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PROJECTED DYNAMICAL SYSTEMS AND VARIATIONAL INEQUALITIES EQUIVALENCE RESULTS

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ABSTRACT. We present some results of equivalence between projected dynamical systems and variational or quasi-variational inequalities. In particular we prove the result in strictly convex and smooth Banach Spaces, providing also some equivalent formulations in terms of Unilateral differential inclusions. Moreover we obtain for Hilbert spaces, using the Clarke tangent cone, a more general equivalence result with extension to non convex subsets.

1. INTRODUCTION

As it is well known in Hilbert spaces the critical points of the projected dynamical system based on general tangent cone of a convex set are equivalent to the equilibrium points of a variational inequality. In particular in [7] Cojocaru, Daniele and Nagurney introduced in the framework of Hilbert spaces the operator

$$\Pi_C(x, -F(x)) = \lim_{\lambda \to 0} \frac{P_C(x - \lambda F(x)) - x}{\lambda} = P_{T_C(x)}(-F(x)) \tag{1.1}$$

where P_C denotes the standard projection on a closed convex subset C of an Hilbert space. The authors applied this new operator to the study of the following class of differential equations called Projected Dynamical Equations

$$\frac{dx}{dt} = \lim_{\lambda \to 0} \frac{P_C(x - \lambda F(x)) - x}{\lambda} = P_{T_C(x)}(-F(x))$$

and to the study of Variational Inequalities (VI).

In the present paper we improve this result, providing an equivalence theorem in strictly convex and smooth Banach Spaces and we give some equivalent formulations in terms of Unilateral differential inclusions. The result is obtained using new effective concepts of projection.

Moreover we achieve for Hilbert spaces a more general equivalence result with extension to nonconvex subsets and in this way we prove an equivalence theorem between the extended definition of Projected Dynamical systems and a quasi variational inequality. The idea we are going to develop is to define on an appropriate convex and close cone a projected Dynamical system which represents an extension of the theory developed by Cojocaru, Daniele, Isac, Nagurney and Raciti (see [14], [15], [6], [8], [7], [19]). Precisely we use the Clarke tangent cone (see [18]). Also in this case a crucial step is the introducing of new concepts of projection.

Objects of future research will be the study of existence of critical points of the "extended" projected dynamical systems and the develop of extended definitions of more computable projected dynamical systems.

Finally let us remark that the equivalence result in Banach spaces presented in the international conference *Variational Analysis and Partial dierential equations* is contained in [12]

2. Preliminary Results

We denote by X a Banach space with dual space X^* and by $\|.\|$ and $\|.\|_*$ the respective norms. We denote also the duality pairing between X^* and X by $\langle f, x\rangle$ for $f \in X^*$ and $x \in X$, $\langle x, f\rangle$ the duality pairing between X and X^* for $f \in X^*$ and $x \in X$.

We define the duality mapping $J : X \to X^*$ by

$$J(x) = \{f \in X^* : \langle f, x\rangle = \|f\|_*^2 = \|x\|^2\},\ \forall x \in X$$

In the same manner we have the duality mapping $J^* : X^* \to X$ defined by:

$$J^*(f) = \{x \in X : \langle x, f\rangle = \|x\|^2 = \|f\|_*^2\},\ \forall f \in X^*$$

The existence of J and J^* is a corollary of the Hahn-Banach analytic form (see for instance [5]).

Remark 2.1. If X is an Hilbert space, we have $J = Id_X = J^*$.

Example 2.2. If $X = L^p(\Omega, \mathbb{R})$ with $1 < p < \infty$ then

$$J(x) = \|x\|^{2-p}|x|^{p-1}sgn(x)$$

and

$$J^*(x) = \|x\|^{\frac{p-2}{p-1}}|x|^{\frac{1}{1-p}}sgn(x)$$

where $sgn(x) = \chi_{[x>0]} - \chi_{[x<0]}$. This result could be usefully applied to Time Dependent Traffic Equilibria problems (see [10]).

Now we recall two definitions we need in the sequel.

Definition 2.3 (see [11])**.** A space $(X, \|.\|)$ is strictly convex if

$$\forall x \in X,\ \forall y \in X : \|x\| = \|y\| = 1,\ x \neq y \Rightarrow \|tx + (1-t)y\| < 1, \forall t \in]0, 1[.$$

Let us denote by $S(X) = \{x \in X : \|x\| = 1\}$.

Definition 2.4 (see [11])**.** A Banach space X is said to be smooth at $x_0 \in S(X)$ whenever there exists a unique $f \in S(X^*)$ such that $f(x_0) = 1$. If X is smooth at each point of $S(X)$ then we say that X is smooth.

From [11] we have also the following characterization criteria: A Banach space $(X, \|.\|)$ is smooth if and only if the norm $\|.\|$ admits a Gâteaux derivative in each direction.

Remark 2.5. Hilbert spaces and L^p spaces ($1 < p < \infty$) are reflexive, strictly convex and smooth.

From [4] we know that if we have X reflexive, strictly convex and smooth then J, J^* are one-to-one single-valued operators and $J^{-1} = J^*$. More precisely we have:

- X is reflexive if and only if J is surjective;
- X is smooth if and only if J is single-valued;
- X is strictly convex if and only if J is injective.

Besides the notion of projection operator in Hilbert space, it is possible to give an effective projection operator definition in a more general framework. Let us recall the following definition of metric projection operator (for more details see for instance [22]).

Definition 2.6 (see [22]). Let X be a Banach space and C a closed convex subset of X. We call the metric projection operator from X on C the set valued mapping $\pi(C|.) : X \to C$ defined by

$$x \to \pi(C|x) = \{y \in C : \|x - y\| = d_C(x)\}$$

where $d_C(x) = \inf_{z \in C} \|x - z\|$.

Note that for $x \in C$, $\pi(C|x)$ is the set of optimal solution of the following minimization problem:

$$\inf_{y \in C} \|x - y\|^2 \tag{2.1}$$

From now on and unless otherwise stated, we make the following assumptions: X Banach space, reflexive, strictly convex, and smooth. Then these additional assumptions ensure that $\pi(C|.) = P_C(.)$ is single valued and P_C is called the best approximate operator. Moreover we have the following characterization of $P_C(x)$:

$$\bar{x} = P_C(x) \Leftrightarrow \langle J(x - \bar{x}), y - \bar{x} \rangle \leq 0, \ \forall y \in C \tag{2.2}$$

As an extension of what we have on Hilbert spaces, (2.2) is called the basic variational principle for P_C in X. This characterization plays a fundamental role for our application.

Another possibility to generalize the notion of projection is to use, as done by Alber in [2], the *Lyapunov function.* The Lyapunov function is the strictly convex function in y, $V(x, y)$ given by:

$$V(x, y) := \|x\|^2 - 2\langle J(x), y \rangle + \|y\|^2$$

We remark that if C is a closed convex subset of X and if $x \in C$ then the problem

$$\min_{y \in C} V(x, y)$$

is uniquely solvable (apply for instance [5],Corollary III.20), then we can give the following definition:

Definition 2.7 (see [2] or [22]). We call generalized projection of x on C the following value:

$$\Pi_C(x) := \arg\min_{y \in C} V(x, y)$$

Remark 2.8 (see [2]).

- The operator $\Pi_C : X \to C \subset X$ is the identity on C, i.e. for every $x \in C, \Pi_C(x) = x$.
- In a Hilbert space, $V(x, y) = \|x - y\|^2$, Π_C coincides with the projection operator P_C.

As stated in [3] we have the following characterization of $\Pi_C(x)$.

Lemma 2.9. *Assume that C is a closed convex subset of X, then:*

$$\hat{x} = \Pi_C(x) \Leftrightarrow \langle J(x) - J(\hat{x}), y - \hat{x}\rangle \leq 0,\ \forall y \in C \tag{2.3}$$

Here again the variational characterization plays a fundamental role for our application.

From Corollary 1, page 22, [11] we know that if X is reflexive then:

$$X \text{ strictly convex} \Leftrightarrow X^* \text{ smooth},$$
$$X \text{ smooth} \Leftrightarrow X^* \text{ strictly convex}.$$

Definition (2.6) applies also to X^* and to convex and closed subset $\Gamma \subset X^*$, and we have the following variational principle:

$$\bar{f} = P_\Gamma(f) \Leftrightarrow \langle J^*(f - \bar{f}), g - \bar{f}\rangle \leq 0,\ \forall g \in \Gamma \tag{2.4}$$

We can introduce also the Lyapunov function on $X^* \times X^*$:

$$V^*(f,g) = \|f\|_* - 2\langle J^*(f), g\rangle + \|g\|_*$$

and then the following definition:

Definition 2.10. We call generalized projection of f on $\Gamma \subset X^*$ the following value:

$$\Pi_\Gamma(f) := \arg\min_{g \in \Gamma} V^*(f,g)$$

We have the following variational principle:

$$\hat{f} = \Pi_\Gamma(f) \Leftrightarrow \langle J^*(f) - J^*(\hat{f}), g - \hat{f}\rangle \leq 0,\ \forall g \in \Gamma \tag{2.5}$$

Finally we remind some classical results regarding Tangent and Normal Cones, please refer to [21] for more details.

Definition 2.11. Let be $C \subset X$ convex, we call **General Tangent Cone** to C at $\bar{x}$ the set given by:

$$T_C(\bar{x}) = \limsup_{\lambda \to 0} \frac{1}{\lambda}(C - \bar{x})$$

Remark 2.12. The definition 2.11 is valid also if C is non convex. If C is a convex subset of X, the definition 2.11 is equivalent to:

$$T_C(\bar{x}) = \overline{\bigcup_{\lambda > 0} \lambda(C - \bar{x})}$$

Definition 2.13. We call **Regular Tangent Cone** to C at $\bar{x}$ the set given by:

$$\hat{T}_C(\bar{x}) = \liminf_{\lambda \to 0,\ x \to \bar{x},\ x \in C} \frac{1}{\lambda}(C - \bar{x}) \tag{2.6}$$

This cone is also called **Clarke Tangent Cone**.

Remark 2.14. We always have $\hat{T}_C(\bar{x}) \subset T_C(\bar{x})$. If C is convex then $\hat{T}_C(\bar{x}) = T_C(\bar{x})$.

Definition 2.15. We call **Regular Normal Cone** to C at $\bar{x}$ the set given by:

$$\hat{N}_C(\bar{x}) = \{v | \langle v, x - \bar{x}\rangle \leq \circ(\|x - \bar{x}\|)\ per\ x \in C\} \tag{2.7}$$

Where $\|.\|$ is the norm on X and '$\circ$' means

$$\limsup_{x \to \bar{x}, x \in C, x \neq \bar{x}} \frac{\langle v, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0 \tag{2.8}$$

Definition 2.16. We call **General Normal Cone** to C at $\bar{x}$ the set given by:

$$N_C(\bar{x}) = \{v|\ \exists x^\nu \in C,\ v^\nu \in \bar{N}_C(x^\nu),\ con\ (x^\nu, v^\nu) \to (\bar{x}, v)\} \tag{2.9}$$

Note: As done in [18] we use ν indexes to indicate the elements of a suite.

Definition 2.17. We call **Clarke normal Cone** the set given by:

$$\bar{N}_C(\bar{x}) = \ Closed\ convex\ hull\ of\ N_C(\bar{x}) \tag{2.10}$$

Remark 2.18. $\bar{N}_C(\bar{x})$ and $N_C(\bar{x})$ are closed and convex. $\hat{N}_C(\bar{x})$ is convex if C is convex. The following inclusions are always true:

$$\hat{N}_C(\bar{x}) \subset N_C(\bar{x}) \subset \bar{N}_C(\bar{x}) \tag{2.11}$$

Proposition 2.19. *We have:*

$$\bar{N}_C(\bar{x}) = \{v|\langle v, w\rangle \leq 0,\ \forall w \in \hat{T}_C(\bar{x})\}, \tag{2.12}$$

$$\hat{T}_C(\bar{x}) = \{w|\langle v, w\rangle \leq 0,\ \forall v \in \bar{N}_C(\bar{x})\} \tag{2.13}$$

We recall for readers utility the following basic definitions and properties.

Definition 2.20. Let $C \subset X$ be convex, we call Normal cone to C in x the set given by:

$$N_C(x) = \{\xi \in X^*, \langle \xi, y - x\rangle \leq 0, \forall y \in C\}$$

Definition 2.21. Let M be a cone of X, the polar set of M, noted M^0 is defined by:

$$M^0 = \{\xi \in X^*, \langle \xi, x\rangle \leq 0, \forall x \in M\}$$

If $\boldsymbol{X}$ **is reflexive**, then the following relationships hold:

$$(T_C(x))^0 = N_C(x), \forall x \in C$$
$$(N_C(x))^0 = T_C(x), \forall x \in C.$$

T_C and N_C are always closed and if C is nonempty and convex they are nonempty and convex. The following one is a very important result due to Albert.

Theorem 2.22 ([3], Theorem 2.4)**.** *Assume that X is a real reflexive strictly convex and smooth Banach space, and K a non-empty, closed and convex cone of X then: $\forall x \in X$ and $\forall f \in X^*$ the decompositions*

$$\begin{aligned} x &= P_K(x) + J^*\Pi_{K^0}J(x)\ \ and\ \ \langle \Pi_{K^0}J(x), P_K(x)\rangle = 0 \\ f &= P_{K^0}(f) + J\Pi_K J^*(f)\ \ and\ \ \langle P_{K^0}(f), \Pi_K J^*(f)\rangle = 0 \end{aligned} \tag{2.14}$$

hold.

Remark 2.23. If X is an Hilbert space the decomposition $x = P_K(x) + J^*\Pi_{K^0}J(x)$ reduces to $x = P_K(x) + P_{K^0}(x)$.

3. Projected Dynamical Systems for non Convex subsets in Hilbert spaces

Let us start introducing the following concepts of projected dynamical system for non convex subsets of an Hilbert space.

Definition 3.1. We call the Clarke Generalized Projected-Dynamical System the operator

$$\Lambda^g_C : C \times X^* \to X$$

defined by setting:

$$\Lambda^g_C(x, h) = \Pi_{\hat{T}_C(x)}(J^*(h))$$

Definition 3.2. We call Generalized Projected Dynamical System (g-PDS), the discontinuous right hand side differential equation given by:

$$\frac{dx}{dt} = \Lambda^g_C(x, -F(x)) = \Pi_{\hat{T}_C(x)}(J^*(-F(x))) \tag{3.1}$$

The associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda^g_C(x, -F(x)) = \Pi_{\hat{T}_C(x)}(J^*(-F(x))), \ x(0) = x_0 \in C \tag{3.2}$$

Remark 3.3. If C is convex then $\hat{T}_C(x) = T_C(x)$ and we obtain the Projected Dynamical system defined in [12] and if in addition X is an Hilbert Space then (3.1) is the Projected dynamical system used in [14], [15], [6], [8], [7], and [19].

We also introduce a quasi-variational inequality or using a common used denomination (see [20]) a quasi-complementarity system.

Definition 3.4. We call Quasi-Complementarity System based on Clarke tangent cone, the problem given by a subset of a real Hilbert space $\mathbb{H}$, a closed subset C and the set value mapping $D : C \to 2^{\mathbb{H}}$ such that :

$$D(x) = x + \hat{T}_C(x)$$

and the following quasi-Variational inequality:

$$x \in C : \langle F(x), y - x \rangle \geq 0, \ \forall y \in D(x) \tag{3.3}$$

where F is a mapping from $C \to \mathbb{H}$.

Then we may obtain the following equivalence results.

Theorem 3.5. *Assume that X is an Hilbert Space. If (3.3) and (3.2) admits a solution then each equilibrium point of (3.3) is a critical point of (3.2) and, if (3.2) admits critical points then they are equilibrium points of (3.3).*

Proof. If x^* is an equilibrium point of (3.3), then we get:

$$x^* \in C : \langle x^* - \lambda F(x*) - x^*, x - x^* \rangle \leq 0, \ \forall x \in x^* + \hat{T}_C(x^*), \ \forall \lambda > 0$$

which can be written in the following way

$$x^* = P_{x^* + \hat{T}_C(x*)}(x^* - \lambda F(x^*)), \ \forall \lambda > 0$$

but as $x^* \in x^* + \hat{T}_C(x^*)$ we deduce that $P_{\hat{T}_C(x^*)}(-F(x^*)) = 0$.□
Now suppose that x^* is a critical point of (3.2), using Moreau's theorem we can write that

$$-F(x^*) = P_{\hat{T}_C(x^*)}(-F(x^*) + P_{\bar{N}_C(x^*)}(-F(x^*) = P_{\bar{N}_C(x^*)}(-F(x^*)$$

If $F(x^*) = 0$ then (3.3) is trivially verified. Now we suppose that $F(x^*) \neq 0$. Then as $-F(x^*) = P_{\bar{N}_C(x^*)}(-F(x^*))$ we get $-F(x^*) \in \bar{N}_C(x^*)$ which means by polarity

$$\langle -F(x^*), \omega \rangle \leq 0, \ \forall \omega \in \hat{T}_C(x^*)$$

and this is (3.3). □

4. Projected Dynamical Systems in Banach Spaces

In the paper [12] the authors provided two equivalence theorems: the first one based on the metric projection operator and the second one based on the generalized projection operator. We remind without proof some definitions and results:

Definition 4.1. We call the Metric Projected Dynamical System the operator

$$\Lambda_C^m : C \times X^* \to X$$

defined by setting:

$$\Lambda_C^m(x, h) = P_{T_C(x)}(J^*(h))$$

So we can define as done in [17] and in [7] the differential equation with a discontinuous right hand side.

Definition 4.2. We call M-Projected Dynamical System (m-PDS), the discontinuous right hand side differential equation given by:

$$\frac{dx}{dt} = \Lambda_C^m(x, -F(x)) = P_{T_C(x)}(J^*(-F(x))) \tag{4.1}$$

Consequently the associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda_C^m(x, -F(x)) = P_{T_C(x)}(J^*(-F(x))), \ x(0) = x_0 \in C \tag{4.2}$$

Definition 4.3. We call the Generalized Projected-Dynamical System the operator

$$\Lambda_C^g : C \times X^* \to X$$

defined by setting:

$$\Lambda_C^g(x, h) = \Pi_{T_C(x)}(J^*(h))$$

Definition 4.4. We call Generalized Projected Dynamical System (g-PDS), the discontinuous right hand side differential equation given by:

$$\frac{dx}{dt} = \Lambda_C^g(x, -F(x)) = \Pi_{T_C(x)}(J^*(-F(x))) \tag{4.3}$$

The associated Cauchy problem is given by:

$$\frac{dx}{dt} = \Lambda^g_C(x, -F(x)) = \Pi_{T_C(x)}(J^*(-F(x))),\ x(0) = x_0 \in C \tag{4.4}$$

In a Hilbert Space both (4.1) and (4.3) are equal to (1.1).

We consider now the variational problem given by:

$$x \in C: \ \langle F(x), v - x\rangle \ \geq \ 0, \quad \forall v \in C \tag{4.5}$$

where $F : C \to X^*$.

The following existence results are known.

Definition 4.5. (see [10]) Let E be a real topological vector space, $C \subset E$ convex. Then $F : C \to E^*$ is said to be:

(i) pseudomonotone iff, for all $x, y \in C$, $\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), x - y\rangle \leq 0$;
(ii) hemicontinous iff, for all $y \in C$, the function $\xi \to \langle F(\xi), y - \xi\rangle$ is upper semicontinous on C;
(iii) hemicontinous along line segments iff, for all $x, y \in C$, the function $\xi \to \langle F(\xi), y - x\rangle$ is upper semicontinous on the line segment $[x, y]$.

Then we have the following result.

Theorem 4.6. *(see* [10]*) Let E be a real topological vector space, and let $C \subseteq E$ be convex and nonempty. Let $F : C \to E^*$ be given such that:*

(i) *there exist $A \subseteq C$ compact, and $B \subseteq C$ compact, convex such that, for every $x \in C \setminus A$, there exists $y \in B$ with $\langle F(x), y - x\rangle < 0$;*
either (ii) *or* (iii) *below holds:*
(ii) *F is hemicontinous;*
(iii) *F is pseudomonotone and hemicontinous along line segments.*

Then, there exists $\bar{x} \in A$ such that $\langle F(\bar{x}), y - \bar{x}\rangle \geq 0$, for all $y \in C$.

We have (see [12]) the equivalence theorem given by

Theorem 4.7. *Assume that the hypotheses of Theorems 2.22 and 4.6 hold. Then each equilibrium point of (4.5) is a critical point of (4.1) and, if (4.1) admits critical points then they are equilibrium points of (4.5).*

and

Theorem 4.8. *Assume that the hypotheses of Theorems 2.22 and 4.6 hold. Then each equilibrium point of (4.5) is a critical point of (4.3) and, if (4.3) admits critical points then they are equilibrium points of (4.5)*

5. Projected Dynamical Systems, Unilateral Differential Inclusions

We consider also the two following differential inclusions:

$$-\dot{x} \in J^*(F(x) + N_{T_C(x)}(\dot{x})) \tag{5.1}$$

$$-\dot{x} \in J^*(F(x) + N_C(x)) \tag{5.2}$$

Proposition 5.1. *Let K be a non empty closed convex cone of X. For any s and v in X the following relations are equivalent:*

$$s = \Pi_K(v) \tag{5.3}$$

$$J(v) - J(s) \in N_K(s) \tag{5.4}$$

$$s \in K,\ J(v) - J(s) \in K^o,\ \langle J(v) - J(s), s\rangle = 0 \tag{5.5}$$

$$J(v) - J(s) \in K^o,\ \text{and } \forall \nu \in K^o,\ \|s\|^2 \leq \langle J(v) - \nu, s\rangle \tag{5.6}$$

Proof. Using the variational characterization of the generalized projection operator (see [12]) we get that (5.3) is equivalent to:

$$s \in K,\ \langle J(v) - J(s), y - s\rangle \leq 0, \forall y \in K$$

and by definition of a normal cone we get (5.4). Before the next step, first let us prove that $N_K(s) = K^o \cap \{s\}^\perp$.
By definition of $N_K(s)$, K^o and $\{s\}^\perp$ we get immediately that $K^o \cap \{s\}^\perp \subset N_K(s)$. Now suppose that $y \in N_K(s)$ then we have

$$\langle y, \eta - s\rangle \leq 0,\ \forall \eta \in K$$

If $\langle y, \eta\rangle > 0$, as K is a cone, we get $\forall \lambda > 0$, $\langle y, \lambda\eta\rangle \leq \langle y, s\rangle$ which implies a contradiction. Then $\langle y, \eta\rangle \leq 0$ and $y \in K^o$. As $s \in K$ we get $\langle y, s\rangle \leq 0$ and as $0 \in K$ we conclude that $\langle y, s\rangle = 0$ and $y \in \{s\}^\perp$. From the previous result we can conclude that

$$J(v) - J(s) \in N_K(s) \Leftrightarrow s \in K,\ J(v) - J(s) \in K^o,\ \langle J(v) - J(s), s\rangle = 0$$

Now suppose that (5.5) holds, take $\nu \in K^o$, as $\langle \nu, s\rangle \leq 0 = \langle J(v) - J(s), s\rangle$ we get $\langle \nu, s\rangle \leq \langle J(v), s\rangle - \langle J(s), s\rangle$ and by definition of J we get:

$$\|s\|^2 \leq \langle J(v) - \nu, s\rangle,\ \forall \nu \in K^o$$

Now suppose that (5.6) holds, in particular we get

$$\langle \nu, s\rangle \leq \langle J(v), s\rangle - \|s\|^2, \forall \nu \in K^o$$

If $\langle \nu, s\rangle > 0$ we have a contradiction. In fact $\langle \nu, s\rangle$ is bounded by $\langle J(v), s\rangle - \|s\|^2$ and K^o is a cone, so we get that $\langle \nu, s\rangle \leq 0,\ \forall \nu \in K^o$
But $J(v) - J(s) \in K^o$ then $\langle J(v) - J(s), s\rangle \leq 0$ if we take $\nu = 0$ in (5.6) we get exactly (5.5). □

Remark 5.2. A proof of the previous result in $\mathbb{R}^n$ space can be found in [1].

Corrollary 5.3. *The following statements are equivalent:*

$$\dot{x} = \Pi_{T_C(x)}(J^*(-F(x))) \tag{5.7}$$

$$-\dot{x} \in J^*(F(x) + N_{T_C(x)}(\dot{x})) \tag{5.8}$$

$$\left\{ \begin{array}{c} -\dot{x} \in J^*(F(x) + N_C(x)) \\ -\dot{x} = J^*(F(x) + P_{N_C(x)}(-F(x)) \\ -\dot{x} = J^*(P_{N_C(x)+F(x)}(0)) \end{array} \right. \tag{5.9}$$

Proof. We apply Proposition 5.1 with $K = T_C(x)$, $v = J^*(-F(x))$ and $s = \dot{x}$, so we get immediately (5.7) from (5.3). From (5.4) we get

$$JJ^*(-F(x)) - J(\dot{x}) \in N_{T_C(x)}(\dot{x})$$

As $JJ^* = Id_{X^*}$ we have the equivalence with (5.8). □

From Albert's theorem we deduce that (5.7) is equivalent to

$$\dot{x} = J^*(-F(x) - P_{N_C(x)}(-F(x)))$$

so using the variational principle for metric projection we get:

$$\langle J^*(-F(x) + J(\dot{x}) + F(x)), y + J(\dot{x}) + F(x)\rangle \leq 0, \ \forall y \in N_C(x)$$

and this is equivalent to

$$-\dot{x} = J^*(P_{N_C(x)+F(x)}(0))$$

And this means that the vector $J(-\dot{x})$ is of minimum norm in $(F(x) + N_C(x))$.

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