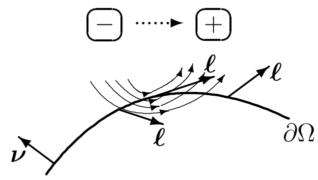
Popivanov and Palagachev [20] and Paneah [16]. The problem (1) has been studied in the framework of Sobolev spaces $H^s(\equiv H^{s,2})$ assuming C^∞-smooth data and this naturally involved techniques from the pseudo-differential calculus.

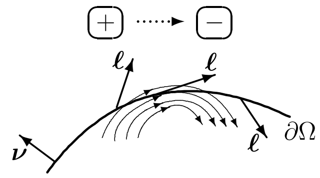
The simplest case arises when $\gamma := \boldsymbol{\ell} \cdot \boldsymbol{\nu}$, even if zero on $\mathcal{E}$, conserves the sign on $\partial\Omega$. Then $\mathcal{E}$ and $\boldsymbol{\ell}$ are of *neutral* type (a terminology coming from the physical interpretation of (1) in the theory of Brownian motion, see [20]) and (1) is a problem of Fredholm type (cf. [2]). Assume now that γ changes the sign from "$-$" to "$+$" in positive direction along the $\boldsymbol{\ell}$-integral curves passing through the points of $\mathcal{E}$. Then $\boldsymbol{\ell}$ is of *emergent* type and $\mathcal{E}$ is called *attracting* manifold. The new effect appearing now is that the *kernel* of (1) is *infinite-dimensional* ([6]) and to get a well-posed problem one has to modify (1) by prescribing the values of u on $\mathcal{E}$ (cf. [2]). Finally, suppose the sign of γ changes from "$+$" to "$-$" along the $\boldsymbol{\ell}$-trajectories. Now $\boldsymbol{\ell}$ is of *submergent* type and $\mathcal{E}$ corresponds to a *repellent* manifold. The problem (1) has *infinite-dimensional cokernel* ([6]) and Maz'ya and Paneah [9] were the first to propose a relevant modification of (1) by violating the boundary condition at the points of $\mathcal{E}$. As consequence, a Fredholm problem arises, but the restriction $u|_{\partial\Omega}$ has a finite jump at $\mathcal{E}$. What is the common feature of the degenerate problems, independently of the type of $\boldsymbol{\ell}$, is that the solution "loses regularity" near the set of tangency from the data of (1) in contrast to the non-degenerate case when any solution gains two derivatives from f and one derivative from φ. Roughly speaking, that loss of smoothness depends on the *order of contact* between $\boldsymbol{\ell}$ and $\partial\Omega$ and is given by the *subelliptic* estimates obtained for the solutions of degenerate problems (cf. [4, 5, 6, 9]). Precisely, if $\boldsymbol{\ell}$ has a contact of order k with $\partial\Omega$ then the solution of (1) gains $2 - k/(k+1)$ derivatives from f and $1 - k/(k+1)$ derivatives from φ.



(a) neutral vector field $\boldsymbol{\ell}$



(b) emergent vector field $\boldsymbol{\ell}$



(c) submergent vector field $\boldsymbol{\ell}$

For what concerns the geometric structure of $\mathcal{E}$, it was supposed initially to be a submanifold of $\partial\Omega$ of codimension one. Melin and Sjöstrand [10] and Paneah [15] were the first to study the Poincaré problem (1) in a more general situation when $\mathcal{E}$ is a massive subset of $\partial\Omega$ with positive surface measure, allowing $\mathcal{E}$ to contain arcs of $\boldsymbol{\ell}$-trajectories of *finite* length. Their results were extended by Winzell ([21, 22]) to the framework of Hölder's spaces who studied (1) assuming $C^{1,\alpha}$-smoothness of the coefficients of $\mathcal{L}$. It is worth noting that $\boldsymbol{\ell}$ has automatically an *infinite* order of contact with $\partial\Omega$ when $\mathcal{E}$ is a massive subset of the boundary.

To deal with non-linear Poincaré problems, however, we have to dispose of precise information on the linear problem (1) with coefficients less regular than C^∞ (see [11, 18, 19, 20]). Indeed, *a priori* estimates in $W^{2,p}$ for solutions to (1) would imply easily pointwise estimates for u and Du for suitable values of $p > 1$ through the Sobolev imbeddings. This way, we are naturally led to consider the problem (1) in a *strong* sense, that is, to searching for solutions lying in $W^{2,p}$ which satisfy $\mathcal{L}u = f$ almost everywhere (a.e.) in Ω and $\mathcal{B}u = \varphi$ holds in the sense of trace on $\partial\Omega$.

In the papers [4, 5] by Guan and Sawyer solvability and precise subelliptic estimates have been obtained for (1) in $H^{s,p}$-spaces ($\equiv W^{s,p}$ for integer s!). However, [4] treats operators with C^∞-coefficients and this determines the technique involved and the results obtained, while in [5] the coefficients are $C^{0,\alpha}$-smooth, but the field $\boldsymbol{\ell}$ is of finite type, that is, it has a *finite* order of contact with $\partial\Omega$.

The main goal of this lecture is to derive *a priori* estimates in Sobolev's classes $W^{2,p}(\Omega)$ with *any* $p \in (1,\infty)$ for the solutions of the Poincaré problem (1), weakening both Winzell's assumptions on $C^{1,\alpha}$-regularity of the coefficients of $\mathcal{L}$ and these of Guan and Sawyer on the *finite type* of $\boldsymbol{\ell}$. We are dealing with the simpler case when γ preserves its sign on $\partial\Omega$ which means the field $\boldsymbol{\ell}$ is of *neutral type.* Of course, the loss of smoothness mentioned, imposes some more regularity of the data near the set $\mathcal{E}$. We assume the coefficients of $\mathcal{L}$ to be Lipschitz continuous near $\mathcal{E}$ while only continuity (and even discontinuity controlled in VMO) is allowed away from $\mathcal{E}$. Similarly, $\boldsymbol{\ell}$ is a Lipschitz vector field on $\partial\Omega$ with Lipschitz continuous first derivatives near $\mathcal{E}$, and *no restrictions* on the order of contact with $\partial\Omega$ are required. Regarding the tangency set $\mathcal{E}$, it may have positive surface measure and is restricted only to a sort of *non-trapping* condition that all trajectories of $\boldsymbol{\ell}$ through the points of $\mathcal{E}$ are non-closed and leave $\mathcal{E}$ in a finite time.

The technique adopted is based on a dynamical system approach employing the fact that $\partial u/\partial\boldsymbol{\ell}$ is a local strong solution, near $\mathcal{E}$, to a Dirichlet-type problem with right-hand side depending on the solution u itself. Application of the L^p-estimates for such problems leads to the functional inequality (26) for suitable $W^{2,p}$-norms of u on a family of subdomains which, starting away from $\mathcal{E}$, evolve along the $\boldsymbol{\ell}$-trajectories and exhaust a sort of their tubular neighbourhoods. Fortunately, that is an inequality with advanced argument and the desired $W^{2,p}$-estimate follows by iteration with respect to the curvilinear parameter on the trajectories of $\boldsymbol{\ell}$. Another advantage of this approach is the *improving-of-integrability* property obtained for the solutions of (1). Roughly speaking, it asserts that the problem (1), even if a *degenerate* one, behaves as an *elliptic* problem for what concerns the degree p of integrability. In other words, the second derivatives of any solution to (1) will have the same rate of integrability as f and φ. We refer the reader to the paper [14] for outgrowths of the $W^{2,p}$-*a priori* estimates, such as uniqueness in $W^{2,p}(\Omega)$, $\forall\, p > 1$, of the strong solutions to (1) as well as its Fredholmness.

Concluding this introduction, we should mention the article [13] where similar results have been obtained by different technique in the particular case when the tangency set $\mathcal{E}$ contains trajectories of $\boldsymbol{\ell}$ with positive, but *small enough* lengths.

1. Hypotheses and the Main Result

Hereafter $\Omega \subset \mathbb{R}^n$, $n \geq 3$, will be a bounded domain with reasonably smooth boundary and $\boldsymbol{\nu}(x) = \big(\nu^1(x), \ldots, \nu^n(x)\big)$ stands for the unit *outward* normal to $\partial\Omega$ at $x \in \partial\Omega$. Consider a unit vector field $\boldsymbol{\ell}(x) = \big(\ell^1(x), \ldots, \ell^n(x)\big)$ on $\partial\Omega$ and let $\boldsymbol{\ell}(x) = \boldsymbol{\tau}(x) + \gamma(x)\boldsymbol{\nu}(x)$, where $\boldsymbol{\tau}\colon \partial\Omega \to \mathbb{R}^n$ is the projection of $\boldsymbol{\ell}(x)$ on the hyperplane tangent to $\partial\Omega$ at $x \in \partial\Omega$ and $\gamma\colon \partial\Omega \to \mathbb{R}$ is the inner product $\gamma(x) := \boldsymbol{\ell}(x) \cdot \boldsymbol{\nu}(x)$. The set of zeroes of γ,

$$\mathcal{E} := \big\{x \in \partial\Omega\colon\ \gamma(x) = 0\big\},$$

is indeed the subset of $\partial\Omega$ where the field $\boldsymbol{\ell}(x)$ becomes tangent to it.

Fix $\mathcal{N} \subset \overline{\Omega}$ to be a closed neighbourhood of $\mathcal{E}$ in $\overline{\Omega}$. We suppose $\mathcal{L}$ is a uniformly elliptic operator with measurable coefficients, satisfying

$$\lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{a.a. } x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \qquad a^{ij}(x) = a^{ji}(x) \tag{2}$$

for some positive constant λ. Regarding the regularity of the data, we assume

$$\begin{cases} a^{ij} \in VMO(\Omega) \cap C^{0,1}(\mathcal{N}), \\ \partial\Omega \in C^{1,1}, \quad \partial\Omega \cap \mathcal{N} \in C^{2,1}, \quad \ell^i \in C^{0,1}(\partial\Omega) \cap C^{1,1}(\partial\Omega \cap \mathcal{N}) \end{cases} \tag{3}$$

with $VMO(\Omega)$ being the Sarason class of functions of vanishing mean oscillation and $C^{k,1}$ denotes the space of functions with Lipschitz continuous k-th order derivatives. Let us point out that (2), (3) and the Rademacher theorem give $a^{ij} \in L^\infty(\Omega) \cap W^{1,\infty}(\mathcal{N})$. For what concerns the boundary operator $\mathcal{B}$, we assume

$$\begin{cases} \gamma(x) = \boldsymbol{\ell}(x)\cdot\boldsymbol{\nu}(x) \geq 0 \quad \forall\, x \in \partial\Omega, \qquad \text{and} \\ \text{the arcs of the } \boldsymbol{\ell}\text{-trajectories lying in } \mathcal{E} \text{ (which coincide with these of } \boldsymbol{\tau}) \\ \text{are all } \textit{non-closed} \text{ and of } \textit{finite lengths.} \end{cases} \tag{4}$$

The first assumption simply means that $\boldsymbol{\ell}(x)$ is either tangential to $\partial\Omega$ or is directed outwards Ω, that is, the field $\boldsymbol{\ell}$ is of *neutral type* on $\partial\Omega$, while the second one is a sort of *non-trapping* condition on the tangency set $\mathcal{E}$. It implies that the $\boldsymbol{\ell}$-integral curves *leave* $\mathcal{E}$ in a *finite time* in both directions.

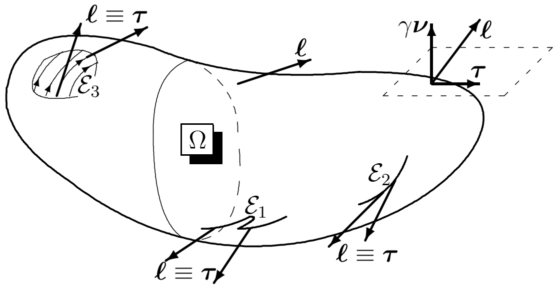


Figure 1. The set of tangency $\mathcal{E}$ is the union $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where $\operatorname{codim}_{\partial\Omega}\mathcal{E}_1 = \operatorname{codim}_{\partial\Omega}\mathcal{E}_2 = 1$ while $\operatorname{meas}_{\partial\Omega}\mathcal{E}_3 > 0$. The vector field $\boldsymbol{\ell}$ is transversal to $\mathcal{E}_1$ and tangent to $\mathcal{E}_2$. Actually, $\mathcal{E}_2$ consists of an arc of $\boldsymbol{\tau}$-trajectory, whereas $\mathcal{E}_3$ is union of such arcs.

Throughout the text $W^{k,p}$ stands for the Sobolev class of functions with L^p-summable weak derivatives up to order $k \in \mathbb{N}$ while $W^{s,p}(\partial\Omega)$ with $s > 0$ non-integer and $p \in (1, +\infty)$, is the Sobolev space of fractional order on $\partial\Omega$. Further, we use the standard parameterization $t \mapsto \boldsymbol{\psi}_{\boldsymbol{L}}(t; x)$ for the *trajectory* (equivalently, *phase curve, maximal integral curve*) of a given vector field $\boldsymbol{L}$ passing through a point x, that is, $\partial_t \boldsymbol{\psi}_{\boldsymbol{L}}(t; x) = \boldsymbol{L}\big(\boldsymbol{\psi}_{\boldsymbol{L}}(t; x)\big)$ and $\boldsymbol{\psi}_{\boldsymbol{L}}(0; x) = x$.

We will employ below an extension of the field $\boldsymbol{\ell}$ near $\partial\Omega$ which preserves therein its regularity and geometric properties. All the results and proofs in the sequel work for such an *arbitrary* $\boldsymbol{\ell}$-extension but, in order to make more evident some geometric constructions, we prefer to introduce a *special* extension as follows. For

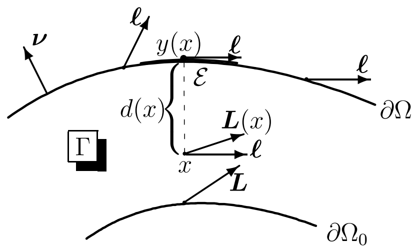


FIGURE 2

each $x \in \mathbb{R}^n$ near $\partial\Omega$ set $d(x) = \mathrm{dist}\,(x, \partial\Omega)$ and define $\Gamma := \{x \in \mathbb{R}^n\colon\ d(x) \le d_0\}$ with small $d_0 > 0$. Letting $\Omega_0 := \Omega \setminus \Gamma$ and $y(x) \in \partial\Omega$ for the unique point closest to $x \in \Gamma$, we have (see [3, Chapter 14]) $y(x) \in C^{0,1}(\Gamma)$ while $y(x) \in C^{1,1}$ near $\mathcal{E}$. Regarding the distance function $d(x) = |x - y(x)|$, it is Lipschitz continuous in Γ and inherits the regularity of $\partial\Omega$ at $y(x)$ when considered on the parts of Γ lying in/out Ω, but its normal derivative has a finite jump on $\partial\Omega$. Anyway, it is a routine to check $\big(d(x)\big)^2 \in C^{1,1}(\Gamma)$. Setting $\boldsymbol{L}(x)$ for the normalized representative of $\boldsymbol{\ell}(y(x)) + \big(d(x)\big)^2 \boldsymbol{\nu}(y(x))$ $\forall x \in \Gamma$, it results $|\boldsymbol{L}(x)| = 1$, $\boldsymbol{L}|_{\partial\Omega} = \boldsymbol{\ell}$, $\boldsymbol{L}|_{\mathcal{E}} = \boldsymbol{\tau}$ and $\boldsymbol{L} \in C^{0,1}(\Gamma) \cap C^{1,1}(\Gamma \cap \mathcal{N})$. Moreover, the field $\boldsymbol{L}$ is strictly transversal to $\partial\Omega_0$.

As consequence of the non-trapping condition (4), the compactness of $\mathcal{E}$ and the semi-continuity properties of the lengths of the $\boldsymbol{\tau}$-maximal integral curves, it is not hard to get that (see [22, Proposition 3.1] and [20, Proposition 3.2.4]) *under the hypotheses* (3) *and* (4), *there is a finite upper bound* κ_0 *for the arclengths of the* $\boldsymbol{\tau}$*-trajectories lying in* $\mathcal{E}$. *Moreover, each point of* Γ *can be reached from* $\partial\Omega_0$ *by an* $\boldsymbol{L}$*-integral curve of length at most* $\kappa = \mathrm{const} > 0$.

In what follows, the letter C will denote a generic constant depending on known quantities defined by the data of (1), that is, on n, p, λ, the respective norms of the coefficients of $\mathcal{L}$ and $\mathcal{B}$ in Ω and $\mathcal{N}$, the regularity of $\partial\Omega$ and the constants κ_0 and κ.

In order to control precisely the regularity of u near the tangency set $\mathcal{E}$, we have to introduce the appropriate functional spaces. For, take an arbitrary $p \in (1, \infty)$ and define the Banach spaces

$$\mathcal{F}^p(\Omega, \mathcal{N}) := \{f \in L^p(\Omega)\colon\ \partial f/\partial \boldsymbol{L} \in L^p(\mathcal{N})\}$$

equipped with norm $\|f\|_{\mathcal{F}^p(\Omega,\mathcal{N})} := \|f\|_{L^p(\Omega)} + \|\partial f/\partial \boldsymbol{L}\|_{L^p(\mathcal{N})}$, and

$$\Phi^p(\partial\Omega,\mathcal{N}) := \left\{\varphi \in W^{1-1/p,p}(\partial\Omega):\ \varphi \in W^{2-1/p,p}(\partial\Omega\cap\mathcal{N})\right\}$$

normed by $\|\varphi\|_{\Phi^p(\partial\Omega,\mathcal{N})} := \|\varphi\|_{W^{1-1/p,p}(\partial\Omega)} + \|\varphi\|_{W^{2-1/p,p}(\partial\Omega\cap\mathcal{N})}$.

Our main result asserts that the couple $(\mathcal{L},\mathcal{B})$ *improves the integrability* of solutions to (1) for any p in the range $(1,\infty)$ and, moreover, provides for an *a priori estimate* in the L^p-Sobolev scales for any such solution.

Theorem 1. *Under the hypotheses* (2)–(4) *let* $u \in W^{2,p}(\Omega)$ *be a strong solution of the problem* (1) *with* $f \in \mathcal{F}^q(\Omega,\mathcal{N})$ *and* $\varphi \in \Phi^q(\partial\Omega,\mathcal{N})$ *where* $1 < p \le q < \infty$.

Then $u \in W^{2,q}(\Omega)$ *and there is an absolute constant* C *such that*

$$\|u\|_{W^{2,q}(\Omega)} \le C\left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})}\right). \tag{5}$$

Let us point out reader's attention that the directional derivative $\partial u/\partial \boldsymbol{L}$ of each $W^{2,p}$-solution to (1) belongs to $W^{2,p}(\mathcal{N})$. For, $\partial u/\partial \boldsymbol{L} \in W^{1,p}(\mathcal{N})$ and taking the difference quotients in (1) in the direction of $\boldsymbol{L}$ (cf. [3, Chapter 8 and Lemma 7.24]) gives that $\partial u/\partial \boldsymbol{L} \in W^{2,p}(\mathcal{N})$ is a strong local solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}\left(\dfrac{\partial u}{\partial \boldsymbol{L}}\right) = \dfrac{\partial f}{\partial \boldsymbol{L}} + 2a^{ij}D_jL^kD_{ki}u + a^{ij}D_{ij}L^kD_ku - \dfrac{\partial a^{ij}}{\partial \boldsymbol{L}}D_{ij}u & \text{a.e. } \mathcal{N}, \\ \dfrac{\partial u}{\partial \boldsymbol{L}} = \varphi & \text{on } \partial\Omega\cap\mathcal{N} \end{cases} \tag{6}$$

where $\boldsymbol{L}(x) = (L^1(x),\dots,L^n(x)) \in C^{1,1}(\mathcal{N})$. Therefore, once having proved $u \in W^{2,q}(\Omega)$ and the estimate (5), we have

$$\|\partial u/\partial \boldsymbol{L}\|_{W^{2,q}(\widetilde{\mathcal{N}})} \le C'\left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})}\right)$$

for any *closed* neighbourhood $\widetilde{\mathcal{N}}$ of $\mathcal{E}$ in $\overline{\Omega}$, $\widetilde{\mathcal{N}} \subset \mathcal{N}$, by means of the L^p-theory of uniformly elliptic equations (see [1] or [3, Chapter 9]). In other words, *if a strong solution* u *to* (1) *belongs to* $W^{2,q}(\Omega)$ *then* $\partial u/\partial \boldsymbol{L} \in W^{2,q}(\mathcal{N})$ *automatically, provided* $f \in \mathcal{F}^q(\Omega,\mathcal{N})$ *and* $\varphi \in \Phi^q(\partial\Omega,\mathcal{N})$.

2. Proof of Theorem 1

Fix hereafter $\mathcal{N}' \subset \mathcal{N}'' \subset \mathcal{N}$ to be closed neighbourhoods of $\mathcal{E}$ in $\overline{\Omega}$ with $\mathcal{N}''$ so "narrow" that $\mathcal{N}'' \subset \Omega \setminus \Omega_0$ (see Figure 3). The next result is an immediate consequence of $\gamma(x) > 0\ \forall x \in \partial\Omega\setminus\mathcal{N}'$ and the L^p-theory of *regular* oblique derivative problems for uniformly elliptic operators with VMO principal coefficients (cf. [7, Theorem 2.3.1]).

Proposition 2. *Assume* (2), (3) *and* $\gamma(x) > 0\ \forall x \in \Omega \setminus \mathcal{E}$, *and let* $u \in W^{2,p}(\Omega)$ *be a solution to* (1) *with* $f \in L^q(\Omega)$ *and* $\varphi \in W^{1-1/q,q}(\partial\Omega)$, *where* $1 < p \le q < \infty$.

Then $u \in W^{2,q}(\Omega \setminus \mathcal{N}')$ *and there is a constant such that*

$$\|u\|_{W^{2,q}(\Omega\setminus\mathcal{N}')} \le C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\varphi\|_{W^{1-1/q,q}(\partial\Omega)}\right). \tag{7}$$

To derive the improving-of-integrability near the tangency set $\mathcal{E}$, we consider any solution of the problem (1) for which $a^{ij},\ \partial a^{ij}/\partial \boldsymbol{L} \in L^\infty(\mathcal{N})$ in view of (3)[1] and f, $\partial f/\partial \boldsymbol{L} \in L^q(\mathcal{N})$ and $\varphi \in W^{2-1/q,q}(\partial\Omega \cap \mathcal{N})$ by hypotheses.

Lemma 3. *Under the assumptions of Theorem* 1, *the solution* u *of* (1) *belongs to* $u \in W^{2,q}(\mathcal{N}'')$ *and there is a constant such that*

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\left(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})}\right). \tag{8}$$

Proof. Take an arbitrary point $x_0 \in \mathcal{E}$. According to (4), the $\boldsymbol{L}$-trajectory through x_0 leaves $\mathcal{E}$ in both directions for a finite time, that is, $\boldsymbol{\psi_L}(t^-;x_0) \in \mathcal{N}'' \setminus \mathcal{N}'$, $\boldsymbol{\psi_L}(t^+;x_0) \in \mathbb{R}^n \setminus \overline{\Omega}$ (see Figure 3) for suitable $t^- < 0 < t^+$.

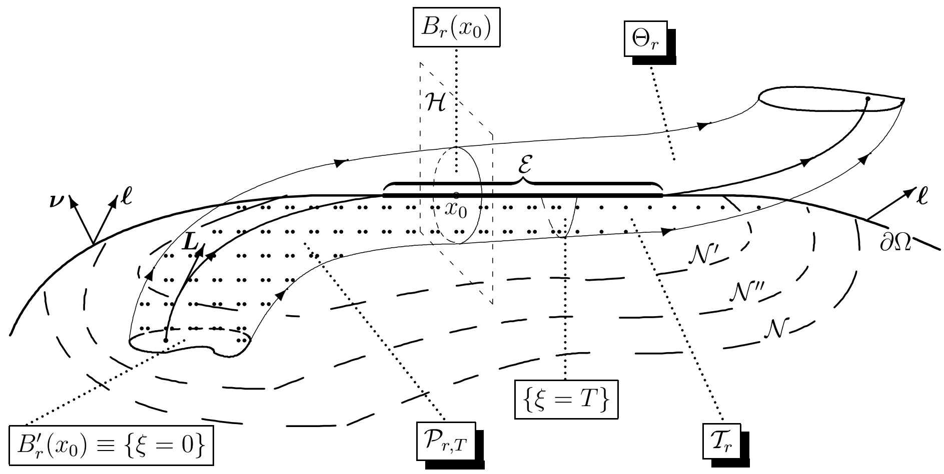


FIGURE 3. $\mathcal{T}_r$ is the dotted set, while the double-dotted one is $\mathcal{P}_{r,T}$.

Set $\mathcal{H}$ for the $(n-1)$-dimensional hyperplane through x_0 and orthogonal to $\boldsymbol{L}(x_0)$, and define

$$B_r(x_0) := \{x \in \mathcal{H}\colon \quad |x - x_0| < r\}$$

with $r > 0$ to be chosen later. It follows from the Picard inequality[2] that if r is small enough, then the flow of $B_r(x_0)$ along the $\boldsymbol{L}$-trajectories at time t^-,

$$B'_r(x_0) := \boldsymbol{\psi_L}(t^-; B_r(x_0)) := \left\{\boldsymbol{\psi_L}(t^-;y)\colon \quad y \in B_r(x_0)\right\}$$

is *entirely contained* in $\mathcal{N}'' \setminus \mathcal{N}'$ whence $B'_r(x_0) \cap \mathcal{E} = \emptyset$. The set

$$\Theta_r := \left\{\boldsymbol{\psi_L}(t;x')\colon \quad x' \in B'_r(x_0), \quad t \in (0, t^+ - t^-)\right\}$$

is an n-dimensional neighbourhood of the $\boldsymbol{L}$-trajectory through x_0 and defining

$$\mathcal{T}_r := \Theta_r \cap \Omega,$$

[1]It will be clear from the considerations given below that instead of Lipschitz continuity of the coefficients of $\mathcal{L}$ in $\mathcal{N}$ as (3) asks, it suffices to have essentially bounded their directional derivatives with respect to the field $\boldsymbol{L}$.

[2]$|\boldsymbol{\psi_L}(t;x') - \boldsymbol{\psi_L}(t;x'')| \leq e^{t\|\boldsymbol{L}\|_{C^1(\mathcal{N})}}|x' - x''|$ for all $x',\ x'' \in \mathcal{N}$.

the boundary $\partial\mathcal{T}_r$ is composed of the "base" $B'_r(x_0)$ and the "lateral" components $\partial_1\mathcal{T}_r := \partial\mathcal{T}_r \cap \partial\Omega$ and $\partial_2\mathcal{T}_r := (\partial\mathcal{T}_r \cap \Omega) \setminus B'_r(x_0)$. Indeed, $\mathcal{T}_r \subset \mathcal{N}''$ if $r > 0$ is small enough.

We will derive (8) in $\mathcal{T}_r$ after that the desired estimate will follow by covering the compact $\mathcal{E} \subset \partial\Omega$ by a finite number of sets like $\overline{\mathcal{T}_r}$. Our strategy is based on a representation of $u(x)$ in $\mathcal{T}_r$ by means of $u(x')$ with $x' = \boldsymbol{\psi}_{\boldsymbol{L}}(-\xi(x); x) \in B'_r(x_0)$ for some $\xi(x) > 0$, and the integral of $\partial u/\partial\boldsymbol{L}$ along the $\boldsymbol{L}$-trajectory joining x' with x. Thus the Sobolev norm of u will be expressed by the respective norm of $\partial u/\partial\boldsymbol{L}$ and that of u itself near $B'_r(x_0)$ where we dispose of (7). Concerning $\partial u/\partial\boldsymbol{L}$, it is a local solution of Dirichlet problem near $\mathcal{E}$ with right-hand side depending on u.

Let $\mu\colon \mathcal{H} \to \mathbb{R}^+$ be a C^∞ cut-off function such that

$$\mu(y) = \begin{cases} 1 & y \in B_{r/2}(x_0), \\ 0 & y \in \mathcal{H} \setminus B_{3r/4}(x_0) \end{cases} \tag{9}$$

and extend it to $\mathbb{R}^n$ as constant on the $\boldsymbol{L}$-trajectory through $y \in \mathcal{H}$. The function $U(x) := \mu(x)u(x)$ is a $W^{2,p}(\mathcal{N})$-solution of

$$\begin{cases} \mathcal{L}U = F(x) := \mu f + 2a^{ij}D_j\mu D_i u + ua^{ij}D_{ij}\mu & \text{a.e. } \mathcal{T}_r, \\ \partial U/\partial\boldsymbol{L} = \Phi := \begin{cases} \mu\varphi & \text{on } \partial_1\mathcal{T}_r, \\ 0 & \text{near } \partial_2\mathcal{T}_r, \\ \mu\partial u/\partial\boldsymbol{L} & \text{on } B'_r(x_0) \subset \mathcal{N}'' \setminus \mathcal{N}'. \end{cases} \end{cases} \tag{10}$$

Indeed, $u \in W^{2,p}(\mathcal{N})$ implies $Du \in L^{np/(n-p)}$ if $p < n$ and $Du \in L^s$ $\forall s > 1$ when $p \geq n$, whence $F \in L^{q'}(\mathcal{N})$ with

$$q' := \begin{cases} \min\left\{q, \dfrac{np}{n-p}\right\} & \text{if } p < n, \\ q & \text{if } p \geq n. \end{cases} \tag{11}$$

Further, $\partial F/\partial\boldsymbol{L} \in L^{q'}(\mathcal{N}'')$ as consequence of (6), $\partial u/\partial\boldsymbol{L} \in W^{2,q}(\mathcal{N}'' \setminus \mathcal{N}')$ by Proposition 2 whence $\Phi \in W^{2-1/q,q}(\partial\mathcal{T}_r)$. Thus (2), (3), $\mathcal{T}_r \subset \mathcal{N}''$ and (6) give that

$$V(x) := \partial U/\partial\boldsymbol{L}$$

is a $W^{2,p}(\mathcal{T}_r)$-solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}V = \partial F/\partial\boldsymbol{L} + 2a^{ij}D_jL^kD_{ik}U + a^{ij}D_{ij}L^kD_kU - \frac{\partial a^{ij}}{\partial\boldsymbol{L}}D_{ij}U & \text{a.e. } \mathcal{T}_r, \\ V = \Phi \quad \text{on } \partial\mathcal{T}_r. \end{cases} \tag{12}$$

Now we pass from $x \in \Theta_r$ into the new variables (x', ξ) with $x' = \boldsymbol{\psi}_{\boldsymbol{L}}(-\xi(x); x) \in B'_r(x_0)$ and $\xi\colon\ \Theta_r \to (0, t^+ - t^-)$, $\xi(x) \in C^{1,1}(\Theta_r)$. The transform $x \mapsto (x', \xi)$ defines a $C^{1,1}$-diffeomorphism because the field $\boldsymbol{L}$ is transversal to $B'_r(x_0)$. Moreover, $\partial/\partial\boldsymbol{L} \equiv \partial/\partial\xi$, $\boldsymbol{\psi}_{\boldsymbol{L}}(t; x') = (x', t)$ and $V(x', \xi) = \partial U(x', \xi)/\partial\xi$ as $(x', \xi) \in \mathcal{T}_r$. Since $V(x', \xi)$ is an absolutely continuous function in ξ for a.a. $x' \in B'_r(x_0)$) (after redefining it, if necessary, on a set of zero measure) we get

$$U(x', \xi) = U(x', 0) + \int_0^\xi V(x', t)dt \qquad \text{for a.a. } (x', \xi) \in \mathcal{T}_r, \tag{13}$$

where the point $(x',0) \in B'_r(x_0)$ lies in $\mathcal{N}''\backslash\mathcal{N}'$ and $U(x',0) \in W^{2,q}$ there by Proposition 2, the Fubini theorem and [12, Remark 2.1]. Passing to the new variables (x',ξ) in (12), taking the derivatives of (13) up to second order and substituting them into the right-hand side of (12), this last reads

$$\begin{cases} \mathcal{L}'V = F_1(x',\xi) + \displaystyle\int_0^\xi \mathcal{D}_2(\xi)V(x',t)dt & \text{a.e. } \mathcal{T}_r, \\ V = \Phi & \text{on } \partial\mathcal{T}_r, \end{cases} \tag{14}$$

where $\mathcal{L}'$ is the operator $\mathcal{L}$ in terms of $(x',\xi) = (x'_1,\dots,x'_{n-1},\xi)$,

$$\begin{aligned} F_1(x',\xi) &:= \partial F/\partial \boldsymbol{L} + \mathcal{D}_1 V(x',\xi) + \mathcal{D}'_1 U(x',\xi) + \mathcal{D}'_2 U(x',0), \\ \mathcal{D}_2(\xi)V(x',t) &:= \sum_{i,j=1}^{n-1} A^{ij}(x',\xi) D_{x'_i x'_j} V(x',t), \qquad A^{ij} \in L^\infty, \end{aligned} \tag{15}$$

$\mathcal{D}_1$, $\mathcal{D}'_1$, $\mathcal{D}'_2$ are linear differential operators with L^∞-coefficients, $\operatorname{ord}\mathcal{D}_1 = \operatorname{ord}\mathcal{D}'_1 = 1$, $\operatorname{ord}\mathcal{D}'_2 = 2$. The Sobolev imbedding theorem implies $F_1 \in L^{q'}(\mathcal{T}_r)$ with q' given by (11) as consequence of $\partial F/\partial\boldsymbol{L} \in L^{q'}(\mathcal{N}'')$, $U(x',0) \in W^{2,q}(B'_r(x_0))$ and $U,\ V \in W^{2,p}(\mathcal{N}'')$. Nevertheless the second-order operator $\mathcal{D}_2(\xi)$ has a quite rough characteristic form which is neither symmetric nor sign-definite, the improving-of-integrability holds for (14) thanks to the particular structure of $\mathcal{T}_r$ as union of $\boldsymbol{L}$-trajectories through $B'_r(x_0)$. Actually, we will show that if $V \in W^{2,q'}$ on a subset of $\mathcal{T}_r$ with $\xi < T$, then V remains a $W^{2,q'}$-function on a larger subset with $\xi < T + r$ for small enough r, after that the higher integrability of U will follow from Proposition 2 and (13). For, take an arbitrary $T \in (0, t^+ - t^-)$ and define

$$\mathcal{P}_{r,T} := \left\{(x',\xi) \in \mathcal{T}_r\colon \quad \xi < T\right\}.$$

For a fixed $r > 0$, $\left\{\mathcal{P}_{r,T}\right\}_{T\geq 0}$ is a non-decreasing family of domains exhausting $\mathcal{T}_r$ and $\mathcal{P}_{r,T} \equiv \mathcal{T}_r$ for values of T greater than the *maximal exit-time*

$$T_{\max} := \sup_{x'\in B'_r(x_0)} \sup\left\{t > 0\colon \quad \boldsymbol{\psi}_{\boldsymbol{L}}(t;x') \in \Omega,\ x' \in B'_r(x_0)\right\}.$$

Proposition 4. *Let $T \in (0, t^+ - t^-)$ and consider the solution $V \in W^{2,p}(\mathcal{T}_r)$ of the problem* (14). *Suppose $V \in W^{2,q'}(\mathcal{P}_{r,T})$ where q' is given by* (11).

There exists an $r_0 > 0$ such that $V \in W^{2,q'}(\mathcal{P}_{r,T+r})$ for all $r < r_0$.

Proof. There are three possible cases to be distinguished.
Case A: $T + 3r < T_{\max}$. We have $\mathcal{P}_{r,T} \subset \mathcal{P}_{r,T+3r} \subset \mathcal{T}_r \equiv \mathcal{P}_{r,T_{\max}}$ and consider the C^∞-function $\eta\colon\ \mathbb{R} \to [0,1]$ such that

$$\eta(\xi) = \begin{cases} 1 & \text{as } \xi \in (-\infty, T+r], \\ \text{strictly decreases} & \text{as } \xi \in (T+r, T+2r), \\ 0 & \text{as } \xi \geq T+2r. \end{cases} \tag{16}$$

Setting $\widetilde{V}(x',\xi) := \eta(\xi)V(x',\xi)$, it follows $\mathcal{L}'\widetilde{V} = \eta(\mathcal{L}'V) + \mathcal{L}_1 V$ where $\mathcal{L}_1$ is a first-order differential operator with L^∞-coefficients depending on these of $\mathcal{L}'$ and on the derivatives of η. Therefore,

$$\mathcal{L}'\widetilde{V} = \eta F_1 + \mathcal{L}_1 V + \eta(\xi)\int_0^\xi \mathcal{D}_2(\xi)V(x',t)dt \tag{17}$$
$$= \eta F_1 + \mathcal{L}_1 V + \int_0^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt$$

because $\mathcal{D}_2(\xi)$ is a second-order operator acting in the x'-variables only.

We set $\Omega_r \subset \mathcal{P}_{r,T+3r}\setminus\mathcal{P}_{r,T-3r}$ for a $C^{1,1}$-smooth domain containing $\mathcal{P}_{3r/4,T+2r}\setminus \mathcal{P}_{3r/4,T-2r}$ and such that

$$r^{-1}\Omega_r := \left\{(\widetilde{y}',\widetilde{\xi}):\quad \widetilde{y}' = x'/r,\ \widetilde{\xi} = (\xi-T)/r,\ (x',\xi)\in\Omega_r\right\}\in C^{1,1}$$

uniformly in r. The boundary $\partial\Omega_r$ consists of the “lateral” parts $\partial_1\Omega_r := \partial\Omega_r\cap\partial\Omega$ and $\partial_2\Omega_r := \partial\Omega_r\cap\Omega\cap\{\xi\in(T-2r,T+2r)\}\subset\big(\mathcal{P}_{r,T+2r}\setminus\mathcal{P}_{r,T-2r}\big)\setminus\big(\mathcal{P}_{3r/4,T+2r}\setminus \mathcal{P}_{3r/4,T-2r}\big)$, and of two $C^{1,1}$-smooth components $\partial\Omega_r^\pm$ lying in $\mathcal{P}_{r,T+3r}\setminus\mathcal{P}_{r,T+2r}$ and $\mathcal{P}_{r,T-2r}\setminus\mathcal{P}_{r,T-3r}$, respectively. The properties of μ (cf. (9)) ensure $U\equiv 0$, $V\equiv 0$, $\widetilde{V}\equiv 0$ on $\mathcal{T}_r\setminus\mathcal{T}_{3r/4}$ whence $\widetilde{V}\equiv 0$ near $\partial_2\Omega_r$.

For an arbitrary $(x',\xi)\in\Omega_r$, the factor $\eta(\xi)/\eta(t)$ in (17) vanishes when $\xi\geq T+2r$ while $\eta(\xi)/\eta(t)\leq 1$ because η decreases in $(T+r,T+2r)$. Moreover, $|\xi-T|<3r$ for $(x',\xi)\in\Omega_r$ and

$$\int_0^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt = \int_0^T \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt + \int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt$$
$$= \eta(\xi)\int_0^T \mathcal{D}_2(\xi)V(x',t)dt + \int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt$$

by means of (15) and since $\eta(t)=\eta(T)=1$ as $t\leq T$.

We get from (14) and (17) that $\widetilde{V}\in W^{2,p}(\Omega_r)$ solves the Dirichlet problem

$$\begin{cases} \mathcal{L}'\widetilde{V} = F_2(x',\xi) + \displaystyle\int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)\widetilde{V}(x',t)dt & \text{a.a. } (x',\xi)\in\Omega_r,\\ \widetilde{V} = \widetilde{\Phi} := \eta\Phi = \begin{cases} \eta\mu\varphi\in W^{2-1/q,q} & \text{on } \partial_1\Omega_r \quad \text{(by (10))},\\ 0 & \text{on } \partial_2\Omega_r \quad \text{(by (10))},\\ 0 & \text{on } \partial\Omega_r^+ \quad \text{(by (16))},\\ V\in W^{2-1/q',q'} & \text{on } \partial\Omega_r^- \quad \text{(since } \xi<T-2r \text{ and } V\in W^{2,q'}(\mathcal{P}_{r,T})) \end{cases} \end{cases} \tag{18}$$

where, recalling $V\in W^{2,q'}(\mathcal{P}_{r,T})$, we have

$$F_2(x',\xi) := \eta F_1 + \mathcal{L}_1 V + \eta(\xi)\int_0^T \mathcal{D}_2(\xi)V(x',t)dt \in L^{q'}(\Omega_r). \tag{19}$$

We are going to prove now that $\widetilde{V}\in W^{2,q'}(\Omega_r)$ for small enough $r>0$, whence it will follow $V\in W^{2,q'}(\mathcal{P}_{r,T+r})$ in view of (16) and $V\equiv 0$ near $\partial_2\Omega_r$. The claim is obvious if $q'=p$ because $V\in W^{2,p}(\mathcal{T}_r)$. Otherwise, take an arbitrary $s\in[p,q']$ and denote by $W^{2,s}_*(\Omega_r)$ the Sobolev space $W^{2,s}(\Omega_r)$ normed with

$$\|u\|_{W^{2,s}_*(\Omega_r)} := \|u\|_{L^s(\Omega_r)} + r\|Du\|_{L^s(\Omega_r)} + r^2\|D^2u\|_{L^s(\Omega_r)}.$$

Define now the operator $\mathfrak{F}\colon W^{2,s}_*(\Omega_r) \to W^{2,s}_*(\Omega_r)$ as follows: for any $w \in W^{2,s}_*(\Omega_r)$ the image $\mathfrak{F}w \in W^{2,s}_*(\Omega_r)$ is the *unique* solution of the Dirichlet problem

$$(20)\qquad \begin{cases} \mathcal{L}'(\mathfrak{F}w) = F_2 + \displaystyle\int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)w(x',t)dt \in L^s(\Omega_r) & \text{a.a. } (x',\xi)\in\Omega_r,\\ \mathfrak{F}w = \widetilde{\Phi} \in W^{2-1/s,s}(\partial\Omega_r) & \text{on } \partial\Omega_r. \end{cases}$$

We will prove that $\mathfrak{F}$ is a contraction for small values of r. For this goal, take arbitrary $w_1,\ w_2 \in W^{2,s}_*(\Omega_r)$. The difference $\mathfrak{F}w_1 - \mathfrak{F}w_2$ solves

$$(21)\qquad \begin{cases} \mathcal{L}'(\mathfrak{F}w_1 - \mathfrak{F}w_2) = \displaystyle\int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)(w_1-w_2)(x',t)dt & \text{a.a. } (x',\xi)\in\Omega_r,\\ \mathfrak{F}w_1 - \mathfrak{F}w_2 = 0 & \text{on } \partial\Omega_r. \end{cases}$$

In order to apply the L^s-*a priori* estimates from [1] or [3] for the solutions of (21), we have to control the dependence on r therein. For, we recall that $r^{-1}\Omega_r \in C^{1,1}$ uniformly in r and apply a standard approach consisting of dilation of Ω_r onto $r^{-1}\Omega_r$, reduction of the problem (21) to a new one in variables $(\widetilde{y}', \widetilde{\xi}) \in r^{-1}\Omega_r$, application of the L^s-estimates from [3, Theorem 9.17] and finally turning back to (21) (see the Proof of Lemma 2.2, Eq. (2.12) in [12]). This way, one gets

$$(22)\qquad \|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{W^{2,s}_*(\Omega_r)} \le Cr^2 \left\| \int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)(w_1-w_2)(x',t)dt \right\|_{L^s(\Omega_r)}$$

where the constant C is independent of r. Jensen's integral inequality yields

$$r^2 \left\| \int_T^\xi \frac{\eta(\xi)}{\eta(t)}\mathcal{D}_2(\xi)(w_1-w_2)(x',t)dt \right\|_{L^s(\Omega_r)} \le C \max_{(x',\xi)\in\Omega_r} |\xi - T| \|w_1 - w_2\|_{W^{2,s}_*(\Omega_r)}$$

and thus (22) rewrites into

$$\|\mathfrak{F}w_1 - \mathfrak{F}w_2\|_{W^{2,s}_*(\Omega_r)} \le C \max_{(x',\xi)\in\Omega_r} |\xi - T| \|w_1 - w_2\|_{W^{2,s}_*(\Omega_r)}.$$

We have $\max_{(x',\xi)\in\Omega_r}|\xi - T| < 3r$, C is independent of r and therefore $\mathfrak{F}$ will be really a contraction from $W^{2,s}_*(\Omega_r)$ into itself for any $s \in [p,q']$ if $r \le r_0$ with r_0 under control and small enough. Fixing $r = r_0/2$, there is a unique fixed point of $\mathfrak{F}$ in $W^{2,s}_*(\Omega_r)$ for all $s\in[p,q']$. However, $\widetilde{V} \in W^{2,p}(\Omega_r)$ is already a fixed point of $\mathfrak{F}$ since it solves (18) and therefore $\widetilde{V} \in W^{2,q'}(\Omega_r)$. It follows $V \in W^{2,q'}(\mathcal{P}_{r,T+r})$ by means of $V \in W^{2,q'}(\mathcal{P}_{r,T})$, $\widetilde{V} \equiv 0$ on $\mathcal{T}_r \setminus \mathcal{T}_{3r/4}$ and the properties of $\eta(\xi)$.

Case B: $T < T_{\max} \le T + 3r$. We have $\mathcal{T}_r \setminus \mathcal{P}_{r,T} \ne \emptyset$, $\mathcal{P}_{r,T+3r} \equiv \mathcal{T}_r$ now and we do not need anymore the cut-off function η because $V = \partial U/\partial \boldsymbol{L} \equiv 0$ near the points of $\partial_2\mathcal{T}_r$ where $\xi > T$ (cf. (9)). Thus, it suffices to repeat the above arguments with $\eta(\xi) \equiv 1\ \forall\xi\in\mathbb{R}$ and $\Omega_r \in C^{1,1}$ defined as before when $\xi \le T$ while $\mathcal{T}_{3r/4}\setminus\mathcal{P}_{3r/4,T} \subset \big(\Omega_r \cap \{\xi > T\}\big) \subset \mathcal{T}_r\setminus\mathcal{P}_{r,T}$ (cf. (9)). We have anyway a problem like (18) for $V \equiv \widetilde{V}$ with boundary condition

$$V = \partial U/\partial\boldsymbol{L} = \begin{cases} \mu\varphi \in W^{2-1/q,q} & \text{on } \partial_1\Omega_r = \partial\Omega_r \cap \partial\Omega,\\ 0 & \text{on } \partial_2\Omega_r = \partial\Omega_r\cap\Omega\cap\{\xi > T - 3r\},\\ V \in W^{2-1/q',q'} & \text{on } \partial\Omega_r^- \quad \text{(by hypothesis)}. \end{cases}$$

Therefore, the procedure from *Case A* gives $V \in W^{2,q'}(\mathcal{P}_{r,T+3r})$.
Case C: $T_{\max} \le T$. We have $\mathcal{P}_{r,T+r} \equiv \mathcal{P}_{r,T} \equiv \mathcal{T}_r$ now and thus the claim. □

Proposition 5. *Suppose* $r < r_0$ *with* r_0 *given in Proposition* 4. *Then the solution* V *of the problem* (14) *lies in* $W^{2,q}(\mathcal{T}_r)$ *and satisfies the estimate*

$$(23)\quad \|V\|_{W^{2,q}(\mathcal{T}_r)} \le C\Big(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{T}_r)}\Big).$$

Proof. We note that $V \in W^{2,q} \subseteq W^{2,q'}$ near $B'_r(x_0)$ in view of $B'_r(x_0) \subset \mathcal{N}'' \setminus \mathcal{N}'$, Proposition 2 and (6). Therefore, successive applications of Proposition 4 with increasing values of T will give $V \in W^{2,q'}(\mathcal{T}_r)$, $q' > p$. After that, in order to get $V \in W^{2,q}(\mathcal{T}_r)$, it suffices to put q' in the place of p in (11) and to repeat finitely many times the above arguments until $q' = q$.

To obtain (23), we take $T \in (0, t^+ - t^-)$ to be arbitrary, fix $r = r_0/2$, and consider the domains Ω_r defined in the proof of Proposition 4. Let $\widetilde{V} = \eta V \in W^{2,q}(\mathcal{T}_r)$ solve (18) with η given by (16) in *Case A* and $\eta \equiv 1$ in *Case B.* Since $\widetilde{V}$ is a fixed point of the mapping $\mathfrak{F}\colon\ W^{2,q}(\Omega_r) \to W^{2,q}(\Omega_r)$, $\mathfrak{F}\widetilde{V} = \widetilde{V}$, we get

$$\|D^2\widetilde{V}\|_{L^q(\Omega_r)} = \|D^2(\mathfrak{F}\widetilde{V})\|_{L^q(\Omega_r)} \le \|D^2(\mathfrak{F}\widetilde{V} - \mathfrak{F}0)\|_{L^q(\Omega_r)} + \|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)},$$

while

$$\|D^2(\mathfrak{F}w_1 - \mathfrak{F}w_2)\|_{L^q(\Omega_r)} \le \theta\|D^2(w_1 - w_2)\|_{L^q(\Omega_r)} \quad \forall w_1,\ w_2 \in W^{2,q}(\Omega_r), \quad \theta < 1$$

because $\mathfrak{F}$ is a contraction, (22) and the fact that $\mathcal{D}_2(\xi)$ is a homogeneous second-order operator (cf. (15)). This way, $\|D^2(\mathfrak{F}\widetilde{V} - \mathfrak{F}0)\|_{L^q(\Omega_r)} \le \theta\|D^2(\widetilde{V} - 0)\|_{L^q(\Omega_r)} = \theta\|D^2\widetilde{V}\|_{L^q(\Omega_r)}$ and therefore

$$(24)\qquad \|D^2\widetilde{V}\|_{L^q(\Omega_r)} \le C\|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)}$$

with $\mathfrak{F}0 \in W^{2,q}(\Omega_r)$ being the unique solution of the Dirichlet problem

$$\Big\{\mathcal{L}'(\mathfrak{F}0) = F_2 \quad \text{a.e. } \Omega_r, \qquad \mathfrak{F}0 = \widetilde{\Phi} \quad \text{on } \partial\Omega_r$$

(see (20)), for which the L^p-theory (cf. [3, Chapter 9]) gives

$$(25)\qquad \|D^2(\mathfrak{F}0)\|_{L^q(\Omega_r)} \le \|\mathfrak{F}0\|_{W^{2,q}(\Omega_r)} \le C\left(\|F_2\|_{L^q(\Omega_r)} + \|\widetilde{\Phi}\|_{W^{2-1/q,q}(\partial\Omega_r)}\right).$$

Direct applications, based on (19) and (15), yield

$$\begin{aligned}\|F_2\|_{L^q(\Omega_r)} &= \left\|\eta F_1 + \mathcal{L}_1 V + \eta(\xi)\int_0^T \mathcal{D}_2(\xi)V(x',t)dt\right\|_{L^q(\Omega_r)} \\ &\le C\Big(\|\partial F/\partial \boldsymbol{L}\|_{L^q(\Omega_r)} + \|U\|_{W^{2,q}(\mathcal{N}''\setminus\mathcal{N}')} + \|U\|_{W^{1,q}(\mathcal{T}_r)} + \|V\|_{W^{1,q}(\mathcal{T}_r)} \\ &\quad + \|D^2V\|_{L^q(\mathcal{P}_{r,T})}\Big) \\ &\le C\Big(\|\partial f/\partial \boldsymbol{L}\|_{L^q(\mathcal{N})} + \|u\|_{W^{2,q}(\mathcal{N}''\setminus\mathcal{N}')} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{T}_r)} \\ &\quad + \|D^2V\|_{L^q(\mathcal{P}_{r,T})}\Big)\end{aligned}$$

in view of (7), (10), $U = \mu u$, $V = \partial U/\partial \boldsymbol{L}$ and (9). Moreover,

$$\begin{aligned}\|\widetilde{\Phi}\|_{W^{2-1/q,q}(\partial\Omega_r)} &\le C\left(\|\varphi\|_{W^{2-1/q,q}(\partial\Omega\cap\mathcal{N})} + \|V\|_{W^{2,q}(\mathcal{P}_{r,T})}\right)\\ &\le C\left(\|\varphi\|_{W^{2-1/q,q}(\partial\Omega\cap\mathcal{N})} + \|V\|_{W^{1,q}(\mathcal{T}_r)} + \|D^2V\|_{L^q(\mathcal{P}_{r,T})}\right)\\ &\le C\left(\|\varphi\|_{W^{2-1/q,q}(\partial\Omega\cap\mathcal{N})} + \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{T}_r)} + \|D^2V\|_{L^q(\mathcal{P}_{r,T})}\right)\end{aligned}$$

by (18) and $\partial\Omega_r^- \subset \mathcal{P}_{r,T}$. Further on, $\widetilde{V} = V$ on $\mathcal{P}_{r,T+r}$, whence

$$\|D^2V\|_{L^q(\mathcal{P}_{r,T+r})} \le \|D^2V\|_{L^q(\mathcal{P}_{r,T})} + \|D^2\widetilde{V}\|_{L^q(\Omega_r)}.$$

Therefore, setting $\zeta(T) := \|D^2V\|_{L^q(\mathcal{P}_{r,T})}$ and $K := \|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{T}_r)} + \|\partial u/\partial\boldsymbol{L}\|_{W^{1,q}(\mathcal{T}_r)}$, it follows from (24), (25) and Proposition 2 that

$$\zeta(T+r) \le C\left(K + \zeta(T)\right) \qquad \forall T \in (0, t^+ - t^-). \tag{26}$$

To get (23), we let m to be the least integer such that $T_{\max} \le mr$ and iterate (26) in order to obtain

$$\begin{aligned}\|D^2V\|_{L^q(\mathcal{T}_r)} &= \|D^2V\|_{L^q(\mathcal{P}_{r,T_{\max}})} = \zeta(T_{\max}) = \zeta(mr) = \zeta((m-1)r + r)\\ &\le C\big(K + \zeta((m-1)r)\big) = C\big(K + \zeta((m-2)r + r)\big)\\ &\le K(C + C^2) + C^2\zeta\big((m-2)r\big)\\ &\ \ \vdots\\ &\le K\sum_{j=1}^{m} C^j + C^m\zeta(0) = K\sum_{j=1}^{m} C^j\end{aligned}$$

This proves (23). □

Remark 6. It is important to note that the constant C in Proposition 5 depends on m through $T_{\max}$, and therefore on the point $x_0 \in \mathcal{E}$. Actually, that constant will have the very same value for each other point of $\mathcal{E}$ lying on the same $\boldsymbol{L}$-trajectory as x_0.

Moreover, if the *improving-of-integrability property* asserted in Propositions 4 and 5 holds on a set $S \subset \overline{\Omega}$ then it is guaranteed, on the base of (13), on any other set which can be reached from S along $\boldsymbol{L}$-trajectories.

To complete the proof of Lemma 3, we select a finite set $\{\mathcal{T}_r^j\}_{j=1}^N$ of neighbourhoods covering the *compact* $\mathcal{E}$, each of the type $\mathcal{T}_r$ above with $r = r_0/2$, and such that $\mathcal{T} :=$ closure $\left(\bigcup_{j=1}^N \mathcal{T}_{r/2}^j\right) \subset \mathcal{N}''$ is a closed neighbourhood of $\mathcal{E}$ in $\overline{\Omega}$. It is clear that Proposition 2 remains true with $\mathcal{T}$ instead of $\mathcal{N}'$ and then (7) rewrites into

$$\|u\|_{W^{2,q}(\Omega\setminus\mathcal{T})} \le C\left(\|u\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)} + \|\varphi\|_{W^{1-1/q,q}(\partial\Omega)}\right). \tag{27}$$

The *improving-of-integrability* claimed in Lemma 3 then follows from (13), Proposition 5 and (27) (recall $U = u$ on $\mathcal{T}_{r/2}^j$). Similarly, (13), (27) and (23) yield

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \le \|u\|_{W^{2,q}(\mathcal{T})} + \|u\|_{W^{2,q}(\mathcal{N}''\setminus\mathcal{T})} \tag{28}$$

$$\leq C\big(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N})} + \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{N})}\big).$$

Later on, $\mathcal{N} \setminus \mathcal{N}'' \subset \Omega \setminus \mathcal{N}'$ and

$$\|u\|_{W^{1,q}(\mathcal{N})} \leq \|u\|_{W^{1,q}(\mathcal{N}'')} + \|u\|_{W^{1,q}(\mathcal{N}\setminus\mathcal{N}'')} \leq \varepsilon\|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\big(\|u\|_{L^q(\Omega)} + \|u\|_{W^{2,q}(\Omega\setminus\mathcal{N}')}\big)$$

in view of the interpolation inequality for the $W^{2,q}(\mathcal{N}'')$-norms with $\varepsilon > 0$ under control[3]. In the same manner,

$$\|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{N})} \leq \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{N}')} + \|\partial u/\partial \boldsymbol{L}\|_{W^{1,q}(\mathcal{N}\setminus\mathcal{N}')} \leq \varepsilon\|\partial u/\partial \boldsymbol{L}\|_{W^{2,q}(\mathcal{N}')} + C(\varepsilon)\big(\|\partial u/\partial \boldsymbol{L}\|_{L^q(\mathcal{N}')} + \|u\|_{W^{2,q}(\Omega\setminus\mathcal{N}')}\big),$$

while

$$\|\partial u/\partial \boldsymbol{L}\|_{W^{2,q}(\mathcal{N}')} \leq C\big(\|u\|_{W^{2,q}(\mathcal{N}'')} + \|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})}\big)$$

by means of the local *a priori* estimates ([3, Theorem 9.11]) for the problem (6).

A substitution of the above expressions into (28) and (7) give

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\big(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})} + \varepsilon\|u\|_{W^{2,q}(\mathcal{N}'')} + C(\varepsilon)\|\partial u/\partial \boldsymbol{L}\|_{L^q(\mathcal{N}')}\big)$$

whence, choosing $\varepsilon > 0$ small enough, we get

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\big(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})} + \|u\|_{W^{1,q}(\mathcal{N}')}\big).$$

Similarly, another application of the interpolation inequality yields

$$\|u\|_{W^{1,q}(\mathcal{N}')} \leq \|u\|_{W^{1,q}(\mathcal{N}'')} \leq \delta\|u\|_{W^{2,q}(\mathcal{N}'')} + C(\delta)\|u\|_{L^q(\mathcal{N}'')}$$

and thus

$$\|u\|_{W^{2,q}(\mathcal{N}'')} \leq C\big(\|u\|_{L^q(\Omega)} + \|f\|_{\mathcal{F}^q(\Omega,\mathcal{N})} + \|\varphi\|_{\Phi^q(\partial\Omega,\mathcal{N})}\big).$$

for small $\delta > 0$. The proof of Lemma 3 is completed. □

The statement of Theorem 1 follows from Proposition 2 and Lemma 3.

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