Journal of Nonlinear and Convex Analysis
Volume 7, Number 3, 2006, 465–482

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

# REGULARITY FOR A CLASS OF SUBELLIPTIC OPERATORS

M. K. VENKATESHA MURTHY

Abstract. We are concerned with the regularity of solutions of subelliptic equations of the form

$$\mathcal{L}_{\mathfrak{X}} u = \sum_{j=1}^{m} X_j^* X_j u + X_0 u + a(x) u = f(x) \quad G$$

where $\mathfrak{X} = \{X_1, \cdots, X_m\}$ and $X_0$ are real $C^\infty$ vector fields defined in an open neighbourhood $\Omega$ of a bounded domain $G$ with smooth boundary $\partial G$ in $\mathbb{R}^N$ and $a(x),\ f(x) \in C^\infty(\Omega)$. Suppose that the system of vector fields $\mathfrak{X}$ satisfies the finite type brackets condition of Hörmander except on a union of smooth surfaces $\Sigma$ which are non characteristic for the system $\mathfrak{X}$ and that an a priori estimate of Sobolev type with a logarithmic weight holds. Then any weak solution of the subelliptic equation $\mathcal{L}_{\mathfrak{X}} u = f(x)$ in $G$ belongs to $C^\infty(G \setminus \Sigma)$.

The class of operators considered includes certain infinitely degenerate elliptic type operators $\mathcal{L}_{\mathfrak{X}}$. Since the components of $\Sigma$ are in general, hypersurfaces (one-codimensional submanifolds) suitable microlocal conditions have to be assumedon the symbols of the vector fields of the system $\mathfrak{X}$ and their commutators in order that the logarithmic a priori estimate holds.

If further the boundary is $C^\infty$ and is not characteristic with respect to the system of vector fields $\mathfrak{X}$ then any weak solution of the Dirichlet problem for $\mathcal{L}_{\mathfrak{X}}$ with $C^\infty$ data is $C^\infty$ upto the boundary except on $\Sigma$; that is, the solution belongs to $C^\infty(G \setminus \Sigma) \cap C^0(\overline{G} \setminus \Sigma)$.

## 1. Introduction

We are concerned with $C^\infty$-regularity of solutions to a class of second order *subelliptic* equations of degenerate type with $C^\infty$-coefficients. Such a study goes back to the work of Kolmogorov in 1934, who proved the hypoellipticity for the equation

$$\frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} = f$$

by explicitely constructing a fundamental solution. Hörmander in his famous paper of Acta Mathematica of 1969 studied the problem of hypoellipticity of general second order equations in detail. For operators of arbitrary orders of degenerate elliptic type the problem of $C^\infty$ and Gevrey hypoellipticity was considered in a paper of Baouendi and Goulaouic in 1971.

It is known from Hörmander's paper that for a second order equation $P(x, D)u = f$ with real principal symbol to be hypoelliptic the principal symbol $p_2(x, \xi)$ should necessarily be a semidefinite quadratic form in $\xi$. Since the work of Hörmander several authors have obtained important results on hypoellipticity of second order operators with $C^\infty$-coefficients, to mention only a few authors, we have J.-M. Bony, Y. Morimoto and T. Morioka, S. Wakabayashi, R. Wheeden, C.-J. Xu, C. Zuily.

For a second order operator, in any open set where the rank of the matrix of the second order coefficients is constant, in a suitable coordinate system the operator can be written in the form

$$\sum_{j=1}^{m} X_j^2 + X_0 + a(x)$$

where $X_0, X_1, \cdots, X_m$ are real $C^\infty$-vector fields and $a(x)$ is a $C^\infty$-real valued function (see Hörmander, Acta Mathematatica, 119 (1969)).

The results described here are motivated by examining the following standard examples in two and three dimensions.

**Some model examples of operators of degenerate elliptic type.** With the standard notations $x = (x_1, \cdots, x_N)$ as a coordinate system in $\mathbb{R}^N$, $N = 2, 3$, some examples are given by the following degenerate elliptic operators:

$$\frac{\partial^2}{\partial x_1^2} + x_1^k \frac{\partial^2}{\partial x_2^2}, \quad \text{with } k \text{ a positive integer;}$$

$$\frac{\partial^2}{\partial x_1^2} + \exp[-2|x_1|^{\frac{-2}{s}}] \frac{\partial^2}{\partial x_2^2}, \quad \text{with a real number } s > 0;$$

$$\frac{\partial^2}{\partial x_1^2} + \exp[-2(x_1^2 \sin^2(\frac{\pi}{x_1})^{\frac{-1}{s}}] \frac{\partial^2}{\partial x_2^2}, \quad \text{with a real number } s > 0;$$

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \exp[-2|x_1|^{\frac{-2}{s}}] \frac{\partial^2}{\partial x_3^2}, \quad \text{with a real number } s > 0;$$

$$\frac{\partial^2}{\partial x_1^2} + x_1^{2m} \frac{\partial^2}{\partial x_2^2} + \exp[-|x_1|^{-s}] \frac{\partial^2}{\partial x_3^2},$$

with an integer $m > 0$ and a real number $0 < s < m + 1$;

$$\frac{\partial^2}{\partial x_1^2} + \exp[\frac{-2}{|x_1|}] \frac{\partial^2}{\partial x_2^2} + \exp[-|x_1|^{-s} \exp(\frac{1}{|x_1|})] \frac{\partial^2}{\partial x_3^2},$$

with a real number $0 < s < 2$.

Here, while the first example is a degenerate elliptic type operator of finite order degeneracy at $x_1 = 0$ the others are all degenerate elliptic type operators of infinite order degeneracy at $x_1 = 0$.

We shall review here some recent progress on the $C^\infty$-regularity theory for subelliptic operators of this type wherein the vector fields may degenerate to infinite orders along smooth surfaces which are non characteristic with respect to the family of vector fields (see section 2 for definitions). The results of this survey were presented at the International Conference on Recent Advances in Partial Differential Equations in memory of Filippo Chiarenza held at Messina in December 2005.

## 2. Notation and function spaces

Suppose $G$ is a bounded domain in $\mathbb{R}^N$ with $C^\infty$-boundary $\partial G$ and suppose $\mathfrak{X} = \{X_1, \cdots, X_m\}$ and $X_0$ are real $C^\infty$-vector fields defined in an open neighbourhood $\Omega$ of $\overline{G}$. Let $X_j^*$ denote the formal adjoint of $X_j$, $\quad j = 1, \cdots, m$. Consider the subelliptic second order equation associated to the system of vector fields $\mathfrak{X}$ and $X_0$:

$$\mathcal{L}_{\mathfrak{X}} u = \sum_{j=1}^{m} X_j^* X_j u + X_0 u + a(x)u = f(x) \quad \text{in } G$$

where $a, f \in C^\infty(\Omega)$. We also denote the principal part by

$$\mathcal{L}_{\mathfrak{X}}^0 = \sum_{j=1}^{m} X_j^* X_j$$

**Function spaces.** We introduce the following natural spaces of distributions associated to the system of vector fields $\mathfrak{X}$, similar to the classical Sobolev spaces:

$$M^1(\Omega; \mathfrak{X}) = \{u \in L^2(\Omega);\ X_j u \in L^2(\Omega), \quad \text{for} \quad j = 1, \cdots, m\}$$

which is a Hilbert space with the natural scalar product and norm, namely,

$$(u, v)_{M^1(\Omega;\mathfrak{X})} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^{m} (X_j u, X_j v)_{L^2(\Omega)}$$

$$||u; M^1(\Omega; \mathfrak{X})||^2 = (u, u)_{M^1(\Omega;\mathfrak{X})} = ||u; L^2(\Omega)||^2 + \sum_{j=1}^{m} ||X_j u; L^2(\Omega)||^2$$

The corresponding local space $M^1_{loc}(\Omega; \mathfrak{X})$ is the space of all distributions $u \in \mathcal{D}'(\Omega)$ such that $\varphi u \in M^1(\Omega; \mathfrak{X})$ for all test functions $\varphi \in \mathcal{D}(\Omega)$.

We denote as usual by $M^1_0(\Omega; \mathfrak{X})$ the closure of $\mathcal{D}(\Omega)$ in $M^1(\Omega; \mathfrak{X})$ and its dual space by $M^{1'}(\Omega; \mathfrak{X})$.

The elements of the dual space $M^{1'}(\Omega; \mathfrak{X})$ can be represented (in a non unique way) as the space of distributions of the form

$$F = f_0 + \sum_{j=1}^{m} X_j^* f_j, \quad \text{where} \quad f_0, f_1, \cdots, f_m \in L^2(\Omega)$$

with the natural duality pairing

$$\langle F, u \rangle = \int_\Omega (f_0 u + \sum_{j=1}^{m} f_j X_j u) \mathrm{d}x$$

**Definition 1.** A $C^\infty$-submanifold of co-dimension one (a hypersurface) $\Sigma$ in $\Omega$ is said to be non characteristic with respect to the system of vector fields $\mathfrak{X}$ if
for any point $x_0 \in \Sigma$ there exists atleast one vector field $X_j$ of the system $\mathfrak{X}$ which is transversal to $\Sigma$at $x_0$; i.e. $X_j(x_0) \notin T_{x_0}\Sigma$.

**Trace of $M^1(\Omega; \mathfrak{X})$ on $\Sigma$.** If $\Sigma$ is a hypersurface in $\Omega$ which is not characteristic with respect to the system of vector fields $\mathfrak{X}$ then we can define, for any $u \in M^1(\Omega; \mathfrak{X})$, the trace $u|_\Sigma$ in the following standard manner:

Suppose $x_0 \in \Sigma$ and that the vector field $X_j$ of $\mathfrak{X}$ is transversal to $\Sigma$ at $x_0$. Then we use the classical method of localizing using a $C^\infty$-cut off function and a local $C^\infty$-diffeomorphism on a small open neighbourhood $U$ of $x_0$ in $\Omega$ to flatten the portion $U \cap \Sigma$ to a subset of the hyperplane $\{y = (y', y_N) \in \mathbb{R}^N;\ y_N = 0\}$ so that the vector field $X_j$ is transformed to the normal vector field $\dfrac{\partial}{\partial y_N}$ and the remaining vector fields are transformed to vector fields $Y_1, \cdots, Y_{m-1}$. The system of vector fields $\mathfrak{X}$ is transformed to the system $\mathfrak{Y} = \{Y_1, \cdots, Y_{m-1}, \frac{\partial}{\partial y_N}\}$. Then the classical method is adapted to define the trace operator $u \to u|_\Sigma$ on $M^1(\Omega; \mathfrak{X})$.

We shall assume from now onwards that

*the boundary $\partial G$ of the domain $G$ is a $C^\infty$-hypersurface which is non characteristic with respect to the system of vector fields $\mathfrak{X}$.*

When the boundary $\partial G$ is a $C^\infty$-hypersurface which is non characteristic with respect to the system of vector fields $\mathfrak{X}$, the trace $u|_{\partial G} = \gamma_0 u$ is well defined. Then we have

$$M^1_0(G; \mathfrak{X}) = \{u \in M^1(\Omega; \mathfrak{X});\ \gamma_0 u = 0 \quad \text{on} \quad \partial G\}$$

which is a Hilbert space with the induced scalar product and obviously $\mathcal{D}(G)$ is dense in $M^1_0(G; \mathfrak{X})$ thus defined.

**Weak solution of $\mathcal{L}_\mathfrak{X} u = F$.**

**Definition 2.** A weak solution of the equation $\mathcal{L}_\mathfrak{X} u = F$ in G for $F \in M^{1'}(G; \mathfrak{X})$ is defined as

$$u \in M^1_{loc}(G; \mathfrak{X}), \quad \int_G \{\sum_{j=1}^m (X_j u)(X_j v) + (X_0 u)v + a(x)uv\}\mathrm{d}x = F(v)$$

$$\text{for all} \quad v \in M^1_0(G; \mathfrak{X})$$

We shall denote by $\mathfrak{G}(\mathfrak{X})$ the Lie algebra generated by the system of vector fields $X_0$ and $\mathfrak{X}$ with the standard bracket operation $[X, Y] = XY - YX$; i.e. $\mathfrak{G}(\mathfrak{X})$ is the smallest $C^\infty(\Omega)$-submodule containing $\mathfrak{X}$ which is closed under the bracket operation.

**Definition 3.** The rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ at a point $x \in \Omega$ is the dimension of the vector space generated by all the vectors $Z(x)$; $Z \in \mathfrak{G}(X)$.

We have the following classical result of Hörmander:

**Theorem 1** (Hörmander)**.** *If the rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ at every point $x \in G$ is equal to the dimension of $G = N$ then the differential operator $\mathcal{L}_\mathfrak{X}$ is hypoelliptic in $G$.*

i.e. For any point $x \in G$ among the commutators

$$X_1, X_2, \cdots, X_m, [X_{j_1}, X_{j_2}], \cdots, [X_{j_1}, [X_{j_2}, \cdots, [X_{j_{k-1}}, X_{j_k}]]], \cdots,$$

where $J = \{j_1, j_2, \cdots, j_k\} \subset \{1, \cdots, m\}$ there exist N commutators which generate the tangent space $T_x(G)$ then $\mathcal{L}_\mathfrak{X}$ is hypoelliptic in $G$.

**Definition 4** (Condition of Hörmander)**.** The system of vector fields $\mathfrak{X}$ is said to satisfy the condition of Hörmander in a subset $\omega$ of $\Omega$ if the rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ at every point of the subset $\omega$ is equal to $N$, the dimension of $\Omega$.

*Remark.* If the rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ is a constant $< N$ then the operator $\mathcal{L}_\mathfrak{X}$ is not hypoelliptic.

We shall make use of the standard notation: if $J = \{j_1, \cdots, j_k\}$ with $j_1, \cdots, j_k \in \{1, \cdots, m\}$ we denote by $|J| = k$ the length of the commutator

$$X_J = [X_{j_1}, [X_{j_2}, \cdots [X_{j_{k-1}}, X_{j_k}]]]$$

We also need the following definition:

**Definition 5.** Suppose $\Sigma = \cup_{j\in J}\Sigma_j$ is a union of $C^\infty$ one- codimensional hypersurfaces in $\Omega$. $\Sigma$ is said to be non characteristic for the system of vector fields $\mathfrak{X}$ if for any point $x_0 \in \Sigma$ there exist at least one vector field $X_i \in \mathfrak{X}$ such that the vector $X_i(x_0)$ is transversal to all the hypersurfaces $\Sigma_j$ passing through the point $x_0$; i.e. $X_i(x_0)$ is transversal to every $\Sigma_j$, $\quad j \in J(x_0) = \{j \in J; x_0 \in \Sigma_j\}$.

This definition is motivated by the following example in $\mathbb{R}^2$:
Suppose

$$\mathcal{L}_{\mathfrak{X}} = -\frac{\partial^2}{\partial x_1^2} - \exp[-\{x_1^2 \sin(\frac{\pi}{x_1})\}^{\frac{-1}{s}}]\frac{\partial^2}{\partial x_2^2}$$

where $\mathfrak{X} = \{X_1, X_2\}$ with $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = \exp[-\{x_1^2 \sin(\frac{\pi}{x_1})\}^{\frac{-1}{2s}}]\frac{\partial}{\partial x_2}$

Here $\Sigma_0 = \{x_1 = 0\}$, $\Sigma_j = \{x_1 = \frac{1}{j}\}$ for all $j \in \mathbb{Z} \setminus 0$ and $\Sigma = \cup_{j\in\mathbb{Z}}\Sigma_j$.

The vector field $X_1 = \frac{\partial}{\partial x_1}$ is transversal to every $\Sigma_j$, $j \in \mathbb{Z}$ while $X_2$ vanishes on $\Sigma = \cup_{j\in\mathbb{Z}}\Sigma_j$ to infinite order.

## 3. The main result

We shall use the notation: $\langle\xi\rangle^2 = \mathrm{e}^2 + |\xi|^2$ and, for $s > 0$,

$$(\log\langle D\rangle)^s v = (Op(\log\langle\xi\rangle)^s)v = (2\pi)^{-N}\int (\log\langle\xi\rangle)^s\hat{v}(\xi)\exp(i\langle x, \xi\rangle)\mathrm{d}\xi$$

Hence using the Parseval formula, we have

$$||(\log\langle D\rangle)^s v; L^2(\mathbb{R}^N)||^2 = \int (\log\langle\xi\rangle)^{2s}|\hat{v}(\xi)|^2\mathrm{d}\xi$$

Then the main result is the following interior regularity theorem

**Theorem 2.** *Suppose the system of vector fields $\mathfrak{X} = \{X_1, \cdots, X_m\}$ satisfy the following hypothesis:*

*(i) there exists a union of $C^\infty$ hypersurfaces $\Sigma = \cup_{j\in J}\Sigma_j$ which is non characteristic with respect to the system of vector fields $\mathfrak{X}$, where $\mathfrak{X}$ satisfies the condition of Hörmander in $\Omega$ except on $\Sigma$, i.e the rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ at every point $x \in \Omega \setminus \Sigma$ is equal to the dimension of $\Omega = N$*

*(ii) there exists an $s > \frac{3}{2}$ such that the following logarithmic estimate holds: there exists a constant $C > 0$ such that*

$$||(\log\langle D\rangle)^s v; L^2(\Omega)|| \leq C||v; M^1(\Omega; \mathfrak{X})||, \quad \text{for all} \quad v \in \mathcal{D}(\Omega)$$

*Then, for $f \in C^\infty(G)$, any locally bounded weak solution*

$$u \in M^1_{loc}(G; \mathfrak{X}) \cap L^\infty_{loc}(G) \quad \text{of} \quad \mathcal{L}_{\mathfrak{X}}u = f \in C^\infty(G)$$

*belongs to $C^\infty(G \setminus \Sigma)$.*

i.e. $\mathcal{L}_{\mathfrak{X}}$ is hypoelliptic in $G$ except on the union of hypersurfaces $\Sigma = \cup_{j\in J}\Sigma_j$.

*Remark.* The logarithmic estimate with $s = 1$, namely

$$||(\log\langle D\rangle)v; L^2(\Omega)||^2 \leq C\{\sum_{j=1}^{m} ||X_j v; L^2(\Omega)||^2 + ||v; L^2(\Omega)||^2\}, \ \ \forall v \in \mathcal{D}(\Omega)$$

is not enough to prove the hypoellipticity of $\mathcal{L}_{\mathfrak{X}}$.

However, an estimate of the form: if $\forall\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

$$||(\log\langle D\rangle)v; L^2(\Omega)||^2 \leq \epsilon \sum_{j=1}^{m} ||X_j v; L^2(\Omega)||^2 + C_\epsilon ||v; L^2(\Omega)||^2$$

would be enough to prove the hypoellipticity. But this can not be derived as a consequence of the $s = 1$ logarithmic estimate.

When $s > 1$, using the fact that the vector fields are homogeneous differential operators of order one, by a standard interpolation argument it is possible to prove that the logarithmic estimate with $s > 1$ implies that $\forall\epsilon > 0$ there exists a constant $C_{\epsilon,s} > 0$ such that

$$||(\log\langle D\rangle)^s v; L^2(\Omega)||^2 \leq \epsilon^{2s} \sum_{j=1}^{m} ||X_j v; L^2(\Omega)||^2 + C_{\epsilon,s}||v; L^2(\Omega)||^2, \quad \text{for all } v \in \mathcal{D}(\Omega).$$

The result of theorem 1 can be extended to a $C^\infty$-regularity result upto the boundary for solutions of the Dirichlet problem as follows:

**Theorem 3.** *Suppose the system of vector fields $\mathfrak{X}$ satisfies the hypothesis (i) and (ii) of the theorem 2. Further assume that*

*(iii) the boundary $\partial G$ of the bounded domain $G$ is a $C^\infty$-smooth one codimensional manifold which is non characteristic with respect to the system $\mathfrak{X}$. Then, given functions $f \in C^\infty(G)$ and $g \in C^\infty(\partial G)$ any bounded weak solution $u \in M^1(G; \mathfrak{X}) \cap L^\infty(G)$ of the Dirichlet problem*

$$\mathcal{L}_{\mathfrak{X}} u = f \quad \text{in } G, \quad \gamma_0 u = u|_{\partial G} = g \quad \text{on } \partial G$$

*belongs to $C^\infty(G \setminus \Sigma) \cap C^0(\overline{G} \setminus \Sigma)$.*

*Remark.* Under suitable sufficient conditions on the system $\mathfrak{X}$ and the commutators $X_J$ the logarithmic Sobolev type estimate holds and since the $\Sigma_j$ are general hypersurfaces (one-codimensional submanifolds of $\Omega$ these involve conditions of microlocal nature on the symbols of the commutators $X_J$ (see the Appendix).

## 4. Logarithmic Sobolev type spaces

In view of the assumption (i) in the above theorems it is necessary to study the properties of the corresponding spaces of distributions. This is done as is customary in Harmonic Analysis by means of an appropriate decomposition of Littlewood - Paley type for functions on the *Phase space* or the Fourier transform space.

We shall denote, for $s > 0$, by

$$E_s^{\log} = E_s^{\log}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N); \ \ (\log\langle D\rangle)^s u \in L^2(\mathbb{R}^N)\}$$

with its natural scalar product and the corresponding norm

$$(u,v)_{E_s^{\log}} = (u,v)_{L^2(\mathbb{R}^N)} + ((\log\langle D\rangle)^s u, (\log\langle D\rangle)^s v)_{L^2(\mathbb{R}^N)}.$$
$$||u, E_s^{\log}||^2 = ||u; L^2(\mathbb{R}^N)||^2 + ||(\log\langle D\rangle)^s u; L^2(\mathbb{R}^N)||^2.$$

In view of Plancheral's theorem we have, by the Fourier transform, the space of functions

$$F_s^{\log} = F_s^{\log}(\mathbb{R}^N) = \{v \in L^2(\mathbb{R}^N); \ \ (\log\langle \xi\rangle)^s v \in L^2(\mathbb{R}^N)\}$$

which is provided with the scalar product and the corresponding norm

$$(v,w)_{F_s^{\log}} = (v,w)_{L^2(\mathbb{R}^N)} + ((\log\langle \xi\rangle)^s u, (\log\langle \xi\rangle)^s v)_{L^2(\mathbb{R}^N)}.$$
$$||v, F_s^{\log}||^2 = ||v; L^2(\mathbb{R}^N)||^2 + ||(\log\langle \xi\rangle)^s v; L^2(\mathbb{R}^N)||^2.$$

The logarithmic Sobolev type regularity hypothesis (ii) in the above theorems (see sec. 3) can be reformulated as follows: For the system of vector fields $\mathfrak{X}$ there is a continuous linear mapping from $M^1(G;\mathfrak{X})$ to $E_s^{\log}(\mathbb{R}^N)$.

## 5. Littlewood-Paley decomposition of functions on the phase space

For the study the properties of the function spaces $E_s^{\log}$ and $F_s^{\log}$ and the estimates needed for the proof of the main result we make use of a technique from Harmonic Analysis which consists of decomposition of functions in the phase space.

We write $\mathbb{R}_+ = (0, +\infty)$ as a union $\cup_{k=-1}^{+\infty} I_k$ where

$$I_{-1} = (0, \mathrm{e}^2), I_0 = (\mathrm{e}, \mathrm{e}^3), I_1 = (\mathrm{e}^2, \mathrm{e}^4), \cdots, I_k = (\mathrm{e}^{k+1}, \mathrm{e}^{k+3}) = \mathrm{e}^k I_0, \cdots$$

and in correspondence with this we obtain a decomposition of the phase space

$$\mathbb{R}_\xi^N = \cup_{k=-1}^{+\infty} \Gamma_k$$

where

$$\Gamma_k = \{\xi \in \mathbb{R}^N; \langle\xi\rangle \in I_k\}, \quad \text{for all} \quad k = -1, 0, 1, \cdots.$$

We introduce a $C^\infty$-partition of unity: let $\varphi_{-1} \in \mathcal{D}(I_{-1}) = \mathcal{D}((0, \mathrm{e}^2))$ and $\varphi_0 = \varphi \in \mathcal{D}(I_0) = \mathcal{D}((\mathrm{e}, \mathrm{e}^3))$ be two test functions such that

$$\varphi_{-1}(\langle\xi\rangle) + \sum_{k=0}^{+\infty} \varphi(\mathrm{e}^{-k}\langle\xi\rangle) = 1$$

in the sense that any $f \in L^2(\mathbb{R}_\xi^N)$ can be decomposed as (a convergent series in $L^2(\mathbb{R}^N)$)

$$f = \varphi_{-1}(\langle D\rangle)(f) + \sum_{k=0}^{+\infty} \varphi(\mathrm{e}^{-k}\langle D\rangle)(f) = \sum_{k=-1}^{\infty} \Phi_k(f)$$

where

$$\Phi_{-1}(f) = \varphi_{-1}(\langle D\rangle)(f) = (2\pi)^{-N} \int \varphi_{-1}(\langle\xi\rangle \hat{f}(\xi)\exp(i\langle x,\xi\rangle)\mathrm{d}\xi$$
$$\Phi_k(f) = \varphi(\mathrm{e}^{-k}\langle D\rangle)(f) = (2\pi)^{-N} \int \varphi(\mathrm{e}^{-k}\langle\xi\rangle)\hat{f}(\xi)\exp(i\langle x,\xi\rangle)\mathrm{d}\xi$$

The properties of the function space $E_s^{\log}(\mathbb{R}^N)$ are characterized by the following propositions:

**Proposition 1** (Estimates for the components $\Phi_k(u)$ for $u \in E_s^{\log}(\mathbb{R}^N)$)**.** *If $u \in E_s^{\log}(\mathbb{R}^N)$ then we have the estimate*

$$||\Phi_k(u); L^2(\mathbb{R}^N)|| \leq c_k k^{-s}, \quad \textit{where} \quad \sum c_k^2 < +\infty$$

$$\textit{and} \quad ||\{c_k\}; l^2||^2 = \sum c_k^2 \leq ||u; E_s^{\log}(\mathbb{R}^N)||^2$$

*Infact, we have,*

$$c_k^2 = \int_{\Gamma_k} (\log\langle\xi\rangle)^{2s} \varphi(\mathrm{e}^{-k}\langle\xi\rangle)^2 |\hat{u}(\xi)|^2 \mathrm{d}\xi$$

*Conversely, if $u \in L^2(\mathbb{R}^N)$ is such that there exists a positive sequence $\{c_k\} \in l^2$ such that*

$$||\Phi_k(u); L^2(\mathbb{R}^N|| \leq c_k k^{-s}$$

*then $u \in E_s^{\log}(\mathbb{R}^N)$ and we also have the following estimate: for any $\rho \geq 1$ there exist positive constants $C_1$, $C_2$ such that*

$$\rho^{2s} ||(\log\langle D\rangle)^s u; L^2(\mathbb{R}^N)||^2 \leq C_1 s^{2s} ||u; L^2(\mathbb{R}^N)||^2 + C_2^\rho \rho^{2s} ||\{c_k\}; l^2||^2$$

**Proposition 2** (Reconstruction of $u$ from its Littlewood-Paley components $u_k = \Phi_k(u)$)**.** *Suppose we have a sequence $\{u_k\}$ in $L^2(\mathbb{R}^N)$ such that the Fourier transform $\hat{u_k}$ of $u_k$ has*

$$\textit{supp } \hat{u_k} \subset B(0, h\mathrm{e}^k) \quad \textit{for some constant} \quad h > 0$$

*and there exist constants $c_k > 0$ with $\{c_k\} \in l^2$ such that, for $s > \frac{1}{2}$,*

$$||u_k; L^2(\mathbb{R}^N)|| \leq c_k k^{-s}$$

*then the series $\sum u_k$ converges in $L^2(\mathbb{R}^N)$ and the sum $u = \sum u_k$ belongs to $E_{s-\frac{1}{2}}^{\log}(L^2(\mathbb{R}^N))$. Moreover, for any $\rho \geq 1$*

$$\rho^{2s-1} ||(\log\langle D\rangle)^{(s-\frac{1}{2})} u; L^2(\mathbb{R}^N)||^2$$
$$\leq C_1 (s - \frac{1}{2})^{2s-1} ||u; L^2(\mathbb{R}^N)||^2 + C_2^\rho \rho^{2s-1} (2s-1) ||\{c_k\}; l^2||^2$$

We observe that since $2s > 1$, $\sum k^{-2s} < +\infty$ and then by the Cauchy-Schwarz inequality

$$\sum c_k . k^{-s} \leq (\sum c_k^2)^{\frac{1}{2}} . (\sum k^{-2s})^{\frac{1}{2}} < \infty$$

and hence $\sum ||u_k; L^2(\mathbb{R}^N)||^2 < +\infty$.

We can also construct distributions in $E_{s-\frac{1}{2}}^{\log}(\mathbb{R}^N)$ starting from a sequence of $C^\infty$-functions provided that their derivatives satisfy appropriate growth conditions. More precisely, we have

**Proposition 3.** *Let $\{u_k\}$be a sequence of $C^\infty$-functions on $\mathbb{R}^N$ such that, for any $s > \frac{1}{2}$ there exists a function $v \in E_s^{\log}(\mathbb{R}^N)$ and for any $\alpha \in \mathbb{N}^N$ there is a constant $A_{|\alpha|}$ so that we have*

$$||D^\alpha u_k; L^2(\mathbb{R}^N)|| \leq A_{|\alpha|} \mathrm{e}^{k|\alpha|} ||\Phi_k(v); L^2(\mathbb{R}^N)||$$

*then* $u = \sum u_k$ *converges and belongs to* $E^{\log}_{s-\frac{1}{2}}(\mathbb{R}^N)$.

Moreover we have an estimate similar to the one in the previous proposition.

*Remark.* Roughly speaking the above propositions imply that the distributions in the space $E^{\log}_s(\mathbb{R}^N)$ are characterized in terms of their Littlewood-Paley components by requiring that

$$\sum k^{2s}||\Phi_k(u); L^2(\mathbb{R}^N)||^2 < +\infty.$$

## 6. A sketch of proof for the interior regularity

If $u \in M^1_{loc}(G; \mathfrak{X}) \cap L^\infty_{loc}(G)$ is a weak solution of the subelliptic equation $\mathcal{L}_{\mathfrak{X}} u = f \in C^\infty(G)$ in $G$ then the classical method consists in using as a test function in the definition of the weak solution a localization $\alpha u$ of $u$ with $\alpha \in \mathcal{D}(G)$. However, we only know that $\alpha u$ belongs to the space $M^1_0(G; \mathfrak{X})$ and so in order to apply the logarithmic Sobolev type estimate and in particular, to apply the operator $(\log(\langle D\rangle))^s$ we need an additional regularity of $\alpha u$. This is achieved making use of a technique already used in Hörmander's paper, namely we regularize $\alpha u$ by means of the fundamental solution $G_\delta$ of $I - \delta^2\Delta$ given by the pseudo-differential operator

$$\begin{aligned} G_\delta v &= (I - \delta^2\Delta)^{-1}(v) = Op((1 + \delta^2|\xi|^2)^{-1})(v) \\ &= (2\pi)^{-N} \int (1 + \delta^2|\xi|^2)^{-1}\hat{v}(\xi)\mathrm{e}^{i\langle x,\xi\rangle}\mathrm{d}\xi, \quad \text{with } \delta > 0 \end{aligned}$$

and let $\delta \to 0$.

Then $\{G_\delta\}_{0<\delta\leq 1}$ is a uniformly bounded family of operators on the classical Sobolev spaces $H^m(\mathbb{R}), \forall m \in \mathbb{R}$. This follows from the fact that the fundamental solution $G = \mathcal{F}^{-1}(1 + |\xi|^2)^{-1}$ of $(I - \Delta)$ together with all its derivatives $(\frac{\partial}{\partial x})^\alpha G$ decay exponentially as $|x| \to +\infty$. Hence $G_\delta$ and all its derivatives decay faster than any power of $\delta$ as $\delta \to 0$ and this in turn implies that $G_\delta(v) \to v$ in $L^2(\mathbb{R}^N)$, as $\delta \to 0$.

**Step 1.** Let $s > \frac{3}{2}$ and the system of vector fields satisfy the following logarithmic estimate: for all $\epsilon > 0$ there is a constant $C_{\epsilon,s} > 0$ such that

$$||(\log\langle D\rangle)^s v; L^2||^2 \leq \epsilon^{2s} \sum_{j=1}^m ||X_j v; L^2||^2 + C_{\epsilon,s}||v; L^2||^2, \quad \forall v \in \mathcal{D}(\Omega)$$

Then, for $\alpha \in \mathcal{D}(G \setminus \Sigma)$ we take $\alpha u$ where $u \in M^1_{loc}(G; \mathfrak{X}) \cap L^\infty(G)$ is a weak solution of $\mathcal{L}_{\mathfrak{X}} u = f$ in $G$. Then we have

**Proposition 4.** *For any integer* $\imath \geq 0$ *and and* $\rho \geq 1$ *we have the estimate*

$$||(\log\langle D\rangle^\rho)^l G_\delta(\alpha u); L^2(\mathbb{R}^N)|| \leq (c_0 l)^l l^{m_\rho} c_\rho$$

*where the constant* $c_0 = c_0(\text{supp } \alpha)$ *depends only on the* supp $\alpha$ *and the constants* $m_\rho$ *and* $c_\rho$ *are independent of* $\delta > 0$ *and* $\imath$.

For the proof we consider the weak solution u and we take for the test function $v$ in the definition of the weak solution an appropriate regularization of the localized $u$; more precisely, we take

$$v = \beta G_\delta \alpha G_\delta(\beta u) \in H^1_0(G)$$

where $\beta \in \mathcal{D}(G \setminus \Sigma)$ such that $\beta = 1$ on supp $\alpha$. We estimate using the Cauchy-Schwarz inequality together with the logarithmic $\epsilon$-estimate in the hypothesis of the proposition to prove the required assertion.

Under the same hypothesis as in the above proposition, we use a similar method to obtain estimates also for the following commutators applied to the weak solution:

$$[X_j, (\log\langle D\rangle^{\rho})^s G_\delta \alpha]u$$

and

$$[X_j, [X_k, (\log\langle D\rangle^{\rho})^s G_\delta \alpha]]u$$

**Step 2.** We can write

$$\langle\xi\rangle^2 = \exp(\log\langle\xi\rangle^2) = \sum_{k=0}^{\infty} \frac{1}{k!}(\log\langle\xi\rangle^2)^k$$

and hence

$$\begin{aligned} ||\langle D\rangle^2 G_\delta(\alpha u); L^2|| &= ||\langle\xi\rangle^2 [\widehat{G_\delta(\alpha u)}](\xi); L^2|| \\ &= \sum_k \frac{1}{k!}||(\log\langle\xi\rangle^2)^k [\widehat{G_\delta(\alpha u)}](\xi); L^2|| \\ &= \sum_k \frac{1}{k!}(\frac{2}{\rho})^k ||(\log\langle\xi\rangle^{\rho})^k [\widehat{G_\delta(\alpha u}](\xi); L^2|| \\ &= \sum_k \frac{1}{k!}(\frac{2}{\rho})^k c_0^k k^k . k^{m_\rho} c_\rho \end{aligned}$$

Taking $\rho = 4\mathrm{e}c_0$ in the proposition to get

$$\begin{aligned} &= \sum_k \frac{1}{k!}(\frac{1}{2\mathrm{e}})^k k^k . k^{m_\rho} c_\rho, \quad \text{since } \frac{2c_0}{\rho} = \frac{1}{2\mathrm{e}} \\ &\leq \sum_k \frac{1}{2^k}(\frac{k^k}{k!\mathrm{e}^k}).k^{m_\rho} c_\rho \\ &\leq ||\alpha u; L^2|| + c_\rho \sum_{k=1}^{\infty} \frac{1}{2^k}.k^{m_\rho} \quad \text{since } \frac{k^k}{k!} < \mathrm{e}^k \end{aligned}$$

which proves that $\{G_\delta(\alpha u); \quad 0 < \delta \leq 1\}$ is a uniformly bounded set in the classical Sobolev space $H^2(\mathbb{R}^N)$. Hence is a weakly relatively compact subset and therefore admits a weakly convergent subsequence in $H^2(\mathbb{R}^N)$; i.e. $G_{\delta_\nu}(\alpha u) \to \alpha u$ weakly in $H^2(\mathbb{R}^N)$ for a subsequence $\delta_\nu \to 0$, This proves that $\alpha u \in H^2(\mathbb{R}^N)$. This implies once again by regularization that $\{G_\delta(\alpha u) \in H^4(\mathbb{R}^N); 0 < \delta \leq 1\}$ is a uniformly bounded set in $H^4(\mathbb{R}^N)$. Repeating the above argument we see that $\alpha u \in H^4(\mathbb{R}^N)$. This bootstrap argument shows that $\alpha u \in H^m(\mathbb{R}^N)$ for all $m > 0$ and then by the classical Sobolev embedding theorem it follows that $\alpha u \in C^\infty(\mathbb{R}^N)$, which completes the proof of the interior regularity of the weak solution.

## 7. The Dirichlet problem and regularity upto the boundary

We assume that the boundary $\partial G$ is non charcteristic wth respect to the system of vector fields $\mathfrak{X}$ and hence we have the trace and extension theorems for the space $M^1(G;\mathfrak{X})$. Given $g \in C^\infty(\partial G)$ take a $C^\infty$ - extension $w$ of $g$ to $\overline{G}$ and take $u - w$ in place of $u$ to transform the probelm to the homogeneous Dirichlet problem:

$$\mathcal{L}_{\mathfrak{X}}(u-w) = f - \mathcal{L}_{\mathfrak{X}} w = h \quad \text{in} \quad G$$

$$u - w \in M_0^1(G;\mathfrak{X})$$

Thus we assume that $u \in L^\infty(G) \cap M_0^1(G;\mathfrak{X})$ and that $u$ satisfies $\mathcal{L}_{\mathfrak{X}} u = h \quad \text{in} \quad G$ in the sense of weak solutions.

Since the boundary $\partial G$ is non-charcteristic with respect to the system $\mathfrak{X}$, if $x_0 \in \partial G$ there is a vector field $X_i$ of the system such that the vector $X_i(x_0)$ is transversal to the boundary at $x_0$. In a neighbourhood $U$ of the point $x_0 \in \partial G$ we localize $u$ using a cut off function $\alpha \in \mathcal{D}(U)$ and then make a $C^\infty$ diffeomorphism to flatten the piece of the boundary $U \cap \partial G$. In the new system of coordiates $y = (y_1, \cdots, y_{N-1}, y_N) = (y', y_N)$ we may assume that the vector field $X_i \in \mathfrak{X}$ which is transversal to $\partial G$ at $x_0$ is transformed to $\frac{\partial}{\partial y_N}$. Thus the system of vector fields $\mathfrak{X} = \{X_1, \cdots, X_m\}$ is transformed to the system $\mathfrak{Y} = \{Y_1, \cdots, Y_{m-1}, \frac{\partial}{\partial y_N}\}$ and the vector field $X_0$ to $Y_0$. Here $Y_1, \cdots, Y_{m-1}$ are the new tangential vector fields on the image of $U \cap \partial G$, contained in $\mathbb{R}^{N-1}$: if $y = (y', y_N)$ then

$$Y_j = \sum_{k=1}^{N-1} a_{jk}(y', y_N) \frac{\partial}{\partial y_k}, \quad \text{for} \quad j = 1, \cdots, m-1$$

The equation $\mathcal{L}_{\mathfrak{X}} u = h$ is transformed to

$$\mathcal{L}_{\mathfrak{Y}} u = -\frac{\partial^2}{\partial y_N^2}(\alpha u) + \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) = \frac{\partial(\tilde{c}\beta u)}{\partial y_N} + Y_0(\beta u) + \tilde{c}\beta u + \tilde{h}(y)$$

for suitable $\beta, \quad \tilde{c} \in \mathcal{D}(\mathbb{R}_+^N)$ and *supp* $\alpha$ and *supp* $\beta$ are contained in a neighbourhood of 0 in $\mathbb{R}_y^N$ with $\beta = 1$ on *supp* $\alpha$.

It can be verified that the transformed system of vector fields $\mathfrak{Y} = \{Y_1, \cdots, Y_{m-1}, \frac{\partial}{\partial y_N}\}$ and $Y_0$ satisfy the logarithmic estimate in a neighbouhood $V$ of 0 in $\mathbb{R}_y^N$.

We use the tangential pseudo-differential operator

$$\langle D' \rangle \text{ where } D' = (\frac{\partial}{\partial y_1}, \cdots \frac{\partial}{\partial y_{N-1}}),$$
$$\eta' = (\eta_1, \cdots, \eta_{N-1}) \text{ and } \langle \eta' \rangle^2 = \mathrm{e}^2 + |\eta'|^2.$$

It follows as before that the logarithmic estimate implies that for any $\epsilon > 0$ there exists a constant $C_{\epsilon,s} > 0$ such that

$$||(\log\langle D'\rangle^s v; L^2(\mathbb{R}^N_y)||^2 \\ \leq \epsilon^{2s}\Big(\sum_{j=1}^{m-1} ||Y_j v; L^2(\mathbb{R}^N)||^2 + ||\frac{\partial v}{\partial y_N}; L^2(\mathbb{R}^N)||^2\Big) + C_{\epsilon,s}||v; L^2(\mathbb{R}^N)||^2 \\ \text{for all} \quad v \in \mathcal{D}(V \cap \mathbb{R}^N_y)$$

and hence also for all $v \in M^1_0(V \cap \mathbb{R}^N_y)$ by the density of $\mathcal{D}(V \cap \mathbb{R}^N_y)$

We introduce the tangential function spaces:

We shall denote, for $s > 0$, by

$$E^{\log}_{0,s} = E^{\log}_{0,s}(\mathbb{R}^N_+) = \{u \in L^2(\mathbb{R}^N); \ \ (\log\langle D', 0\rangle)^s u \in L^2(\mathbb{R}^N)\}$$

and the corresponding function space in the (tangential) phase space

$$F^{\log}_{0,s} = F^{\log}_{0,s}(\mathbb{R}^N_+) = \{v \in L^2(\mathbb{R}^N_+); \ \ (\log\langle \eta', 0\rangle)^s v(\eta) \in L^2(\mathbb{R}^N_+)\}$$

Here we have used the standard notation

$$\mathbb{R}^N_+ = \{(y', y_N) \in \mathbb{R}^N; \ \ y_N > 0, \ \ y' \in \mathbb{R}^{N-1}\}$$

and

$$\eta = (\eta', \eta_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+.$$

Clearly we also have

$$E^{\log}_{0,s}(\mathbb{R}^N_+) = E^{\log}_s(\mathbb{R}^N)|_{\mathbb{R}^N_+}$$

We also make use of the tangential Littlewood-Paley decomposition of functions in $L^2$:

$$\Phi'_{-1}(f) = \varphi_{-1}(\langle D', 0\rangle)(f), \quad \text{and} \quad \Phi'_k(f) = \varphi(\mathrm{e}^{-k}\langle D', 0\rangle)(f), \ \ k \in \mathbb{N}$$

where

$$\mathcal{F}(\varphi_{-1}(\langle D', 0\rangle)(f) = \varphi_{-1}(\langle \eta', 0\rangle)\hat{f}(\eta)$$

and

$$\mathcal{F}(\varphi(\mathrm{e}^{-k}\langle D', 0\rangle)(f) = \varphi(\mathrm{e}^{-k}\langle \eta', 0\rangle)\hat{f}(\eta)$$

Once again the distributions in the space $E^{\log}_{0,s}(\mathbb{R}^N_+)$ are characterized by

$$\sum k^{2s}||\Phi'_k(u); L^2(\mathbb{R}^N_+)||^2 < +\infty$$

We have more precise estimates similar to the ones in the propositions of section 5.

## 8. Idea of the proof for the regularity upto the boundary

As in the proof of the interior regularity, since the system of vector fields $\mathfrak{Y} = \{Y_1, \cdots, Y_{m-1}, \frac{\partial}{\partial y_N}\}$ satisfy the logarithmic estimate, we obtain the estimate

$$||(\log\langle D', 0\rangle^\rho)^l(\alpha v); L^2(\mathbb{R}^N_+)|| \leq l!||\langle D', 0\rangle^\rho(\alpha v); L^2(\mathbb{R}^N_+)||$$

for any integer $l \geq 0$ and $\rho \geq 1$, and $\alpha \in \mathcal{D}(\mathbb{R}^N_+)$ is a localizing cut off function.

Denoting, by

$$\Delta_{y'} = \sum_{j=1}^{N-1} \frac{\partial^2}{\partial y_j^2}$$

the tangential Laplacian on $\mathbb{R}_{y'}^{N-1}$ we take the fundamental solutions

$$G'_\delta = (1 - \delta^2 \Delta_{y'})^{-1}$$

as the regularizing operators. Starting from the weak solution $u$ of the transformed problem for the differential operator

$$\mathcal{L}_{\mathfrak{Y}} u = -\frac{\partial^2}{\partial y_N^2}(\alpha u) + \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) = \frac{\partial(\tilde{c}\beta u)}{\partial y_N} + Y_0(\beta u) + \tilde{c}\beta u + \tilde{h}(y)$$

we take as test function

$$v = \beta G'_\delta (\log\langle D', 0\rangle^\rho)^j \alpha^2 (\log\langle D', 0\rangle^\rho)^j G'_\delta(\beta u)$$

in the trasformed differential equation for $\mathcal{L}_{\mathfrak{Y}}$. Here, in the right hand side

$$\beta G'_\delta (\log\langle D', 0\rangle^\rho)^j \alpha^2 (\log\langle D', 0\rangle^\rho)^j G'_\delta \beta$$

is a tangential pseudo-differenial operator on $\mathbb{R}_{y'}^{N-1}$. Here the integration by parts with respect to the variable $y_N$ occurs only once. This proves the required estimate. As in the proof of the interior regularity this estimate implies that

$$\langle D', 0\rangle^l (\alpha u) \in L^2(\mathbb{R}_+^N), \quad \text{for all integers} \quad \imath \geq 0$$

and any $\alpha \in \mathcal{D}(V \cap \mathbb{R}_+^N)$. In particular, $\dfrac{\partial(\alpha u)}{\partial y_N} \in L^2(\mathbb{R}_+^N)$ and so $\alpha u$ belongs to $H^1(\mathbb{R}_+^N)$. Making use of the differential equation

$$\frac{\partial^2(\alpha u)}{\partial y_N^2} = \sum_{j=1}^{m-1} Y_j^* Y_j(\alpha u) + \frac{\partial(\tilde{c}\beta u)}{\partial y_N} + Y_0(\beta u) + \tilde{c}\beta u + \tilde{h}(y)$$

we see that $\alpha u$ belongs to $H^2(\mathbb{R}_+^N)$. By the boot strap argument as before we prove that $\alpha u$ belongs to $H^l(\mathbb{R}_+^N)$ for all integers $\imath \geq 0$. Then by applying the classical Sobolev embedding theorem it follows that $\alpha u \in C^\infty(\overline{\mathbb{R}}_+^N)$. Taking $\alpha = 1$ in a neighbourhood of $0 \in \mathbb{R}^N$ we conclude that $u \in C^\infty(\overline{\mathbb{R}}_+^N \cap \tilde{V})$ for some neighbourhood $\tilde{V}$. Finally by the usual patching up argument we obtain the $C^\infty$-regularity upto the boundary from the interior.

## Appendix—A sufficient condition for the logarithmic estimate

In all our preceding considerations the fundamental hypothesis consists in that the system of vector fields $\mathfrak{X}$ satisfy a logarithmic estimate. Sufficient conditions in order that such an estimate holds involve precise decay assumptions near the degeneracy set on the coefficients of the vector fields which degenerate. In order to illustrate such a sufficient condition we shall begin with the following example in two dimensions:

**(A) A 2-dimensional example.** Suppose

$$\mathfrak{X} = \{X_1, X_2\} = \{\frac{\partial}{\partial x_1}, b(x_1)\frac{\partial}{\partial x_2}\}$$
$$\text{where } b \in C^\infty(\mathbb{R}), \quad b(0) = 0 \quad \text{and} \quad b(x_1) \neq 0 \quad \text{for} \quad x_1 \neq 0$$

i.e. $\mathcal{L}_{\mathfrak{X}} = -(\frac{\partial}{\partial x_1})^2 - a(x_1)(\frac{\partial}{\partial x_2})^2$ where $a(x_1) = b(x_1)^2$ and hence $a \in C^\infty(\mathbb{R})$, $a(0) = 0$ and $a(x_1) > 0$ for $x_1 \in \mathbb{R} \setminus 0$.

A specific example is

$$a(x_1) = \exp[-2|x_1|^{\frac{-1}{s}}], \quad s > 0$$

**Proposition 5.** *Suppose that there exists an $\epsilon \geq 0$ such that*

$$\limsup_{x_1 \to 0} \ |x_1|^{\frac{1}{s}} |\log a(x_1)| \leq \epsilon$$

*then for any compact set $K$ in $\mathbb{R}^2$ there exist constants $c_0 > 0$ and $c_{\epsilon,s} > 0$ such that*

$$||(\log\langle D\rangle)^s u; L^2||^2 \leq c_0 \epsilon^{2s} \sum_{j=1}^{2} ||X_j u; L^2||^2 + c_{\epsilon,s}||u\ L^2||^2, \quad \forall u \in \mathcal{D}(K)$$

This is a particular case of the following result due to Wakabayashi and Suzuki.

**Proposition 6.** *Suppose that $f, a$ be non-negative continuous functions on $\mathbb{R}$ such that*

$$f(t) > 0, \quad a(t) > 0 \quad \textit{in} \quad \mathbb{R} \setminus 0$$

*and there is an $\epsilon \geq 0$ such that*

$$\limsup_{t \to 0} |f(t)^{\frac{-1}{2}} \int_0^t f(\tau) \mathrm{d}\tau|^{\frac{1}{s}} |\log a(t)| \leq \epsilon$$

*then, for any compact set $K$ in $\mathbb{R}^2$, there exist constants $c_0 > 0$ and $c_{\epsilon,s} > 0$ such that*

$$||f(x_1)^{\frac{1}{2}}(\log\langle D\rangle)^s u; L^2||^2 \leq c_0 \epsilon^{2s} \sum_{j=1}^{2} ||X_j u; L^2||^2 + c_{\epsilon,s}||u; L^2||^2, \quad \forall u \in \mathcal{D}(K)$$

*An idea of the proof* - Let $F$ be a primitive of $f$. We shall write $t$ for the variable $x_1$. Since $a$ is continuous and $a(0) = 0$ we can find a $t_0 > 0$ such that $a(t) < 1$ for $|t| < t_0$ and

$$|F(t)[-\log a(t)]^s \leq 2\epsilon^s f(t)^{\frac{1}{2}} \quad \text{for} \quad |t| < t_0$$

Once again since $a(0) = 0, \quad a(t) > 0$ in $\mathbb{R} \setminus 0$ and is continuous, there is a large enough positive number $\lambda_0 > 0$ such that $a(t) \leq \lambda_0^{-1}$, for $|t| < t_0$. Now for any $\lambda > 0$ and since $|\log a(t)| = -\log a(t)$, for any $v \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} |f(t)^{\frac{1}{2}}(\log\lambda)^s v; L^2||^2 &= \int f(t)(\log\lambda)^{2s}|v(t)|^2 \mathrm{d}t \\ &= \int \frac{\mathrm{d}F(t)}{\mathrm{d}t}(\log\lambda)^{2s}|v(t)|^2 \mathrm{d}t \end{aligned}$$

$$= \int [\frac{\mathrm{d}}{\mathrm{d}t}, F(t)](\log\lambda)^{2s}|v(t)|^2 \mathrm{d}t$$
$$\leq 2|(\frac{\mathrm{d}v}{\mathrm{d}t}, F(t)(\log\lambda)^{2s}v)|$$

which by the Cauchy-Schwarz inequality implies

$$\leq 2||\frac{\mathrm{d}v}{\mathrm{d}t}; L^2||.||F(t)(\log\lambda)^{2s}v); L^2||$$
$$\leq 8\epsilon^{2s}||\frac{\mathrm{d}v}{\mathrm{d}t}; L^2||^2 + \frac{1}{8\epsilon^{2s}}||F(t)(\log\lambda)^{2s}v); L^2||^2$$

We estimate the second term of the last inequality as follows:

For $\lambda \geq \lambda_0$ we set

$$E_\lambda = \{t \in \mathbb{R}; a(t)\lambda \leq 1\} \subset \{t \in \mathbb{R}; |t| < t_0\}$$

and writing

$$||F(t)(\log\lambda)^{2s}v); L^2||^2 = \int_{\mathbb{R}} F(t)^2(\log\lambda)^{4s}|v(t)|^2 \mathrm{d}t,$$

we split the integral into a sum of integrals over $E_\lambda$ and its complement $E_\lambda^c$ and estimate the two integrals seperately.

We observe that, in *supp* $v \cap E_\lambda^c$ we have $a(t)\lambda > 1$ for $\lambda \geq \lambda_0$ and hence we can take

$$\frac{1}{8\epsilon^{4s}} F(t)^2(\log\lambda)^{4s} \leq \lambda \leq a(t)\lambda^2$$

(if necessary for a larger $\lambda \geq \lambda_0 > 0$). On the other hand, in the set $supp(v) \cap E_\lambda$ for $\lambda \geq \lambda_0$ we have $a(t)\lambda \leq 1$ so that $\log\lambda < -\log\ a(t)$ and hence using the hypothesis of the proposition we see that

$$F(t)^2(\log\lambda)^{4s} \leq F(t)^2(\log\lambda)^{2s}(-\log\ a(t))^{2s} \leq 4\epsilon^{2s} f(t)(\log\lambda)^{2s}$$

Finally, we obtain

$$\int_{\mathbb{R}} F(t)^2(\log\lambda)^{4s}|v(t)|^2 \mathrm{d}t = \int_{E_\lambda} + \int_{E_\lambda^c}$$
$$\leq 4\epsilon^{2s} \int_{E_\lambda} f(t)(\log\lambda)^{2s}|v(t)|^2 \mathrm{d}t + \epsilon^{2s} \int_{E_\lambda^c} a(t)\lambda^2|v(t)|^2 \mathrm{d}t$$

After substituting this in the estimates at the beginning we conclude that, for $\lambda \geq \lambda_0$, we have

$$||f(t)^{\frac{1}{2}}(\log\langle D\rangle)^s v; L^2||^2 \leq 16\epsilon^{2s}\{||\frac{\mathrm{d}v}{\mathrm{d}t}; L^2||^2 + (a(t)\lambda^2 v, v)_{L^2}\}$$

The required estimate follows from this immediately.

**(B) The general case.** In the special case considered above in part (A), $a(0) = 0$ where $a(x_1)$ is the coefficient of the vector field $X_2$ and hence $X_2$ degenerates along the manifold $\Sigma = \{x_1 = 0\}$ and, by continuity, $a(x_1) > 0$ and remains close to 0 in a small neighbourhood $V$ of $x_1 = 0$, that is, we have

$$0 < a(x_1) < 1, \quad \text{and} \quad |\log a(x_1)|^s = [-\log a(x_1)]^s \leq (2\epsilon)^s |I|^{-1}$$

This latter condition can be interpreted as the local behaviour of the symbol $X_2(x,\xi) = X_2(x_1, x_2; \xi_1, \xi_2) = a(x_1)\xi_2$ of the vector field $X_2$ along the integral curve $t \to \exp(tX_1)(0)$ of the vector field $X_1$ starting from a point $x^0 \in \Sigma$, the manifold where $X_2$ has degeneracy and where the vector field $X_1$ is transversal to the degeneracy manifold $\Sigma = \{x_1 = 0\}$),.

Since the degeneracy set $\Sigma$ is, in general, a union of one-codimensional smooth hypersurfaces we are led to assume a micro-local refinement of the above condition, involving also the commutators $X_J$; $J = \{j_1, \cdots, j_k\}$. This corresponds to taking derivatives of $a(x_1)$ in the spcial case considered in part (A). In order to formulate the condition more precisely we introduce some notation.

**Formal reduction of the problem.** Let $x^0$ be a point of the degeneracy set $\Sigma$. Then by definition of $\Sigma$, we know that there exists a unit co-vector $\xi \in S^{N-1}$ such that the symbols of all orders $X_J(x,\xi)$ vanish at $(x^0, \xi)$.

For an integer $k \geq 1$ consider the symbols $X_J(x,\xi)$ of the commutators $X_J = [X_{j_1}, [X_{j_2}, \cdots [X_{j_{k-1}}, X_{j_k}]]]$, $J = \{j_1, \cdots, j_k\}$ of orders $|J| \leq k$.

Since the condition of Hörmander holds outside the degeneracy set $\Sigma$ we have a subelliptic estimate of the following form: given an integer $k \geq 1$ there exists a real number $0 < h = h(k) \leq \frac{1}{2}$ such that

$$\sum_{|J| \leq k} ||\langle D \rangle^{h-1} X_J u; L^2||^2 \leq C\{(\mathcal{L}_{\mathfrak{X}} u, u) + ||u, L^2||^2\}$$

$\Sigma$ being non characteristic with respect to the system of vector fields $\mathfrak{X}$ there exists a vector field $X_r \in \mathfrak{X}$ transversal to $\Sigma$ at $x^0$. Let $t \to \exp(tX_r)(x^0)$ be the (local) integral curve of the vector field $X_r$ starting from $x^0$. We may now introduce new local coordinates $x = (x', x_N) = (x', t)$ so that $x^0 = (0', 0)$ and the integral curve is straightened and hence is of the form $x'$=a constant vectorin $\mathbb{R}^{N-1}$.

The left hand side of the above subelliptic estimate can now be rewritten as

$$\sum_{|J| \leq k} \int_{\mathbb{R}} \{ \int_{\mathbb{R}^{N-1}} \langle \xi', D_{x_N} \rangle^{2h-2} |X_J(x', x_N; \xi', D_{x_N}) \hat{u}(\xi', x_N)|^2 \mathrm{d}\xi' \} \mathrm{d}x_N$$

where $X_J(x,\xi) = X_J(x', x_N; \xi', \xi_N)$ is the symbol of the commutator $X_J$ and $\hat{u}(\xi', x_N)$ is the partial Fourier transform of $u$ with respect to $x'$. Observe that

$$\langle \xi \rangle^{2h-2} |X_J(x,\xi)|^2$$

is a pseudo-operator symbol of order $2h$. Then

$$p(x', x_n; \xi') = |\xi'|^{-2h}.[|\xi|^{2h-2} |X_J(x,\xi)|^2]|_{\xi_N = 0}$$

is a pseudo differential symbol of order zero on $\mathbb{R}^{N-1}_{x'}$. Thus the subelliptic estimate becomes

$$(p(x', x_N; D')u, u) + ||D_{X_N} u||^2 \leq C\{(\mathcal{L}_{\mathfrak{X}} u, u) + ||u||^2\}$$

**Formulation of the sufficient condition.** We introduce the following functions associated to the symbols $X_J(x,\xi)$, $|J| \leq k$, for all $k \geq 1$, along this integral curve;

$$a(t; x^0, r, k) = min_{\xi \in S^{N-1}} \sum_{|J| \leq k} |X_J(\exp(tX_r)(x^0), \xi)|^2$$

The function $t \rightarrow a(t; x^0, r, k)$ being a non-negative continuous function which vanishes at $t = 0$ (that is, at the initial point $x^0$ of the integral curve of $X_r$), for any $\delta > 0$ there exists a neighbourhood $(-\rho, \rho)$ of the parameter $t = 0$ such that the function $a$ remains small and $> 0$. For intervals $I$ contained in such a neighbourhood $(-\rho, \rho)$ we introduce the mean value

$$[a]_I(x^0; r, k) = |I|^{-1} \int_I a(t; x^0, r, k) \mathrm{d}t$$

Hence, for any $\delta > 0$, there is a $\rho > 0$ such that for any interval $I \subset (-\rho, \rho)$ we have

$$[a]_I(x^0; r, k) < \delta$$

The condition on $a(x_1)$ is refined to the following assumption introduced by C. -J. Xu:

**Hypothesis (A).** *There exist a real number $s > 0$ and an $\epsilon > 0$ such that*

$$\inf \left( sup\{|I|^{1/s} |\log[a]_I(x^0; r, k)|; \quad I \subset (-\rho, \rho) \quad and \quad [a]_I(x^0; r, k) < \delta\} \right) < \epsilon$$

*where the* infimum *is taken over all $\delta > 0$, $k \in \mathbb{N}$, $\rho > 0$, $1 \leq r \leq m$.*

That is, there is an $s > 0$ and an $\epsilon > 0$ such that there exist a vector field $X_r \in \mathfrak{X}$ transversal to $\Sigma$ at $x^0$, an integer $k \geq 1$ and a $\delta > 0$ so that for sufficiently small arcs $\gamma_r$ of the integral curve of $X_r$ through $x^0$ where

$$[a]_{\gamma_r}(x^0; r, k) < \delta$$

and

$$|\log[a]_{\gamma_r}(x^0; r, k)|^{2s} \leq (2\epsilon)^{2s} |l(\gamma_r)|^{-2}.$$

We are now in a position to formulate a sufficient condition which generalizes the condition in the Proposition 5 of Wakabayashi and Suzuki:

**Theorem 4** (C. -J. Xu)**.** *Suppose the system of vector fields $\mathfrak{X} = \{X_1, \cdots, X_m\}$ satisfy the following hypothesis:*

*(i) there exists a union of $C^\infty$ hypersurfaces $\Sigma = \cup_{j \in J} \Sigma_j$ in $\Omega$ such that $\Sigma$ is non characteristic with respect to the system $\mathfrak{X}$, and moreover the system $\mathfrak{X}$ satisfies the condition of Hörmander in $\Omega$ except on $\Sigma$, i.e. the rank of the Lie algebra $\mathfrak{G}(\mathfrak{X})$ at every point $x \in \Omega \setminus \Sigma$ is equal to $N$, the dimension of $\Omega$.*

*Assume further that the system of vector fields $\mathfrak{X}$ satisfy the Hypothesis (A). Then there exist constants $C_0$ (which is independent of $\epsilon > 0$) and a constant $C_{\epsilon,s}$ such that*

$$||(\log\langle D \rangle)^s u; L^2||^2 \leq C_0 \epsilon^{2s} (\mathcal{L}_{\mathfrak{X}} u, u) + C_{\epsilon,s} ||u; L^2||^2$$

A suitable decomposition of Littlewood-Paley type is used in order to estimate the term $(p(x', x_N; D')u, u)$ using the Hypothesis (A) where the method of proof sketched in part (A) can be adapted. For each component in this decomposition one applies the following lemma due to E.Sawyer :

**Lemma 1** (E. Sawyer)**.** *Suppose $m_1(t)$, $m_2(t)$ be two non negative weight functions defined on an interval $I_0 \subset \mathbb{R}$ and belonging to $L^1_{loc}(I_0)$. Then, the weighted estimate*

$$\int_{I_0} |v(t)|^2 m_1(t) \mathrm{d}t \leq C \int_{I_0} \{|v'(t)|^2 + m_2(t)|v(t)|^2\} \mathrm{d}t, \quad \forall v \in C_0^1(I_0)$$

*holds if and only if*

$$[m_1]_I \leq C'\{3[m_2]_{3I} + 2|I|^{-2}\} \quad \textit{for every subinterval} \quad I \ \textit{such that} \ 3I \subset I_0$$

*where $[m_i]_I$ denotes the mean value of $m_i$ over the interval $I$:*

$$[m_i]_I = |I|^{-1} \int_I m_i(t) \mathrm{d}t$$

The proof for a general $u$ is then completed using the standard partition of unity argument with respect to a covering of $\Sigma$ by small neighbourhoods of its points $x^0$. The technical details of the proof of the theorem we refer to the paper of Xu.

## References

[1] M.S. Baouendi and C. Goulaouic, *Études de l'analyticité e de la regularité Gevrey pour une classe d'opérateures elliptiques dégénérès*, Ann. Sci.Ecole Norm. Sup. 4 (1971), 31 - 46.

[2] J. -M. Bony, *Principe de maximum, inégalité de Harnack pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier 19 (1969), 277 - 304.

[3] Lars Hörmander, *Hypoelliptic second order differential equations*, Acta Mathematica 119 (1967), 147 - 171.

[4] Y. Morimoto and T. Morioka, *The positivity of Schrödinger operators and the hypoellipticity of second order degenerate ellptic operators*, Bull. Sc. Math. France 121 (1997), 507-547.

[5] A.N. Kolmogorov, *Zufällige Bewegungen*, Annals of Math. 35(1934), 116-117.

[6] Y. Morimoto, *Hypoellipticity of infinitely degenerate elliptic operators*, Osaka Jr. Math. 24 (1987), 13-25.

[7] E. Sawyer, *A wighted inequality and eigenvalue estiamtes for Schrödinger operators* , Indiana Univ. Jr. 35 (1986), 1-28.

[8] C.J. Xu, *Subelliptic variational problems*, Bull.Soc. Math. France 118 (1990), 147-169.

[9] S. Wakabayashi and M. Suzuki, *Microhypoellipticity for a class of pseudo-differential operators with double characteristics*, Funkciaj Ekvacioj, 36 (1993), 519-556.

*Manuscript received September 5, 2006*
*revised September 14, 2006*

M. K. Venkatesha Murthy