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DEGENERATE GINZBURG-LANDAU FUNCTIONALS

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Abstract. In this note we present the results of the recent paper [14] concerning a class of Ginzburg–Landau functionals E_ε associated with a couple of non–commuting vector fields. We study the asymptotic behavior of the minimizers, showing that it is independent of the topological degree of the boundary datum. Moreover, we prove uniqueness and regularity of the minimizer of the limit problem, in spite of the lack of lifting theorems in the natural function spaces for the limit functional.

1. Introduction

In [14], we carried out an asymptotic analysis for the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + x_1^2 \left|\frac{\partial u}{\partial x_2}\right|^2 \right) dx + \frac{1}{4\varepsilon^2}\int_G (|u|^2-1)^2 dx, \tag{1}$$

in the naturally associated function spaces. Here G be a bounded open subset of $\mathbb{R}^2$ with boundary ∂G that is a smooth simple closed regular curve. The functional E_ε in (1) can be considered as a "degenerate Ginzburg–Landau functional" associated with the family of smooth vector fields $X = \{X_1, X_2\} := \{\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_2}\}$. The asymptotic theory for classical (i.e. elliptic or non degenerate) Ginzburg–Landau functionals in G (see [1] and [2]) describes the behavior as $\varepsilon \to 0^+$ of the minima

$$u_\varepsilon := \min_{u\in W^{1,2}(G)} E_\varepsilon(u)$$

of the variational functionals

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G (|u|^2-1)^2 dx, \tag{2}$$

under the boundary constraint $u = g$ on ∂G. Here $W^{1,2}(G) := W^{1,2}(G,\mathbb{C})$ is the usual space of complex-valued Sobolev functions, and g is a prescribed (say) smooth function $g : \partial G \to \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{C}$.

Roughly speaking, in the elliptic case the asymptotic behavior of u_ε as $\varepsilon \to 0^+$ depends on $d := \deg(g, \partial G)$, the Brouwer degree of g. Indeed, if $d = 0$ then

$$W^{1,2}_g(G,\mathbb{S}^1) := \{u \in W^{1,2}(G,\mathbb{S}^1)\,;\, u = g \text{ on } \partial G)\}$$

is not empty, since it contains at least a smooth extension of g to all of $\bar{G}$, and $u_\varepsilon \to u_0$ in $W^{1,2}(G,\mathbb{S}^1)$, where u_0 is a solution of the minimum problem

$$\min_{u\in W^{1,2}(G,\mathbb{S}^1)} \int_G |\nabla u|^2 dx \tag{3}$$

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(we shall see below that this minimum problem admit a unique smooth solution).

On the contrary, if $d \neq 0$, roughly speaking there is a subsequence u_{ε_k} that converges uniformly on compact sets outside of a finite number of points ($|d|$ points, to be precise) to a limit function u_*, and singularities (the so-called vortices) may appear.

It is well known that there are several properties of the functionals (1) and of the associated Sobolev type spaces that either are already known in the literature, or can be derived more or less mimicking the proofs of the corresponding statements in the elliptic setting. However, we encountered at least two somehow unexpected phenomena that require *ad hoc* arguments.

First of all, as soon as the functional is truly degenerate, i.e. as soon as ∂G intersect the degeneration line $\{x_1 = 0\}$, then the topological degree of the boundary datum g does not affect anymore the asymptotic behavior of the minimizers, that is always akin to that of the case $d = 0$ in the elliptic counterpart. In particular, if for $1 \leq p < \infty$ we set

$$W^{1,p}_X(G, \mathbb{S}^1) := \{u \in L^p(\Omega, \mathbb{S}^1) \, : \, X_1 u, X_2 u \in L^p(G)\},$$

and

$$W^{1,p}_{X,g}(G, \mathbb{S}^1) = \{u \in W^{1,p}_X(G, \mathbb{S}^1) \, ; \, u \equiv g \text{ in } \partial G\},$$

then the space $W^{1,2}_{X,g}(G, \mathbb{S}^1)$ – still defined as the set of functions $u \in W^{1,2}_X(G, \mathbb{S}^1)$ such that $u = g$ on ∂G – is not empty, regardless of the topological degree of g (Theorem 2.11 and Corollary 2.12), and the minimizers of the functionals in (1) converge to a minimizer of E_0 in $W^{1,p}_{X,g}(G, \mathbb{S}^1)$ (see Theorem 2.14).

From the technical point of view, this basically relies on the fact that the sharp trace spaces on the boundary for functions in $W^{1,p}_{X,g}(G)$ allow jump discontinuities when the boundary crosses the degeneration line, and we can always, starting from a smooth function g taking values in $\mathbb{S}^1$, write it in the form $g = \exp(i\tilde{\varphi})$, with $\tilde{\varphi}$ smooth on ∂G, except for a jump discontinuity in an arbitrarily fixed point (discontinuity coming from the jump discontinuity of $z \to \arg z$).

In addition, in the elliptic setting – again if $d = 0$ – the minimizer of (3) is unique and smooth up to the boundary. This fact can be derived writing the Euler equation of (3), that leads to the study of the following Dirichlet problem for the harmonic map equation:

$$\begin{cases} -\Delta u_0 = u_0 |\nabla u_0|^2 & \text{in } G \\ |u_0| \equiv 1 & \text{in } G \\ u_0 = g & \text{in } \partial G. \end{cases} \tag{4}$$

Now, the smoothness and the uniqueness of the solution of (4) can be proved by a lifting argument, i.e. relying on the facts that g can we written in the form $g = \exp(i\tilde{\phi})$ for a smooth function $\tilde{\phi}$ (since $\deg(g, \partial G) = 0$), and that *every* $u \in W^{1,2}(G, \mathbb{S}^1)$ can be written in the form $u = \exp(ih)$, with $h \in W^{1,2}(G)$. In fact, this property holds more generally for $u \in W^{1,p}(G, \mathbb{S}^1)$ when $p \geq 2$, since $G \subset \mathbb{R}^2$: see, e.g., [3] or [4]. An elementary computation yields that h solves the Dirichlet problem

$$
\begin{cases} \Delta h = 0 & \text{in } G \\ h = \tilde{\phi} & \text{in } \partial G. \end{cases} \tag{5}
$$

Thus, uniqueness and regularity of h (hence of u_0) follow.

Unfortunately, a similar lifting statement fails to hold for an arbitrary function $u \in W^{1,2}_X(G, \mathbb{S}^1)$, for a lifting theorem in $W^{1,p}_X(G, \mathbb{S}^1)$ holds if and only if $p > 3$ (tough, also when $1 \le p < 3$, a function $u \in W^{1,p}_X(G, \mathbb{S}^1)$ admits a lifting in larger Sobolev spaces of fractional order). This phenomenon is clearly related to the fact that – roughly speaking – the dimension of the Carnot–Carathéodory space associated with X_1 and X_2 equals 3 (to be slightly more precise: can be estimated by 3), that is larger than 2, the dimension of G as a manifold. It is worth mentioning here that our proof of the positive part of the lifting theorem does not follow the scheme of the proof in usual Sobolev spaces, but rather relies on it, by building two lifted functions $\tilde{\varphi}_\pm$ in $G \cap \{x_1 > 0\}$ and $G \cap \{x_1 < 0\}$ respectively, and showing that their traces on $\{x_1 = 0\}$ fit well, thanks to the trace theorems in the spaces $W^{1,p}_X$ ([8], [6]) and the results of [4] .

In spite of the lack of lifting in $W^{1,2}_X(G, \mathbb{S}^1)$, in [14] we were still able to carry out the proof of the equivalence between the (degenerate) harmonic map equation and a linear equation, by showing (Theorem 2.15 below) that *weak solutions* $u \in W^{1,2}_X(G, \mathbb{S}^1)$ *of* $-\Delta_X u = u|Xu|^2$ can be written in the form $u = \exp(ih)$, with $h \in W^{1,2}_X(G)$. Such a proof relies on a careful description of the singularities of the so–called subelliptic harmonic map equation associated with X. In this Note, after sketching the results of [12], we provide an alternative proof of this equivalence, that still holds in much more general situations.

2. Main results

Definition 2.1. If $1 \le p < \infty$, we can associate with the family X of vector fields $X = (X_1, X_2) = (\partial_1, x_1\partial_2)$ the function space

$$
W^{1,p}_X(G) = W^{1,p}_X(G, \mathbb{R}) := \{u \in L^p(\Omega) \, : \, X_1 u, X_2 u \in L^p(G)\}, \tag{6}
$$

endowed with its natural norm.

By [7], $W^{1,p}_X(G)$ is the completion of $\mathbf{C}^\infty(\bar{G})$ with respect to the $W^{1,p}_X(G)$-norm. As usual, we shall say that $u \in W^{1,p}_{X,\text{loc}}(G)$ if $\varphi u \in W^{1,p}_X(G)$ for all $\varphi \in \mathcal{D}(G)$, and we define the space $\overset{\circ}{W}{}^{1,p}_X(G)$ as the closure of $\mathbf{C}^\infty_0(G)$ in $W^{1,p}_X(G)$. The following Poincaré inequality is well known (see, e.g. [11], [12], [17], [10]).

Theorem 2.2. *If* $1 \le p < \infty$, *there exists* $C > 0$ *such that for any ball* $B = B(\bar{x}, r)$ *with respect to the Carnot–Carathéodory distance associated with* X

$$
\int_B |u - u_B|^p \, dx \ \le C \, r^p \int_B |Xu|^p \, dx \tag{7}
$$

for every $u \in W^{1,p}_X(B)$, *where* $u_B := \fint_B u \, dx$ *denotes the average of* u *on* B.

In particular, for any $p \in [1, \infty)$ there exists $C_{G,p} > 0$ such that

$$\left(\int_G |u|^p \, dx\right)^{1/p} \le C_{G,p} \left(\int_G |Xu|^p \, dx\right)^{1/p} \tag{8}$$

for any $u \in \overset{\circ}{W}{}^{1,p}_X(G)$.

From Poincaré inequality (7) the following compactness theorem follows as usual.

Theorem 2.3. *The space $\overset{\circ}{W}{}^{1,p}_X(G)$ is compactly embedded in $L^p(G)$. Therefore $W^{1,p}_X(G)$ is compactly embedded in $L^p_{\mathrm{loc}}(G)$*

In addition, if $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $W^{-1,p'}_X(G)$ will denote the dual space $(\overset{\circ}{W}{}^{1,p}_X(G))^*$. It is well known that

$$W^{-1,p'}_X(G) = \{\mathrm{div}_X f := X_1 f_1 + X_2 f_2 \text{ where } f = (f_1, f_2), f_i \in L^{p'}(G), i = 1, 2\}, \tag{9}$$

endowed with the usual norm.

More generally, if $k \in \mathbb{N}$, then we set

$$\begin{aligned} W^{k,p}_X(G) := \{u \in L^p(\Omega) \,:\, X_{i_1} \cdots X_{i_\ell} u \in L^p(G) \\ \text{for any choice of } i_1, \dots, i_\ell \in \{1, 2\} \text{ and for } 1 \le \ell \le k\}, \end{aligned} \tag{10}$$

endowed with its natural norm.

Later on, we shall use the function space

$$W^{1,p}_X(G, \mathbb{S}^1) = \{u : G \to \mathbb{S}^1, u = u_1 + i\, u_2, u_j \in W^{1,p}_X(G) \text{ for } j = 1, 2\},$$

where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$.

Finally, we shall denote by $Xu = (X_1 u, X_2 u)$ the intrinsic gradient associated with X, by $|Xu|$ its norm, and by Δ_X the sum-of-squares differential operator $\Delta_X = X_1^2 + X_2^2$ (the so-called subelliptic Laplacian associated with X). By [16], Δ_X is hypoelliptic, since the rank of the Lie algebra generated by X equals 2 at any point of $\mathbb{R}^2$.

By the Poincaré inequality (8), the quadratic form associated with Δ_X is coercive in $\overset{\circ}{W}{}^{1,2}_X(G)$, and hence the following well known result.

Theorem 2.4. *If $f \in W^{-1,2}_X(G)$, the Dirichlet problem*

$$\begin{cases} \Delta_X u = f & \text{in } G \\ u \equiv 0 & \text{in } \partial G, \end{cases} \tag{11}$$

has a unique solution in $\overset{\circ}{W}{}^{1,2}_X(G)$.

If ν is the outward unit normal, we denote by μ the measure on $\partial\Omega$ defined by

$$\mu = \left(\langle X_1, \nu\rangle^2 + (\langle X_2, \nu\rangle^2\right)^{1/2} \mathcal{H}^1 \llcorner \partial G, \tag{12}$$

where $\mathcal{H}^1 \llcorner \partial G$ is the 1-dimensional Hausdorff measure concentrated on ∂G. It is well known ([5]) that, due to the smoothness of ∂G, the measure μ coincides with the perimeter measure $|\partial G|_X$ (see [13], [15]). Thus, following [6], [19], if $p \geq 1$, $0 < s < 1$, and $u \in L^p(\partial G, d\mu)$, we denote by $[u]_{s,p}$ the Besov-type seminorm

$$[u]^p_{s,p} = \int_{\partial G} d\mu(x) \int_{\partial G} d\mu(y) \frac{|u(x) - u(y)|^p}{d(x,y)^{ps}\mu(B(x, d(x,y)))}. \tag{13}$$

Finally, we denote by $B^{s,p}_X(\partial G)$ the linear space of all $u \in L^p(\partial G, d\mu)$ such that $[u]_{s,p} < \infty$, endowed with its natural norm.

We denote by ∂G_X the set of characteristic points of ∂G with respect to X, i.e., if ν is the outward unit normal to ∂G, then

$$\partial G_X := \{x = (x_1, x_2), x_1 = 0, \nu(x) = (0, \pm 1)\}.$$

We assume that

Hypothesis 2.5 (Main geometric assumptions). We assume that $\partial G \cap \{(0, y) \ : \ y \in \mathbb{R}\} \neq \emptyset$. In addition, if $(0, q) \in \partial G \cap \{x_1 = 0\}$ is a characteristic point of ∂G with respect to X, then there exists a bounded open neighborhood $\mathcal{U}$ of $(0, q)$ such that $G \cap \mathcal{U} = \{(x_1, x_2) \in \mathcal{U}; x_2 < f(x_1)\}$, where $f'(x_1) = O(|x_1|)$ as $x_1 \to 0$.

For sake of simplicity, we assume also that we can write

$$\partial G = \bigcup_{j=1}^{m} \Gamma_j \cup \Gamma_0,$$

where, for $j = 1, \ldots, m$, Γ_j is a (connected) arc of ∂G intersecting the x_2-axis and that can be written as a graph. Moreover, $\operatorname{dist}(\Gamma_i, \Gamma_j) \geq \varepsilon_0 > 0$ for $i \neq j$, and $\operatorname{dist}(\Gamma_0, \{x_1 = 0\}) = \varepsilon_0 > 0$.

By Lemma 6 of [19], μ is a 1-Ahlfors measure in the sense of [6], since the domain G satisfies Hypothesis 2.5. More precisely, we have:

Lemma 2.6. *[*[19]*, Lemma 6] If Hypothesis 2.5 holds, then*

$$\mu(B(x,r)) \approx \frac{|B(x,r)|}{r}$$

for $x \in \bar{G}$ and $0 < r \leq r_0$.

Trace theorems on the boundary for the spaces $W^{1,p}_X(G)$ have been described in the last few years in [8], [6], [19]. The following result can be deduced from Theorem 11.9 of [6], keeping in mind

(1) the equality between the perimeter measure that appears therein and our measure μ (see e.g. [6], Remark 2.3.3);
(2) Lemma 2.6 above;
(3) that G is a (ε, δ)-domain ([20] and [18], Theorem 3.3.3).

Theorem 2.7. *[*[6]*, Theorem 11.9] If $p > 1$ there exist two bounded linear operators*

$$\gamma : W^{1,p}_X(G) \to B^{1-1/p,p}_X(\partial G), \quad \mathcal{R} : B^{1-1/p,p}_X(\partial G) \to W^{1,p}_X(G)$$

such that

(1) $\gamma(u) = u\big|_{\partial G}$ *when* $u \in \mathbf{C}^\infty(\bar{G})$*;*
(2) $\gamma \circ \mathcal{R}$ *is the identity on* $B^{1-1/p,p}_X(\partial G)$*.*

In the sequel, the following proposition will provide a key tool.

Theorem 2.8. *Let $Q := (0,q) \in \partial G$, and let $u \in \mathbf{C}(\partial G \setminus \{Q\}) \cap L^\infty(\partial G)$ be Lipschitz continuous on any open arc that is properly contained in $\partial G \setminus Q$. Then $u \in B_X^{1-1/p,p}(\partial G)$ for all $p \in (1,3)$.*

Definition 2.9. Let $G \subset \mathbb{R}^2$ be an open set satisfying the assumptions of Hypothesis 2.5. If $g \in \mathbf{C}^\infty(\partial G, \mathbb{S}^1)$, we put

$$W^{1,p}_{X,g}(G,\mathbb{S}^1) = \{u \in W^{1,p}_X(G,\mathbb{S}^1)\,;\, \gamma(u) = g\},$$

where $\gamma(u) = g$ must understood as an equality of traces in the sense of [19]. We shall see below (see Corollary 2.12) that, if $g \in \mathbf{C}^\infty(\partial G, \mathbb{S}^1)$, then $W^{1,p}_{X,g}(G,\mathbb{S}^1) \neq \emptyset$.

Proposition 2.10. *Il $u \in W^{1,p}_X(G)$, $1 < p < \infty$, and $\gamma(u) = 0$, then $u \in \overset{\circ}{W}{}^{1,p}_X(G)$.*

The following result is a keystone of [14].

Theorem 2.11. *Let $G \subset \mathbb{R}^2$ be an open set satisfying the assumptions of Hypothesis 2.5, and let $g \in \mathbf{C}^\infty(\partial G, \mathbb{S}^1)$. Then, if $1 < p < 3$,*

$$g = \gamma(\exp(i\,\varphi)) \quad \text{for at least one function} \quad \varphi \in W^{1,p}_X(G). \tag{14}$$

Corollary 2.12. *With the assumptions of Theorem 2.11, if $g \in \mathbf{C}^\infty(\partial G, \mathbb{S}^1)$, then $W^{1,p}_{X,g}(G,\mathbb{S}^1) \neq \emptyset$.*

Theorem 2.13. *Let $G \subset \mathbb{R}^2$ be an open set satisfying the assumptions of Hypothesis 2.5, and let $g \in \mathbf{C}^\infty(\partial G, \mathbb{S}^1)$. Then*

i) *there exists $u_0 \in W^{1,2}_{X,g}(G,\mathbb{S}^1)$ that solves the minimum problem*

$$\min_{u \in W^{1,2}_{X,g}(G,\mathbb{S}^1)} \int_G |Xu|^2\,dx.$$

ii) *The minimizer u_0 is a weak solution of the nonlinear Dirichlet problem*

$$\begin{cases} -\Delta_X u_0 = u_0 |Xu_0|^2 & \text{in } G \\ |u_0| \equiv 1 & \text{in } G \\ u_0 = g & \text{in } \partial G. \end{cases} \tag{15}$$

iii) *The harmonic map equation (15) has a unique solution that is smooth in G. In particular, the minimizer of i) is unique.*

Theorem 2.14. *Let $G \subset \mathbb{R}^2$ be an open set satisfying the assumptions of Hypothesis 2.5, and let $g \in C^\infty(\partial G, \mathbb{S}^1)$. For $\varepsilon > 0$ consider the functionals*

$$E_\varepsilon(u) = \frac{1}{2}\int_G |Xu|^2 dx + \frac{1}{4\varepsilon^2}(|u_\varepsilon|^2 - 1)^2 dx. \tag{16}$$

Let u_ε be the minimizer of E_ε in $W^{1,2}_{X,g}(G)$ for $\varepsilon > 0$, and let u_0 be defined in Theorem 2.13. Then

$$u_\varepsilon \to u_0 \text{ as } \varepsilon \to 0$$

strongly in $W^{1,2}_X(G)$.

As anticipated in the Introduction, in spite of the lack of lifting in $W^{1,2}_X(G,\mathbb{S}^1)$, in [14] we were still able to carry out the proof of the equivalence between the (degenerate) harmonic map equation and a linear equation, by showing that *weak solutions* $u \in W^{1,2}_X(G,\mathbb{S}^1)$ *of* $-\Delta_X u = u|Xu|^2$ can be written in the form $u = \exp(ih)$, with $h \in W^{1,2}_X(G)$.

More precisely, the following lifting result holds.

Theorem 2.15. *If* $u_0 \in W^{1,2}_X(G,\mathbb{S}^1)$ *is a weak solution of (15), then there exists* $h \in W^{1,2}_X(G,\mathbb{R})$ *such that*

$$u_0 = \exp ih, \tag{17}$$

and

$$Xu_0 = iu_0 Xh. \tag{18}$$

Theorem 2.16. *Let the assumptions of Theorem 2.13 hold. Then the function* $h \in W^{1,2}_X(G,\mathbb{R})$ *given by Theorem 2.15, that is defined up to an integer multiple of* $2\pi i$*, can be chosen in order to solve the Dirichlet problem*

$$\begin{cases} \Delta_X h = 0 & \text{in } G \\ \gamma(h) = \gamma(\varphi) & \text{in } \partial G, \end{cases} \tag{19}$$

where φ *has been defined in Theorem 2.11.*

Proof. Let h be as in Theorem 2.15. We have

$$\Delta_X h = X_1 f_1 + X_2 g_2 = 0.$$

Thus, we have but to show that $\gamma(h) - \gamma(\varphi) = 2k\pi$ for some $k \in \mathbb{Z}$. We have

$$\begin{aligned} \exp(-i\gamma(\varphi))\exp(i\gamma(h)) &= \exp(i(\gamma(h) - \gamma(\varphi))) = \exp(i\gamma(h-\varphi)) \\ &= \gamma(\exp(ih - i\varphi)) = \gamma(\exp(ih))\gamma(\exp(-i\varphi)) \\ &= \gamma(u_0)g^{-1} = 1, \end{aligned}$$

and the proof is complete. □

Corollary 2.17. *Let* $u_0 \in W^{1,2}_X(G,\mathbb{S}^1)$ *solve the Dirichlet problem (15). Then*

(1) *the solution is unique;*
(2) u_0 *is smooth in* G.

Proof. The second assertion follows from the hypoellipticity of Δ_X ([16]) and Theorem 2.16. Suppose now $u_1 \in W^{1,2}_X(G,\mathbb{S}^1)$ is another solution of (15), and let h_1 be obtained from u_1 as in Theorem 2.15, chosen in order to satisfy (19). If we set $\tilde h := h - h_1$, we obtain that $\gamma(\tilde h) = g - g = 0$, and hence $\tilde h \in \overset{\circ}{W}{}^{1,2}_X(G)$, by Proposition 2.10. Since $\Delta_X \tilde h = 0$, then $h = 0$, by Theorem 2.4. Thus, the assertion is proved. □

Remark 2.18. Since Δ_X is elliptic outside of an arbitrary open neighborhood $\mathcal{V}$ of the axis $\{x_1 = 0\}$, it follows trivially by classical Schauder theory that h (and hence u_0) are smooth up to boundary outside of $\mathcal{V}$. It also evident that we cannot expect continuity up to the boundary for a general Dirichlet datum g, by topological obstructions. On the other hand, suppose the topological degree of g is zero, so that we can choose φ smooth up to the boundary. By (19) and Proposition 2.10, the function $\theta := h - \varphi$ is a weak solution in $\overset{\circ}{W}{}^{1,2}_X(G)$ of the Dirichlet problem

$$\begin{cases} \Delta_X \theta = -\Delta\varphi & \text{in } G \\ \gamma(\theta) = 0 & \text{in } \partial G, \end{cases} \tag{20}$$

Thanks to the smoothness of the right hand side of the equation in (20), we can apply regularity results up to the boundary in domains satisfying condition (S) ([12]) to conclude that θ (and hence h and eventually u_0) is Hölder continuous up to the boundary on all G.

3. The lifting problem in $W^{1,p}_X(G, \mathbb{S}^1)$

In this section, we provide a complete solution of the lifting problem in $W^{1,p}_X(G, \mathbb{S}^1)$. It turns out that the threshold separating the range of p's in $[1, \infty)$ for which a lifting theorem either holds or fails to hold is $p = 3$. As we already stressed, this critical value differs from the critical value $p = 2$ of the usual elliptic Sobolev spaces (see e.g. [3], [4]), reflecting the well know fact that the *homogeneous dimension* of $\mathbb{R}^2$ endowed with the Carnot–Carathédory metric associated with X_1, X_2 equals 3. In fact, this last statement is not fully correct, since the dimension of this Carnot–Carathédory metric is not constant (it equals 3 on the x_2-axis and is 2 away from it), because there is no underlying group structure making X_1, X_2 left invariant.

Theorem 3.1. *If $u \in W^{1,p}_X(G, \mathbb{S}^1)$ with $p \geq 3$, then there exists $\tilde{\phi} \in W^{1,p}_X(G)$ such that*

$$u = \exp(i\tilde{\phi}). \tag{21}$$

Theorem 3.2. *If $1 \leq p < 3$, then there exists $u \in W^{1,p}_X(G, \mathbb{S}^1)$ such that the equation*

$$u = \exp(ih). \tag{22}$$

has no solutions $h \in W^{1,p}_X(G)$.

Proof. Clearly, we can carry on our arguments in a neighborhood of a point of $G \cap \{x_1 = 0\}$, and hence, without loss of generality, we can assume G is the unit ball $B(0, 1)$ for the distance d. Moreover, let us remind that

$$d(x, 0)^\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \quad \text{if and only if} \quad \alpha > -3. \tag{23}$$

It is enough to choose

$$u(x) = (u_1(x), u_2(x)) := \frac{1}{(x_1^4 + x_2^2)^{1/2}} \big(x_1|x_1|, x_2\big)$$

for $x \in G$. $\square$

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