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INTRINSIC LIPSCHITZ GRAPHS IN HEISENBERG GROUPS

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1. Introduction

In the last few years there have been a fairly large amount of work dedicated to the study of intrinsic submanifolds - of various dimension and codimension - inside the Heisenberg groups $\mathbb{H}^n$ or more general Carnot groups. For example intrinsically C^1 surfaces, rectifiable sets, finite perimeter sets, various notions of convex surfaces have been studied. Here and in what follows, *intrinsic* will denote properties defined only in terms of the group structure of $\mathbb{H}^n$ or, equivalently, of its Lie algebra $\mathfrak{h}$.

We postpone complete definitions of $\mathbb{H}^n$ to the next section. Here we remind that $\mathbb{H}^n$, with group operation $\cdot$, is a (connected and simply connected) Lie group identified through exponential coordinates with $\mathbb{R}^{2n+1}$. If $\mathfrak{h}$ denotes the Lie algebra of all left invariant vector fields on $\mathbb{H}^n$, then $\mathfrak{h}$ admits the stratification $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$; $\mathfrak{h}_1$ is called horizontal layer. The horizontal layer defines, by left translation, the horizontal fiber bundle $H\mathbb{H}^n$. Since $H\mathbb{H}^n$ depends only on the stratification of $\mathfrak{h}$, we call 'intrinsic' any notion depending only on $H\mathbb{H}^n$. The stratification of $\mathfrak{h}$ induces, through the exponential map, a family of anisotropic dilations δ_λ for $\lambda > 0$. We refer to δ_λ as intrinsic dilations. A privileged role in the geometry of $\mathbb{H}^n$ is played by *horizontal curves*, i.e. curves tangent at any point to the fiber of $H\mathbb{H}^n$ at that point.

We recall the notions of Carnot-Carathéodory distance and Hausdorff measures in $\mathbb{H}^n$. Once a scalar product is defined in $\mathfrak{h}$, each fiber of the horizontal bundle over a generic point p is consequently endowed with a scalar product $\langle \cdot, \cdot \rangle_p$. We denote also by $|\cdot|_p$ the associated norm. Thus, we can define the (sub-Riemannian) length of a horizontal curve $\gamma : [0, T] \to \mathbb{H}^n$ as $\int_0^T |\gamma'(t)|_{\gamma(t)}\, dt$. Given $p, q \in \mathbb{H}^n$, their *Carnot-Carathéodory distance* $d_c(p, q)$ is the minimal length of horizontal curves connecting p and q.

Intrinsic s-dimensional Hausdorff measures $\mathcal{H}_c^s$ and $\mathcal{S}_c^s$, $s \geq 0$, are obtained from d_c, following Carathéodory construction as in Federer's book [6]. The intrinsic metric (or Hausdorff) dimension $\dim_{\mathbb{H}}(S)$ of a set S is the number $\dim_{\mathbb{H}}(S) \overset{\text{def}}{=} \inf\{s \geq 0 \,:\, \mathcal{H}^s(S) = 0\}$.

Heisenberg groups provide the simplest non-trivial examples of nilpotent stratified, connected and simply connected Lie groups (Carnot groups in most of the recent literature).

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We begin to recall some of the main definitions of intrinsic submanifolds: $\mathbb{H}$-regular submanifolds, $\mathbb{H}$-rectifiable submanifolds, finite perimeter sets.

Intrinsic C^1 surfaces - or $\mathbb{H}$-regular surfaces - were first defined for codimension one, in [8] and later, for general dimensions and codimensions in [11]. The definition is the following one, if $1 \le k \le n$,

k-dimensional $\mathbb{H}$-regular surfaces of $\mathbb{H}^n$ are images of continuously Pansu differentiable functions $\mathcal{V} \to \mathbb{H}^n$, $\mathcal{V}$ open in $\mathbb{R}^k$, with differentials of maximal rank, hence injective;

k-codimensional $\mathbb{H}$-regular surfaces of $\mathbb{H}^n$ are level sets of continuosly Pansu differentiable functions $\mathcal{U} \to \mathbb{R}^k$, $\mathcal{U}$ open in $\mathbb{H}^n$, with Pansu differential of maximal rank, hence surjective.

Notice that no nontrivial geometric object falls under the scope of both definitions. Indeed, for $k > n$, there is no k-dimensional subgroup of the horizontal fibre; hence surfaces having as a tangent space a subgroup of the horizontal fibre are limited to have dimension $\le n$ and, dually, the ones with an horizontal normal space are limited to have codimension $\le n$.

The two families of low dimensional and low codimensional $\mathbb{H}$-regular surfaces contain very different objects. We recall here some of their properties as proved in [11].

k-dimensional $\mathbb{H}$-regular surfaces are Euclidean submanifolds of $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$. For $k = 1$, they are horizontal curves and for $k \le n$ they are submanifolds of Legendrian manifolds. They have topological dimension = metric dimension = Euclidean dimension = k. Locally they have finite $\mathcal{S}_c^k$ measure. Their intrinsic tangent k-planes coincide with their Euclidean tangent k-planes (both are cosets of subgroups of $\mathbb{H}^n$ contained in the horizontal fibre).

Low codimensional $\mathbb{H}$-regular surfaces, on the contrary, can be very irregular and in general these surfaces are not Euclidean $\mathcal{C}^1$ submanifolds, not even locally (see [15]). Nevertheless it can be proved that they have metric dimension $= 2n + 2 - k$, and topological dimension $= 2n + 1 - k$. Locally they have finite $\mathcal{S}_c^{2n+2-k}$ measure. At each point there is a, continuosly varying, intrinsic tangent $(2n + 1 - k)$-plane that is a coset of a subgroup of $\mathbb{H}^n$.

Intrinsic rectifiable sets are defined as countable unions of compact subsets of $\mathbb{H}$-regular surfaces (see [8] and [11]). Precisely, if $1 \le k \le n$, we say that M is a k-dimensional $\mathbb{H}$-rectifiable set if $\mathcal{S}_c^k(M) < \infty$ and $\mathcal{S}_c^k$ almost all of M is contained in the countable union of k-dimensional $\mathbb{H}$-regular surfaces. Analogously, we say that M is a k-codimensional $\mathbb{H}$-rectifiable set - or a $(2n + 2 - k)$-dimensional $\mathbb{H}$-rectifiable set - if $\mathcal{S}_c^{2n+2-k}(M) < \infty$ and $\mathcal{S}_c^{2n+2-k}$ almost all of M is contained in the countable union of k-codimensional $\mathbb{H}$-regular surfaces.

Sets with locally finite $\mathbb{H}$-perimeter or - following De Giorgi - $\mathbb{H}$-Caccioppoli sets were first defined in [13]. Notice that there are several ways of defining intrinsic bounded variation functions and finite perimeter sets in $\mathbb{H}^n$ or in much more general settings. These definitions have been proposed independently by different authors (see [3], [13], [7]) and are in fact equivalent, see [7]).

We say, following [13], that $E \subset \mathbb{H}^n$ has locally finite $\mathbb{H}$-perimeter if for any bounded open set $\Omega \subseteq \mathbb{H}^n$

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup \left\{ \int_E \sum_{j=1}^{n} X_j\phi(p) + Y_j\phi(p) \, d\mathcal{L}_p^{2n+1} \right\} < \infty,$$

where the supremum is taken over all $\phi \in C_0^1(\Omega, H\mathbb{H}^n)$, such that $|\phi(p)|_p \leq 1$. In such a way, $|\partial E|_{\mathbb{H}}$ is a Radon measure in $\mathbb{H}^n$.

Riesz' representation theorem yields the existence of a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $H\mathbb{H}^n$, the generalized inward normal. Then, following De Giorgi (see [5]), we define the $\mathbb{H}$-*reduced boundary* $\partial_* E$ saying that $p \in \partial_* E$ if $|\partial E|_{\mathbb{H}}(B(p,r)) > 0$ for any metric ball $B(p,r)$ and if

$$\left| \lim_{r \to 0} \frac{1}{|\partial E|_{\mathbb{H}}(B(p,r))} \int_{B(P,r)} \nu_E d|\partial E|_{\mathbb{H}} \right| = 1.$$

One of the main results in [8] (see also [10]) states that the reduced boundary of finite perimeter sets is a 1-codimensional $\mathbb{H}$-rectifiable set. This theorem, beyond extending the classical result to Heisenberg groups setting, is a strong support in favour of the previously given definitions of $\mathbb{H}$-regular surfaces and of $\mathbb{H}$-rectifiable sets.

Finally we particularly want to stress two important features:

(i): all these classes of sets and surfaces are *invariant* with respect to *group translations* or *group dilations* of $\mathbb{H}^n$. Precisely , if S is - say - the boundary of a finite perimeter set E, then also any left translated surface $\tau_q S$ is again the boundary of a finite perimeter. The same can be said if S is a $\mathbb{H}$-regular surfaces and, consequently, if S is a $\mathbb{H}$-rectifiable sets .

(ii): the implicit function theorems proved in [8] and [11] yield that $\mathbb{H}$-regular surfaces are, locally, *intrinsic graphs.* By this we mean - see Definition 2.4 - that there are subgroups $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{H}^n$ such that $\mathbb{V} \cap \mathbb{W} = e$, $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ - in short $\mathbb{H}^n$ is the semidirect product of $\mathbb{V}$ and $\mathbb{W}$ - and there is $\varphi : \mathbb{W} \to \mathbb{V}$ such that $S = \text{graph}(\varphi)$, that is $S = \{w \cdot \varphi(w) : w \in E \subset \mathbb{W}\}$.

Then, it is a natural problem to try to understand the classes of functions, acting between subgroups of a given Carnot group such that their graphs are $\mathbb{H}$-regular surfaces or $\mathbb{H}$-rectifiable surfaces. This problem has been addressed for the first time in [1] where the authors characterize real valued functions on 1-codimensional subgroups of $\mathbb{H}^n$ such that their intrinsic graphs are $\mathbb{H}$-regular 1-codimensional surfaces.

Notice that, following from (i), the defining properties of these classes of functions have to be invariant when the graphs of the functions are group translated or dilated. This fact gives origin - quite naturally - to apparently strange definitions of intrinsically Lipschitz functions or of intrinsically differentiable functions, in our notation $\mathbb{H}^n$-Lipschitz functions or $\mathbb{H}^n$-differentiable functions - see Definition 3.1 and Definition 4.4. These notions are different from the usual ones and can be seen to reduce to the usual ones only in some very special situations.

We limit ourselves in this note to study $\mathbb{H}^n$-Lipschitz functions acting between subgroups of $\mathbb{H}^n$. One of our aims is convincing the reader that they are very natural objects inside $\mathbb{H}^n$, enjoying a number of very natural properties: they can be defined equivalently by metric properties, boundedness of intrinsic difference quotients or existence of parallel cones non intersecting their graphs; their graphs have locally finite intrinsic Hausdorff measure and finally when they are 1-codimensional they are boundary of sets with locally finite $\mathbb{H}$-perimeter.

2. Notations and definitions

2.1. **Heisenberg groups.** For a general review on Heisenberg groups and their properties we refer to [21], [14] and to [22]. We limit ourselves to fix some notations.

$\mathbb{H}^n$ is the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p', p_{2n+1})$, with $p' \in \mathbb{R}^{2n}$ and $p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$p \cdot q = (p' + q', p_{2n+1} + q_{2n+1} - \frac{1}{2}\langle Jp', q'\rangle_{\mathbb{R}^{2n}})$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $2n \times 2n$ symplectic matrix. We denote as $p^{-1} := (-p', -p_{2n+1})$ the inverse of p and as e the identity of $\mathbb{H}^n$.

For any fixed $q \in \mathbb{H}^n$ and for any $r > 0$ left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are automorphisms of the group defined as

$$\tau_q(p) := q \cdot p \qquad \text{and as} \qquad \delta_r p := (rp', r^2 p_{2n+1}).$$

We denote as $\mathfrak{h}$ the Lie algebra of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i - \frac{1}{2}(Jp')_i \partial_{2n+1}, \quad Y_i := \partial_{i+n} + \frac{1}{2}(Jp')_{i+n}\partial_{2n+1}, \quad T := \partial_{2n+1}.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

If $p \in \mathbb{H}^n$ we indicate

$$\|p\| := d_\infty(p, e) := \max\{\|(p_1, \cdots, p_{2n})\|_{\mathbb{R}^{2n}}, |p_{2n+1}|^{1/2}\}$$

and

$$d_\infty(p, q) = d_\infty(q^{-1} \cdot p, e) = \left\|q^{-1} \cdot p\right\|.$$

It is well known that d_∞ is equivalent with the Carnot-Caratheodory distance of $\mathbb{H}^n$, moreover

$$d_\infty(z \cdot x, z \cdot y) = d_\infty(x, y) \qquad d_\infty(\delta_\lambda x, \delta_\lambda y) = \lambda d_\infty(x, y)$$

for $x, y, z \in \mathbb{H}^n$ and $\lambda > 0$. We denote by $U(p, r)$ and by $B(p, r)$ the open and the closed ball associated with d_∞.

Definition 2.1. $\mathbb{H}^n$ is the *semidirect product* of the homogeneous subgroups $\mathbb{W}$ and $\mathbb{V}$ and we wright

$$\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$$

if $\mathbb{W} := \exp \mathfrak{w}$, $\mathbb{V} := \exp \mathfrak{v}$, $\mathfrak{w}$ and $\mathfrak{v}$ are homogeneous subalgebras of $\mathfrak{h}$ (see [21] 5.2.4) such that

(i): $\mathfrak{h} = \mathfrak{w} \oplus \mathfrak{v}$;
(ii): $\mathfrak{w} \supset \mathfrak{h}_2$ or, equivalently, $\mathfrak{w}$ is an ideal in $\mathfrak{h}$;

Clearly $\mathbb{W} \cap \mathbb{V} = \{e\}$, moreover (ii) is equivalent to saying that $\mathbb{W}$ is a normal subgroup of $\mathbb{H}^n$.

Notice also that $\mathfrak{v} \subset \mathfrak{h}_1$, indeed $T \notin \mathfrak{v}$ because $T \in \mathfrak{w}$, moreover if $T + V \in \mathfrak{v}$ for some $V \in \mathfrak{v}$, then both $\lambda T + \lambda V \in \mathfrak{v}$ and $\lambda V + \lambda^2 T \in \mathfrak{v}$ yielding that $T \in \mathfrak{v}$. Because $\mathfrak{v}$ is a subalgebra of $\mathfrak{h}_1$ it follows that the linear dimension of $\mathfrak{v}$ is $\leq n$, that $\mathfrak{v}$ is a commutative algebra and consequently that $\mathbb{V} \simeq \mathbb{R}^k$ if $k = \dim \mathfrak{v}$.

Each element $p \in \mathbb{H}^n$ can be written in a (unique) way as $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}$, with $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{V}} \in \mathbb{V}$.

Proposition 2.2. *If $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, each $q \in \mathbb{H}^n$ has unique 'components' $q_{\mathbb{W}} \in \mathbb{W}$, $q_{\mathbb{V}} \in \mathbb{V}$, such that $q = q_{\mathbb{W}} \cdot q_{\mathbb{V}}$. The maps*

$$q \to q_{\mathbb{V}} \text{ and } q \to q_{\mathbb{W}}$$

are continuous and there is a constant $c = c(\mathbb{V}, \mathbb{W}) > 0$ such that

$$c\left(\|q_{\mathbb{V}}\| + \|q_{\mathbb{W}}\|\right) \leq \|q\| \leq \left(\|q_{\mathbb{V}}\| + \|q_{\mathbb{W}}\|\right). \tag{1}$$

Moreover,

$$\begin{aligned} (q^{-1})_{\mathbb{V}} &= (q_{\mathbb{V}})^{-1} \quad \text{and} \quad (q^{-1})_{\mathbb{W}} = q_{\mathbb{V}}^{-1} \cdot (q_{\mathbb{W}})^{-1} \cdot q_{\mathbb{V}} \\ (p \cdot q)_{\mathbb{V}} &= p_{\mathbb{V}} \cdot q_{\mathbb{V}} \quad \text{and} \quad (p \cdot q)_{\mathbb{W}} = p_{\mathbb{W}} \cdot p_{\mathbb{V}} \cdot q_{\mathbb{W}} \cdot p_{\mathbb{V}}^{-1}. \end{aligned} \tag{2}$$

The norm and distance in $\mathbb{W}$ or in $\mathbb{V}$ are the restrictions to $\mathbb{W}$ and to $\mathbb{V}$ of $\|\cdot\|$ and d_∞.

Remark 2.3. The component map

$$\mathbb{H}^n \to \mathbb{W} : \qquad p \mapsto p_{\mathbb{W}}$$

is *not a Lipschitz map* with respect to the previously indicated norms.

If $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$, we denote *system of coordinate planes* the double family $\mathcal{L}_{\mathbb{V}}$ and $\mathcal{L}_{\mathbb{W}}$ of cosets of $\mathbb{V}$ and $\mathbb{W}$, that is

$$\mathcal{L}_{\mathbb{V}}(p) := p \cdot \mathbb{V}, \quad \forall p \in \mathbb{W} \quad \text{and} \quad \mathcal{L}_{\mathbb{W}}(q) := q \cdot \mathbb{W}, \quad \forall q \in \mathbb{V}.$$

Each $p \in \mathbb{H}^n$ belongs exactly to one leaf in $\mathcal{L}_{\mathbb{V}}$ and to one in $\mathcal{L}_{\mathbb{W}}$; the leaves in $\mathcal{L}_{\mathbb{V}}$ (or in $\mathcal{L}_{\mathbb{W}}$) are invariant by translations, that is $x \in \mathcal{L}_{\mathbb{V}}(p) \implies \tau_x \mathcal{L}_{\mathbb{V}}(p) = \mathcal{L}_{\mathbb{V}}(p)$.

For a nonnegative integer k, $\mathcal{L}^k$ denotes the k-dimensional Lebesgue measure. $\mathcal{L}^{2n+1}$ is the bi-invariant Haar measure of $\mathbb{H}^n$, hence, if $E \subset \mathbb{R}^{2n+1}$ is measurable, then $\mathcal{L}^{2n+1}(\tau_p(E)) = \mathcal{L}^{2n+1}(E)$ for all $p \in \mathbb{H}^n$. Moreover, if $\lambda > 0$ then $\mathcal{L}^{2n+1}(\delta_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E)$. We explicitly observe that, $\forall p \in \mathbb{H}^n$ and $\forall r > 0$,

$$\mathcal{L}^{2n+1}(B(p,r)) = r^{2n+2}\mathcal{L}^{2n+1}(B(p,1)) = r^{2n+2}\mathcal{L}^{2n+1}(B(0,1)).$$

Notice also that, if ω_k is the $\mathcal{L}^k$ measure of the unit Euclidean ball in $\mathbb{R}^k$, then $\mathcal{L}^{2n+1}\left(B(e,r)\right) = 2\omega_{2n} r^{2n+2}$ and, if $k := \dim \mathfrak{v} \leq n$,

$$\begin{aligned} &\mathcal{L}^k\left(B(e,r) \cap \mathbb{V}\right) = \omega_k r^k; \\ &\mathcal{L}^{2n+1-k}\left(B(e,r) \cap \mathbb{W}\right) = 2\omega_{2n-k} r^{2n+2-k}. \end{aligned} \tag{3}$$

Related with the distance d_∞, Hausdorff measures are obtained following Carathédory's construction as in [6] Section 2.10.2. For $m \geq 0$, we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measures in $\mathbb{H}^n$, obtained from the distances d_∞. Analogously, $\mathcal{S}^m$ denotes the spherical Hausdorff measure. We have to be more precise about the constants appearing in the definitions. Since explicit computations will be carried out only for m a positive integer, we limit ourselves to this case. For each $A \subset \mathbb{H}^n$ and $\delta > 0$, $\mathcal{H}^m(A) := \lim_{\delta\to 0} \mathcal{H}^m_\delta(A)$, where

$$\mathcal{H}^m_\delta(A) = \inf\left\{\sum_i \zeta(C_i) :\ A \subset \bigcup_i C_i, C_i \text{ closed, } \operatorname{diam}(C_i) \leq \delta\right\}$$

and the evaluation function ζ is

$$\zeta(C) := \begin{cases} \omega_m 2^{-m}\operatorname{diam}(C)^m & \text{if } 1 \leq m \leq n, \\ \omega_{m-1}2^{-m+1}\operatorname{diam}(C)^m & \text{if } m = n+1, \\ \omega_{m-2}2^{-m+1}\operatorname{diam}(C)^m & \text{if } n+2 \leq m. \end{cases} \tag{4}$$

We notice that, due to the lack of an optimal isodiametric inequality in $\mathbb{H}^n$, it is not known if, in general, $\mathcal{H}^m(E) = \mathcal{S}^m(E)$ even for 'nice' subsets of $\mathbb{H}^n$ and for $m = Q$. Related to this point see the recent paper [19] by Severine Rigot.

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (2.1) and we have

$$\mathcal{H}^m(\tau_p A) = \mathcal{H}^m(A) \quad \text{and} \quad \mathcal{H}^m(\delta_r A) = r^m \mathcal{H}^m(A), \tag{5}$$

for $A \subseteq \mathbb{H}^n$, $p \in \mathbb{H}^n$ and $m,r \in [0,\infty)$.

Because the topologies induced by d_∞ and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n+1$. On the contrary the metric dimension (or Hausdorff dimension) of $\mathbb{H}^n$, with respect to d_∞ is $2n+2$.

If $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ as in Definition 2.1, then if k, $1 \leq k \leq n$, is the linear dimension of $\mathfrak{v}$ then the metric dimension of $\mathbb{V}$ is k while the metric dimension of $\mathbb{W}$ is $2n+2-k$. Hence

$$\dim \mathbb{H}^n = \dim \mathbb{V} + \dim \mathbb{W}.$$

2.2. **Graphs.** We assume that $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ is the semidirect product of $\mathbb{W}$ and $\mathbb{V}$.

Definition 2.4. We say that a set $S \subset \mathbb{H}^n$ is a *graph over* $\mathbb{W}$ *along* $\mathbb{V}$ if, for each $\xi \in \mathbb{W}$, $S \cap \mathcal{L}_{\mathbb{V}}(\xi)$ contains at most one point. Equivalently if there is a function $\varphi : E \subset \mathbb{W} \to \mathbb{V}$ such that

$$S = \{w\cdot\varphi(w) : w \in E\}$$

and we say that S *is the graph of* φ, $S = \operatorname{graph}(\varphi)$. Graphs over $\mathbb{V}$ along $\mathbb{W}$ are defined symmetrically, $S = \operatorname{graph}(\psi)$, with $\psi : F \subset \mathbb{V} \to \mathbb{W}$, if

$$S = \{v\cdot\psi(v) : v \in F\}.$$

Observe that the notions of intrinsic graph and of euclidean graph are different ones.

Example 2.5. Let $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$, with $\mathbb{V} = \{x = (x_1, 0, 0)\}$ and $\mathbb{W} = \{w = (0, w_2, w_3)\}$. For $1/2 < \alpha < 1$ let $\varphi : \mathbb{W} \to \mathbb{V}$ be defined as

$$\varphi(0, w_2, w_3) = (|w_3|^\alpha, 0, 0).$$

Then

$$\operatorname{graph}(\varphi) = \{w \cdot \varphi(w) : w \in \mathbb{W}\} = \left\{ \left(|w_3|^\alpha, w_2, w_3 - \frac{1}{2} w_2 |w_3|^\alpha \right) \right\}.$$

It is easy to convince oneself, looking at the sections of $\operatorname{graph}(\varphi)$ in Figure 2, that $\operatorname{graph}(\varphi)$ is not an Euclidean graph in any neighborhood of the origin.

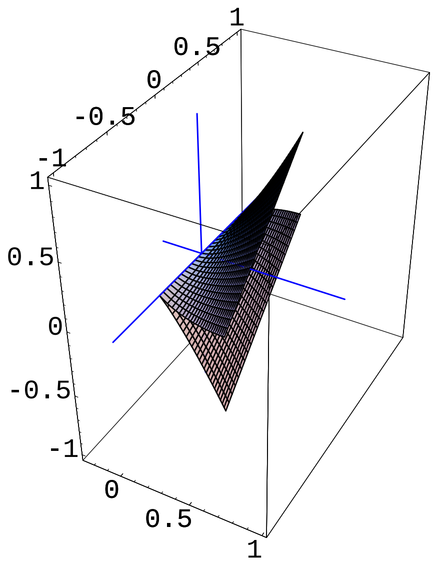

FIGURE 1. The surface $\operatorname{graph}(\varphi) \subset \mathbb{H}^1$ of Example 2.5 when $\alpha = 2/3$

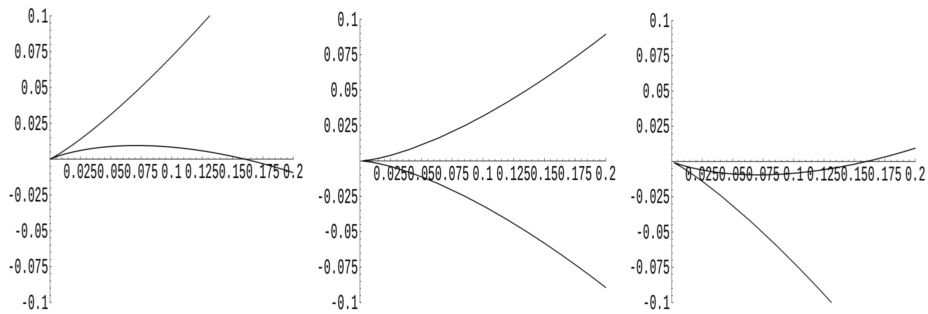

FIGURE 2. Sections of $\operatorname{graph}(\varphi)$ for $x = -.2$, $x = 0$ and $x = .2$

Notice that the bounds on α yield that $\operatorname{graph}(\varphi)$ is a $\mathbb{H}$-regular surface of $\mathbb{H}^1$. For a proof of this fact see Corollary 5.11 of [1].

On the other side, no relatively open neighborhood of the origin in $S := \{(x, y, 0) : x, y \in \mathbb{R}\} \subset \mathbb{H}^1$ is an intrinsic graph while it is an Euclidean graph. More generally no C^2 Euclidean hypersurface of $\mathbb{R}^{2n+1} \equiv \mathbb{H}^n$ is an intrinsic graph in any relatively open neighborhood of a characteristic point of the hypersurface.

A trivial but key feature of so defined graphs is their *invariance with respect to dilations and translations*. That is, if S is a graph (say from $\mathbb{W}$ to $\mathbb{V}$) then also $\delta_\lambda S$ and $\tau_p S$ are graphs from $\mathbb{W}$ to $\mathbb{V}$ - obviously of different functions - and it is possible to write explicitly the analytic form of these new functions.

Proposition 2.6. *Let $S = \{\xi \cdot \varphi(\xi)\}$ with $\varphi : E \subset \mathbb{W} \to \mathbb{V}$. Then the dilated set $\delta_\lambda S$ is the graph of $\varphi_\lambda : \delta_\lambda E \subset \mathbb{W} \to \mathbb{V}$, precisely*

$$\delta_\lambda S = graph(\varphi_\lambda) \quad \textit{with } \varphi_\lambda := \delta_\lambda \circ \varphi \circ \delta_{1/\lambda} : \delta_\lambda E \to \mathbb{V}.$$

The same statement holds interchanging $\mathbb{V}$ and $\mathbb{W}$.

Proof. Trivial: $\delta_\lambda S = \delta_\lambda(\xi \cdot \varphi(\xi)) = \delta_\lambda \xi \cdot \delta_\lambda(\varphi(\xi)) = \delta_\lambda \xi \cdot \varphi_\lambda(\delta_\lambda \xi)$. $\square$

Notice that in the preceding proposition, there is no assumption on $\mathbb{W}$ and $\mathbb{V}$. On the contrary, for translations of graphs, we have to distinguish between graphs on $\mathbb{W}$ and graphs on $\mathbb{V}$. Precisely we have

Proposition 2.7. *Let $S = \{\xi \cdot \varphi(\xi)\}$ be a graph and let $q = q_\mathbb{W} \cdot q_\mathbb{V} \in \mathbb{H}^n$. Then the translated set $\tau_q S$ is again a graph. Precisely*

(i) *If S is a graph over $\mathbb{W}$, that is $\varphi : E \subset \mathbb{W} \to \mathbb{V}$, then $\tau_q S = \{\eta \cdot \varphi_q(\eta) : \eta \in E' := q \cdot E \cdot (q_\mathbb{V})^{-1} \subset \mathbb{W}\}$, where*

$$\varphi_q(\eta) = q_\mathbb{V} \cdot \varphi(q_\mathbb{V}^{-1} \cdot q_\mathbb{W}^{-1} \cdot \eta \cdot q_\mathbb{V}), \qquad \varphi_q : E' \to \mathbb{V};$$

(ii) *If S is a graph over $\mathbb{V}$, that is $\varphi : F \subset \mathbb{V} \to \mathbb{W}$, then $\tau_q S = \{\eta \cdot \varphi_q(\eta) : \eta \in F' := q_\mathbb{V} \cdot F \subset \mathbb{V}\}$, where*

$$\varphi_q(\eta) = \eta^{-1} \cdot q_\mathbb{W} \cdot \eta \cdot \varphi(q_\mathbb{V}^{-1} \cdot \eta), \qquad \varphi_q : F' \to \mathbb{W}.$$

Proof. First case: because $\mathbb{W}$ is a normal subgroup of $\mathbb{G}$ then $E' = q_\mathbb{W} \cdot q_\mathbb{V} \cdot E \cdot q_\mathbb{V}^{-1} \subset \mathbb{W}$. Then $\tau_q S = \{q \cdot \xi \cdot \varphi(\xi) : \xi \in \mathbb{W}\}$ and

$$q \cdot \xi \cdot \varphi(\xi) = q_\mathbb{W} \cdot q_\mathbb{V} \cdot \xi \cdot \varphi(\xi) = q_\mathbb{W} \cdot q_\mathbb{V} \cdot \xi \cdot q_\mathbb{V}^{-1} \cdot q_\mathbb{V} \cdot \varphi(\xi).$$

Observe that $q_\mathbb{W} \cdot q_\mathbb{V} \cdot \xi \cdot q_\mathbb{V}^{-1} \in \mathbb{W}$ then set $\eta := q_\mathbb{W} \cdot q_\mathbb{V} \cdot \xi \cdot q_\mathbb{V}^{-1}$ that is $\xi = q_\mathbb{V}^{-1} \cdot q_\mathbb{W}^{-1} \cdot \eta \cdot q_\mathbb{V}$ and the first part of the proposition follows.

Second case: $\tau_q S = \{q \cdot \xi \cdot \varphi(\xi) : \xi \in \mathbb{V}\}$, then, recalling Proposition 2.2,

$$q \cdot \xi \cdot \varphi(\xi) = q_\mathbb{V} \cdot_\mathbb{W} q \cdot \xi \cdot \varphi(\xi) = q_\mathbb{V} \cdot \xi \cdot \xi^{-1} \cdot_\mathbb{W} q \cdot \xi \cdot \varphi(\xi);$$

here $q_\mathbb{V} \cdot \xi \in \mathbb{V}$ and $\xi^{-1} \cdot_\mathbb{W} q \cdot \xi \cdot \varphi(\xi) \in \mathbb{W}$, then setting $\eta := q_\mathbb{V} \cdot \xi$ and observing that $\xi^{-1} \cdot_\mathbb{W} q \cdot \xi = \eta^{-1} \cdot q_\mathbb{V} \cdot_\mathbb{W} q \cdot q_\mathbb{V}^{-1} \cdot \eta = \eta^{-1} \cdot q_\mathbb{W} \cdot q_\mathbb{V} \cdot q_\mathbb{V}^{-1} \cdot \eta = \eta^{-1} \cdot q_\mathbb{W} \cdot \eta$, we get also the second part of the Proposition. $\square$

Notice that if $q \in \mathbb{W}$ then the formula in (i) of Proposition 2.7 becomes completely similar to euclidean ones

$$\varphi_q(\eta) = \varphi(q^{-1} \cdot \eta), \qquad \varphi_q : \tau_q E \to \mathbb{V}.$$

Analogously if $q \in \mathbb{V}$ the formula in (ii) becomes

$$\varphi_q(\eta) = \varphi(q^{-1} \cdot \eta), \qquad \varphi_q : \tau_q F \to \mathbb{W}.$$

Given that $\mathbb{V}$ and $\mathbb{W}$ are metric spaces, continuous functions $\mathbb{W} \to \mathbb{V}$ or $\mathbb{V} \to \mathbb{W}$ are defined as usual. It is then easy to see that if $g : E \subset \mathbb{W} \to \mathbb{V}$ is a continuous function then also any translated function $g_q : \mathbb{W} \to \mathbb{V}$ is continuous. The same statement holds for a function $g : \mathbb{V} \to \mathbb{W}$.

3. Lipschitz functions and graphs

3.1. **Lipschitz graphs.** As we recalled in the introduction, the implicit function theorem proved in [11] states that if S is a low codimensional $\mathbb{H}$-regular surface, that is if S is a non critical level set of a Pansu differentiable function $f : \mathbb{H}^n \to \mathbb{R}^k$, $1 \le k \le n$, then, given any $p \in S$, there are $r > 0$, a couple of subgroups $\mathbb{V}$, $\mathbb{W}$ such that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ and a function $\varphi : \mathbb{W} \to \mathbb{V}$ such that $S \cap B(p, r) \subset \text{graph}(\varphi)$. Moreover if $p \equiv e \in S$ then there is $L > 0$ such that, $\forall \eta$ in a bounded neighborhood of e,

$$\|\varphi(\eta)\| \le L \|\eta\| . \tag{6}$$

Notice that L depends only on the norm of the Pansu differential of f, on the radius r and on the choice of the subgroups $\mathbb{V}$ and $\mathbb{W}$.

Let $p = \bar{w} \cdot \varphi(\bar{w})$ be a point of an $\mathbb{H}$-regular surface S, then, locally near p, $S = \text{graph}(\varphi)$ and the $\mathbb{H}$-regular translated surface $\tau_{p^{-1}} S$, near e, is the graph of

$$\varphi_{p^{-1}}(\eta) := \varphi(\bar{w})^{-1} \cdot \varphi(\bar{w} \cdot \varphi(\bar{w}) \cdot \eta \cdot \varphi(\bar{w})^{-1})$$

Hence (6) holds for $\varphi_{p^{-1}}$ and, for any η in a neighborhood of e in $\mathbb{W}$, we have

$$\left\|\varphi_{p^{-1}}(\eta)\right\| \equiv \left\|\varphi(\bar{w})^{-1} \cdot \varphi(\bar{w} \cdot \varphi(\bar{w}) \cdot \eta \cdot \varphi(\bar{w})^{-1})\right\| \le L \|\eta\| .$$

Changing variables, setting $w = \bar{w} \cdot \varphi(\bar{w}) \cdot \eta \cdot \varphi(\bar{w})^{-1}$, that is $\eta = \varphi(\bar{w})^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \varphi(\bar{w})$, it follows that, $\forall w, \bar{w} \in \mathbb{W}$,

$$\left\|\varphi(\bar{w})^{-1} \cdot \varphi(w)\right\| \le L \left\|\varphi(\bar{w})^{-1} \cdot (\bar{w}^{-1} \cdot w) \cdot \varphi(\bar{w})\right\| .$$

This we use as a definition of intrinsically Lipschitz function:

Definition 3.1. We say that $\varphi : \mathbb{W} \to \mathbb{V}$ (or $\varphi : \mathbb{V} \to \mathbb{W}$) is $\mathbb{H}^n$-*Lipschitz*, if there is $L > 0$ such that, $\forall p \in \text{graph}(\varphi)$,

$$\left\|\varphi_{p^{-1}}(x)\right\| \le L \|x\| , \qquad \forall x \in \text{domain of } \varphi. \tag{7}$$

Equivalently, recalling Proposition 2.7, we have

(i) $\varphi : \mathbb{W} \to \mathbb{V}$ is $\mathbb{H}^n$-*Lipschitz*, if $\exists L > 0$ such that $\forall w, w' \in \mathbb{W}$,
$$\left\|\varphi(w)^{-1} \cdot \varphi(w')\right\| \le L \left\|\varphi(w)^{-1} \cdot w^{-1} \cdot w' \cdot \varphi(w)\right\| ;$$

(ii) $\varphi : \mathbb{V} \to \mathbb{W}$ is $\mathbb{H}^n$-*Lipschitz*, if $\exists L > 0$ such that $\forall v, v' \in \mathbb{V}$,
$$\left\|v'^{-1} \cdot v \cdot \varphi(v)^{-1} \cdot v^{-1} \cdot v' \cdot \varphi(v')\right\| \le L \left\|v^{-1} \cdot v'\right\| .$$

As usual the Lipschitz constant of φ is the infimum of the numbers L such that (7) holds. As usual the definitions can be localized to subsets of $\mathbb{V}$ or of $\mathbb{W}$.

Notice that there are plenty of non trivial $\mathbb{H}^n$-Lipschitz functions in $\mathbb{H}^n$. Indeed, as explained at the beginning of this section, it follows from the implicit function theorem of [11] and the very definition of $\mathbb{H}^n$-Lipschitz functions that all $\mathbb{H}$-regular surfaces, of low codimension, are locally graphs of $\mathbb{H}^n$-Lipschitz functions.

Remark 3.2. Given that $\mathbb{V}$ and $\mathbb{W}$ are metric spaces, also the usual, seemingly more natural, definition of Lipschitz function between metric spaces is available: we say that $f : \mathbb{W} \to \mathbb{V}$ (or $f : \mathbb{V} \to \mathbb{W}$) is Lipschitz if there is $L > 0$ such that

$$\left\| f(\eta)^{-1} \cdot f(\eta') \right\| \le L \left\| \eta^{-1} \cdot \eta' \right\|, \quad \forall \eta, \eta' \in \mathbb{W}(\text{or } \mathbb{V}). \tag{8}$$

The following example shows firstly that the two properties of being Lipschitz or of being $\mathbb{H}^n$-Lipschitz are independent from each other and secondly that (8) is not invariant with respect to group translation of the graph of the function.

Example 3.3. Consider the subgroups $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{H}^1 \equiv \mathbb{R}^3$ defined as

$$\mathbb{V} = \{x = (x_1, 0, 0)\}, \quad \mathbb{W} = \{x = (0, x_2, x_3)\}.$$

Observe that $\mathbb{W}$ is a normal subgroup and that $\mathbb{H}^1 = \mathbb{W} \cdot \mathbb{V}$ as in Definition 2.1. Moreover, $\forall w = (0, w_2, w_3) \in \mathbb{W}$ and $\forall v = (v_1, 0, 0) \in \mathbb{V}$, $\|w\| = \max\{|w_2|, |w_3|^{1/2}\}$ and $\|v\| = |v_1|$.

Let $f : \mathbb{W} \to \mathbb{V}$ be defined as

$$f(0, w_2, w_3) = \left(1 + |w_3|^{1/2}, 0, 0\right).$$

It is easy to check that (8) holds with $L = 1$. Indeed

$$f(w)^{-1} \cdot f(w') = \left(|w_3'|^{1/2} - |w_3|^{1/2}, 0, 0\right)$$

and

$$\begin{aligned} \left\| f(w)^{-1} \cdot f(w') \right\| &= \left| |w_3'|^{1/2} - |w_3|^{1/2} \right| \\ &\le \left| w_3' - w_3 \right|^{1/2} = \left\| w^{-1} \cdot w' \right\|. \end{aligned}$$

On the contrary, f is not $\mathbb{H}^n$-Lipschitz. To see this we translate the graph of f moving $p := (1, 0, 0) \in \text{graph}(f)$ to the origin e. Following the argument in Proposition 2.7 we see that the translated set is the graph of $f_{p^{-1}} : \mathbb{W} \to \mathbb{V}$ and from (i) of Proposition 2.7 we have

$$f_{p^{-1}}(w) = \left(|w_2 + w_3|^{1/2}, 0, 0\right).$$

Now observe that (7) should be equivalent to the inequality $|w_2 + w_3|^{1/2} \le L \max\{|w_2|, |w_3|^{1/2}\}$ that is, in general, false. This shows also that (8) is not invariant under graph translations.

On the contrary, the function $\psi : \mathbb{W} \to \mathbb{V}$ defined as

$$\psi(w) := \left(1 + |w_3 - w_2|^{1/2}, 0, 0\right)$$

is $\mathbb{H}^n$-Lipschitz but it is not Lipschitz in the sense of (8).

We will indicate in this note that $\mathbb{H}^n$-Lipschitz functions enjoy many nice properties, i.e. properties that are typical of Lipschitz functions in Euclidean spaces. Most of these properties cannot be stated in terms of usual regularity properties as the previous example suggests. We only state the following mild regularity theorem

Proposition 3.4. *$\mathbb{H}^n$-Lipschitz functions are $\frac{1}{2}$-Holder continuous with respect to the Carnot Caratheodory or the d_∞ distance.*

Hence, in particular, $\mathbb{H}^n$-Lipschitz functions are continuous functions.

We give now a very natural equivalent definition of $\mathbb{H}^n$-Lipschitz functions. As it is true for functions between Euclidean spaces, $\mathbb{H}^n$-Lipschitz functions can be characterized in terms of existence of parallel cones non intersecting their graphs. First we give a notion of closed cone.

Definition 3.5. Assume that $\mathbb{H}^n = \mathbb{B}\cdot\mathbb{A}$ is the product of two subgroups $\mathbb{B}$ and $\mathbb{A}$, with $\mathbb{B}\cap\mathbb{A} = \{e\}$. For $q\in\mathbb{H}^n$, $\alpha>0$ we define the *closed cones* $C_{\mathbb{B},\mathbb{A}}(q,\alpha)$ *with axis* $\mathbb{A}$*, base* $\mathbb{B}$*, vertex* q as

$$C_{\mathbb{B},\mathbb{A}}(q,\alpha) := q\cdot C_{\mathbb{B},\mathbb{A}}(e,\alpha)$$

where

$$C_{\mathbb{B},\mathbb{A}}(e,\alpha) := \{p : \|p_{\mathbb{B}}\| \le \alpha\,\|p_{\mathbb{A}}\|\}\,.$$

Clearly, $C_{\mathbb{B},\mathbb{A}}(e,0) = \mathbb{A}$. Moreover

$$\cup_{\alpha>0} C_{\mathbb{B},\mathbb{A}}(e,\alpha) = \mathbb{H}^n\setminus\mathbb{B}\cup\{e\}.$$

If $S=\{x\cdot\varphi(x)\}$ is the graph of $\varphi:\mathbb{W}\to\mathbb{V}$ and $\varphi(e)=e$ then it is trivial to observe that $\|\varphi(x)\| < L\,\|x\|$ if and only if $C_{\mathbb{W},\mathbb{V}}(e,\alpha)\cap S=\{e\}$ for all α, $0\le\alpha<1/L$. In general we have

Proposition 3.6. *A function* $\varphi:\mathbb{W}\to\mathbb{V}$ *is* $\mathbb{H}^n$*-Lipschitz, with Lipschitz constant* $\le L$*, if and only if* $\forall q\in\operatorname{graph}(\varphi)$ *and* $\forall\alpha: 0\le\alpha<1/L$*,*

$$C_{\mathbb{W},\mathbb{V}}(q,\alpha)\cap\operatorname{graph}(\varphi)=\{q\}.$$

Proof. Indeed, if $q\in\operatorname{graph}(\varphi)$, $C_{\mathbb{W},\mathbb{V}}(e,\alpha)\cap\operatorname{graph}(\varphi_{q^{-1}}) = \{e\}$, hence $\{q\} = \tau_q(C_{\mathbb{W},\mathbb{V}}(e,\alpha)\cap\operatorname{graph}(\varphi_{q^{-1}})) = \tau_q(C_{\mathbb{W},\mathbb{V}}(e,\alpha)\cap\tau_{q^{-1}}\operatorname{graph}(\varphi)) = C_{\mathbb{W},\mathbb{V}}(q,\alpha)\cap\operatorname{graph}(\varphi)$. ☐

See the following picture.

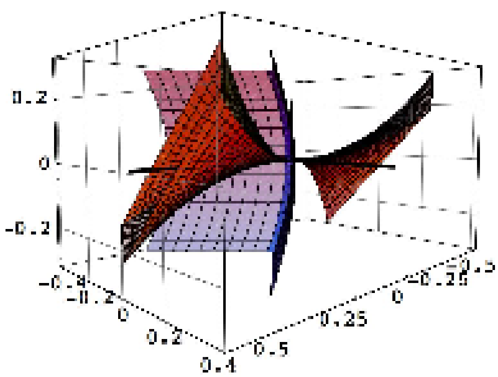

FIGURE 3. The graph of a $\mathbb{H}^1$-Lipschitz function $\mathbb{W}\to\mathbb{V}$ and a cone $C_{\mathbb{W},\mathbb{V}}(e,\alpha)$

3.2. **Difference quotients and directional derivatives.** Another characterization of $\mathbb{H}^n$-Lipschitz functions can be given in terms of boundedness of their difference quotients. Let us begin defining a notion of translation invariant difference quotient.

Definition 3.7. Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$. If $f:\mathbb{W}\to\mathbb{V}$, $w\in\mathbb{W}$, $p := w\cdot f(w)\in \mathrm{graph}(f)$, the $\mathbb{H}^n$*-difference quotient of f, at w along the direction* $Y\in\mathfrak{w}$, is

$$\Delta_Y f(w;t) = \Delta_Y f_{p^{-1}}(e;t) = \delta_{1/t}\left(f_{p^{-1}}(\exp tY)\right).$$

Simmetrically, if $g:\mathbb{V}\to\mathbb{W}$ and $V\in\mathfrak{v}$, $q := v\cdot g(v)$ the $\mathbb{H}^n$*-difference quotient* is

$$\Delta_V g(v;t) = \Delta_V g_{q^{-1}}(e;t) = \delta_{1/t}\left(g_{q^{-1}}(\exp tV)\right).$$

More explicitly, from Proposition 2.7 we obtain that, for $f:\mathbb{W}\to\mathbb{V}$,

$$\Delta_Y f(w;t) = \delta_{1/t}\left(f(w)^{-1}\cdot f(w\cdot f(w)\cdot \exp tY\cdot f(w)^{-1})\right), \tag{9}$$

and for $g:\mathbb{V}\to\mathbb{W}$

$$\Delta_V g(v;t) = \delta_{1/t}\left(\exp tV^{-1}\cdot g(v)^{-1}\cdot \exp tV\cdot g(v\cdot\exp tV)\right). \tag{10}$$

Definition 3.8. Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$. The *directional derivative* $D_Y f(x)$ is defined as

$$D_Y f(x) := \lim_{t\to 0}\Delta_Y f(x;t). \tag{11}$$

Notice that $f(x) = e$ implies $\Delta_Y f(x;t) := \delta_{1/t}\left(f(x\cdot\exp tY)\right)$. Observe that if $D_Y f(x)$ exists then, $\forall\lambda>0$,

$$\begin{aligned} D_{\lambda Y} f(x) &= \lim_{t\to 0}\delta_{1/t}\Delta_{\lambda Y} f(x;t)\\ &= \delta_\lambda \lim_{t\to 0}\delta_{1/\lambda t}\Delta_Y f(x;\lambda t) = \delta_\lambda D_Y f(x). \end{aligned}$$

Clearly, directional derivatives are translation invariant; that is if $p = x\cdot f(x)$,

$$D_Y f(x) = D_Y f_{p^{-1}}(e). \tag{12}$$

Next Proposition gives a characterization of $\mathbb{H}^n$-Lipschitz functions in terms of the boundedness of their difference quotients along *horizontal directions*. We would like to stress that, notwithstanding the similarity of this statement with, e.g. the one characterizing Lipschitz functions $\mathbb{H}^n\to\mathbb{R}$ in terms of the Lipschitzianity along horizontal directions of $\mathbb{H}^n$, this statement is a quite different one. Indeed in general $\mathbb{W}$ is not a Carnot group because its Lie algebra is not generated by the horizontal layer. Think, once more to the example of $\mathbb{H}^1 = \mathbb{W}\cdot\mathbb{V}$, with $\mathbb{W} = \{(0,x_2,x_3)\}$ and $\mathbb{V} = \{(x_1,0,0)\}$. Then $\mathfrak{w}\cap\mathfrak{h}_1$ - i.e. the horizontal subspace of the Lie algebra $\mathfrak{w}$ of $\mathbb{W}$ - is 1-dimensional and it is generated by the vector field $Y_1 = \partial_{x_2} + \frac{1}{2}\partial_{x_3}$ only. We state that, for $f:\mathbb{W}\to\mathbb{V}$, the boundedness just of $\Delta_{Y_1} f$ ensures that f is $\mathbb{H}$-Lipschitz. For a proof see [12].

Proposition 3.9. *Let $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$.*

(i) *If $f:\mathbb{W}\to\mathbb{V}$ is $\mathbb{H}^n$-Lipschitz with Lipschitz constant L then,*

$$\|\Delta_Y f(x;t)\| \le L\,\|\exp Y\|, \quad \forall Y\in\mathfrak{w}.$$

The analogous statement holds if $f:\mathbb{V}\to\mathbb{W}$, with $Y\in\mathfrak{v}$.

(ii) *If* $f : \mathbb{W} \to \mathbb{V}$ *and*

$$\|\Delta_Y f(x;t)\| \le L \|\exp Y\|, \quad \forall Y \in \mathfrak{h}_1 \cap \mathfrak{w},$$

then f *is* $\mathbb{H}^n$*-Lipschitz with Lipschitz constant* $C = C(L, \mathbb{V}, \mathbb{W})$.

3.3. **Surface measure of Lipschitz graphs.** In this section we prove that the graph of a $\mathbb{H}^n$-Lipschitz function f has the same metric dimension as the domain of f and that, if s is this metric dimension, $\mathcal{H}^s(\text{graph}(f) \cap U) < \infty$, for any bounded $U \subset \mathbb{H}^n$.

An interesting, non trivial, corollary of the previous estimate is that 1-codimensional $\mathbb{H}^n$-Lipschitz graphs are boundaries of sets of locally finite $\mathbb{H}^n$-perimeter.

Remember that upper and lower bounds on the Hausdorff measure of a Lipschitz graph are trivially true in Euclidean spaces. Indeed if $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is Lipschitz then the map $\Phi : \mathbb{R}^k \to \mathbb{R}^n$ defined as $\Phi(x) := (x, f(x))$ is a Lipschitz parametrization of the Euclidean graph of f and this gives the upper bound; on the other side the projection $\mathbb{R}^n \equiv \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ is 1 Lipschitz yielding the lower bound.

Such a proof cannot work here because, from one side, projections $\mathbb{H}^n \to \mathbb{W}$, sending $p \mapsto p_{\mathbb{W}}$, are not Lipschitz continuous; on the other side, even if $f : \mathbb{W} \to \mathbb{V}$ is very regular – see Example 3.10 – the 'natural' parametrization of graph(f) given by

$$\Phi : \mathbb{W} \to \text{graph}(f) \subset \mathbb{H}^n, \qquad \Phi(w) = w \cdot f(w)$$

is not a Lipschitz map between metric spaces..

Example 3.10. Consider once more the subgroups $\mathbb{V}$ and $\mathbb{W}$ of $\mathbb{H}^1 \equiv \mathbb{R}^3$ defined as

$$\mathbb{V} = \{x = (x_1, 0, 0)\}, \quad \mathbb{W} = \{x = (0, x_2, x_3)\}$$

and let $f : \mathbb{W} \to \mathbb{V}$ be the constant map $f(w) = (1, 0, 0) \in \mathbb{V}$. Then graph$(f)$ is a vertical plane in $\mathbb{R}^3$ parallel to $\mathbb{W}$. The parametrization Φ acts as

$$\Phi(w) = (1, w_2, w_3 + \frac{1}{2} w_2).$$

Then $\Phi(e) = (1, 0, 0)$ and, if $\bar{w} = (0, \varepsilon, 0) \in \mathbb{W}$, $\Phi(\bar{w}) = (1, \varepsilon, \frac{\varepsilon}{2})$. It is easy to check that $\left\|\Phi(e)^{-1} \cdot \Phi(\bar{w})\right\|$ is comparable with $\varepsilon^{1/2}$ while $\|\bar{w}\|$ is comparable with ε.

Remark 3.11. The situation is completely different for maps $\mathbb{V} \to \mathbb{W}$. Indeed, when $f : \mathbb{V} \to \mathbb{W}$ is $\mathbb{H}^n$-Lipschitz, the map

$$\Phi : \mathbb{V} \to \mathbb{H}^n, \qquad v \mapsto \Phi(v) := v \cdot f(v),$$

is a Lipschitz map between the metric spaces $\mathbb{V}$ and $\mathbb{H}^n$. Indeed, if $v, \bar{v} \in \mathbb{V}$, using (ii) of Definition 3.1, we have

$$\begin{aligned}\left\|f(v)^{-1} \cdot v^{-1} \cdot \bar{v} \cdot f(\bar{v})\right\| &= \left\|v^{-1} \cdot \bar{v} \cdot \bar{v}^{-1} \cdot v \cdot f(v)^{-1} \cdot v^{-1} \cdot \bar{v} \cdot f(\bar{v})\right\| \\ &\le \left\|v^{-1} \cdot \bar{v}\right\| + \left\|\bar{v}^{-1} \cdot v \cdot f(v)^{-1} \cdot v^{-1} \cdot \bar{v} \cdot f(\bar{v})\right\| \\ &\le (1 + L) \left\|v^{-1} \cdot \bar{v}\right\|.\end{aligned}$$

Remark 3.12. It is a, certainly non trivial, open problem to understand if a different Lipschitz continuous parameterization exists. About this, in [18] it has been proved that, if the surface S is somehow more regular than just Lipschitz, then such a parametrization exists. On the contrary, D.Vittone has provided us an example (see [2]) showing that in general bilipschitz parametrizations may not exist.

Theorem 3.13. *Assume that* $\mathbb{H}^n = \mathbb{W}\cdot\mathbb{V}$ *as in Definition 2.1, and let* k, $1 \le k \le n$, *be the dimension of* $\mathbb{V}$. *If* $f : \mathbb{W} \to \mathbb{V}$ *is a* $\mathbb{H}^n$*-Lipschitz function with Lipschitz constant* L, *then* $\mathrm{graph}(f)$ *has metric dimension* $2n+2-k$ *and there is a geometric constant* $c = c(\mathbb{V}, \mathbb{W}) > 0$ *such that*

$$\mathcal{H}^{2n+2-k}\left(\mathrm{graph}(f) \cap B(p,R)\right) \le c(1+L)^{2n+2-k} R^{2n+2-k}. \tag{13}$$

Simmetrically, if $f : \mathbb{V} \to \mathbb{W}$ *then* $\mathrm{graph}(f)$ *has metric dimension* k *and*

$$\mathcal{H}^{k}\left(\mathrm{graph}(f) \cap B(p,R)\right) \le c(1+L)^{k} R^{k}. \tag{14}$$

Proof. The proof follows the same pattern as the Euclidean one when dealing with functions $f : \mathbb{V} \to \mathbb{W}$. Indeed, as observed in Example 3.10 in this case the natural parametrization Φ of $\mathrm{graph}(f)$ is Lipschitz and also the projection $\mathbb{H}^n \to \mathbb{V}$ is a Lipschitz map.

We consider now the only interesting case, that of functions $\mathbb{W} \to \mathbb{V}$.

The lower bound for $\mathcal{H}^{2n+2-k}(\mathrm{graph}(f))$ is a consequence of the following Lemma proved in [20]

Lemma 3.14. *There is* $C = C(\mathbb{V}, \mathbb{W}) > 0$ *such that,* $\forall A \subset \mathbb{H}^n$

$$\mathcal{L}^{2n+1-k}(\Pi(A)) = \mathcal{H}^{2n+2-k}(\Pi(A)) \le \mathcal{S}^{2n+2-k}(A),$$

where $\Pi : \mathbb{H}^n \to \mathbb{W}$ *is the 'projection on the first component' i.e. if* $p = p_{\mathbb{W}} \cdot p_{\mathbb{V}}$ *then* $\Pi p := p_{\mathbb{W}}$.

To get the upper bound, fix $p \in \mathrm{graph}(f)$ and $R > 0$, it is enough to prove that it is possible to cover $\mathrm{graph}(f) \cap B(p,R)$ with less than $N := c\left(\frac{1}{\varepsilon}\right)^{2n+2-k}$ metric balls of radius less than ε. Here c will depend on R, $\mathbb{W}$, $\mathbb{V}$ and L.

Without loss of generality, we can assume that $p = e$. Let $E := \{w \in \mathbb{W} : w \cdot f(w) \in B(e,R)\}$. From (1), it follows $E \subset \{w : \|w\| \le R/c\}$.

Fix ε, $0 < \varepsilon < 1$. Using a Vitali covering argument choose a covering of $\mathrm{graph}(f) \cap B(e,R)$ with metric balls $B(p_i, 5\varepsilon)$, $p_i = \bar{w}_i \cdot f(\bar{w}_i) \in \mathrm{graph}(f)$, such that the concentric smaller balls $B_i := B(p_i, \varepsilon)$ are pairwise disjointed. We estimate the number N of balls B_i in this Vitali covering.

Define $E_i \subset E$ as $E_i := \{w \in \mathbb{W} : w \cdot f(w) \in \mathrm{graph}(f) \cap B_i\}$. Clearly the sets E_i are pairwise disjointed. To get the necessary estimate of N we get an estimate from below of $\mathcal{L}^{2n+1-k}(E_i)$.

For each E_i consider the group translation $\tau_{p_i^{-1}}$ that moves the point p_i to the origin e. Let $\tilde{E}_i := \{w : w \cdot f_{p_i^{-1}}(w) \in B(e,\varepsilon)\}$. Remember that $f_{p_i^{-1}}$ is $\mathbb{H}^n$-Lipschitz with the same constant L of f, that $f_{p_i^{-1}}(e) = e$ hence $\left\| f_{p_i^{-1}}(w) \right\| \le L \|w\|$ and

$\left\| w \cdot f_{p_i^{-1}}(w) \right\| \leq (1 + L) \|w\|$. Hence

$$\mathbb{W} \cap B\left(e, \frac{\varepsilon}{1+L}\right) \subset \tilde{E}_i,$$

and, from (3) , it follows

$$\begin{aligned} &\mathcal{L}^{2n+1-k}(\tilde{E}_i) \\ &\quad \geq \mathcal{L}^{2n+1-k}\left(\mathbb{W} \cap B\left(e, \frac{\varepsilon}{1+L}\right)\right) = 2\omega_{2n-k}\left(\frac{\varepsilon}{1+L}\right)^{2n+2-k}. \end{aligned}$$

Recalling (i) of Proposition 2.7, we have that $\tilde{E}_i = p_i^{-1} \cdot E_i \cdot f(\bar{w}_i)$, that is

$$\tilde{E}_i = \{f(\bar{w}_i)^{-1} \cdot \bar{w}_i^{-1} \cdot w \cdot f(\bar{w}_i) : w \in E_i\}.$$

It is easy to check, by a straightforward computation, that any map $\chi : \mathbb{W} \equiv \mathbb{R}^{2n+1-k} \to \mathbb{W} \equiv \mathbb{R}^{2n+1-k}$, given by

$$w \mapsto \chi(w) := \bar{v}^{-1} \cdot \bar{w}^{-1} \cdot w \cdot \bar{v},$$

has Jacobian determinant equal to 1. Hence

$$\mathcal{L}^{2n+1-k}(E_i) = \mathcal{L}^{2n+1-k}(\tilde{E}_i) \geq 2\omega_{2n-k}\left(\frac{\varepsilon}{1+L}\right)^{2n+2-k}.$$

Since all the E_i are disjointed and contained in $B(e, (R+1)/c)$ we get

$$N \leq \left((1+L)\,\frac{R}{c}\right)^{2n+2-k}\left(\frac{1}{\varepsilon}\right)^{2n+2-k}.$$

When dealing with $f : \mathbb{V} \to \mathbb{W}$ the thesis follows from well known results on the scaling of Hausdorff measures under Lipschitz maps (see e.g.([6]) or ([16])). $\square$

Assume now that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ with $\dim \mathbb{V} = 1$. Notice that it follows the existence of $Y \in \mathfrak{h}_1$ such that $\mathbb{V} = \{\exp tY : t \in \mathbb{R}\}$. Hence it is defined a real valued function $t : \mathbb{H}^n \to \mathbb{R}$ such that

$$p = p_{\mathbb{W}} \cdot \exp\left(t(p)Y\right), \qquad \forall p \in \mathbb{H}^n.$$

Then, given $f : \mathbb{W} \to \mathbb{V}$ it is possible to define the subgraph of f as the set $E(f)$ such that

$$E(f) := \{p \in \mathbb{H}^n : t(p) \leq t(f(p_{\mathbb{W}}))\}.$$

Then the following theorem holds

Theorem 3.15. *Assume* $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ *with* $\dim \mathbb{V} = 1$. *If* $f : \mathbb{W} \to \mathbb{V}$ *is* $\mathbb{H}^n$*-Lipschitz and* $E(f) \subset \mathbb{H}^n$ *is the* subgraph of f *then* $E(f)$ *is a set with locally finite perimeter.*

Proof. The graph of f is the essential boundary of the subgraph. The result then follows from Theorem 3.13 and the characterization of finite perimeter sets proved in [20]. $\square$

3.4. **Rectifiable sets.** We recall the definition of $\mathbb{H}$-rectifiable sets given in [11].

Definition 3.16. Let $1 \le k \le n$ and assume that $M \subset \mathbb{H}^n$ is such that

$$M \subset M_0 \cup \left(\bigcup_{j=1}^{+\infty} S_j \right).$$

We say that M is

k-dimensional $\mathbb{H}$-rectifiable if $\mathcal{S}^k(M \cap U) < \infty$ for any bounded $U \subset \mathbb{H}^n$, $\mathcal{S}^k(M_0) = 0$ and S_j are k-dimensional $\mathbb{H}$-regular surfaces;

k-codimensional $\mathbb{H}$-rectifiable if $\mathcal{S}^{2n+2-k}(M \cap U) < \infty$ for any bounded $U \subset \mathbb{H}^n$, $\mathcal{S}^{2n+2-k}(M_0) = 0$, S_j are k-codimensional $\mathbb{H}$-regular surfaces.

We can give a, possibly, more general definition using the notion of $\mathbb{H}^n$-Lipschitz graphs.

Definition 3.17. Let $1 \le k \le n$ and assume that $E \subset \mathbb{H}^n$ is such that

$$E \subset E_0 \cup \left(\bigcup_{i=1}^{\infty} \operatorname{graph}(f_i) \right).$$

We say that E is

k-dimensional $\mathbb{H}$-rectifiable if $\mathcal{H}^k(E \cap U) < \infty$ for any bounded $U \subset \mathbb{H}^n$, $\mathcal{H}^k(E_0) = 0$, $f_i : A_i \subset \mathbb{V}_i \to \mathbb{W}_i$ are $\mathbb{H}$-Lipschitz and $\dim(\mathbb{V}_i) = k$;

k-codimensional $\mathbb{H}$-rectifiable if $\mathcal{H}^{2n+2-k}(E \cap U) < \infty$ for any bounded $U \subset \mathbb{H}^n$, $\mathcal{H}^{2n+2-k}(E_0) = 0$, $f_i : A_i \subset \mathbb{W}_i \to \mathbb{V}_i$ are $\mathbb{H}$-Lipschitz, $\dim(\mathbb{V}_i) = k$.

Since $\mathbb{H}$-regular surfaces are locally graphs of $\mathbb{H}^n$-Lipschitz functions it follows that the scope of the second definition is larger than the first one. The equivalence of the two definitions should depend on a Rademacher type theorem in the context of $\mathbb{H}^n$-Lipschitz functions.

4. Intrinsic Differentiable Functions

Assume that $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in Definition 2.1. We suggest here a possible definition of intrinsic differentiability for functions $f : \mathbb{W} \to \mathbb{V}$ (or $f : \mathbb{V} \to \mathbb{W}$). We look for a definition that is invariant with respect to translations and dilations of graph(f) in $\mathbb{H}^n$, that is strictly related with the notion of $\mathbb{H}^n$-Lipschitz functions and that mimics Pansu definition of P-differentiability for functions between Carnot groups.

We recall the definition of P-differentiability: let $f : \mathbb{G}_1 \to \mathbb{G}_2$, with $\mathbb{G}_1$ and $\mathbb{G}_2$ Carnot groups. We say that f is P-differentiable in $g \in \mathbb{G}_1$ if there is an H-linear map $L : \mathbb{G}_1 \to \mathbb{G}_2$, such that, $\forall g' \in \mathbb{G}_1$,

$$\left\| L(g^{-1} \cdot g')^{-1} \cdot f(g)^{-1} \cdot f(g') \right\|_{\mathbb{G}_2} = o(\left\| g^{-1} \cdot g' \right\|_{\mathbb{G}_1}),$$

where $o(t)/t \to 0$ as $t \to 0^+$.

We first need a substitute notion for H-linear maps: these will be maps such that their graph is a homogeneous subgroup of $\mathbb{H}^n$ and such that the map from the domain to the graph is an homomorphism in an appropriate sense.

Definition 4.1. We say that

(i): $L : \mathbb{V} \to \mathbb{W}$ is a $\mathbb{H}^n$-linear map when, $\forall v, v' \in \mathbb{V}$ and $\forall \lambda > 0$,

$$L(\delta_\lambda v) = \delta_\lambda(Lv)$$
$$L(v \cdot v') = (v')^{-1} \cdot Lv \cdot v' \cdot Lv'.$$

(ii): $L : \mathbb{W} \to \mathbb{V}$ is a $\mathbb{H}^n$-linear map when $\forall w, w' \in \mathbb{W}$ and $\forall \lambda > 0$,

$$L(\delta_\lambda w) = \delta_\lambda(Lw)$$
$$L(w \cdot w') = L(w) \cdot L(w').$$

Notice that $\mathbb{H}^n$-linear maps $\mathbb{W} \to \mathbb{V}$ are precisely H-linear maps $\mathbb{W} \to \mathbb{V}$. On the contrary the two notions are different for maps $\mathbb{V} \to \mathbb{W}$.

Example 4.2. Consider the subgroups of $\mathbb{H}^1$: $\mathbb{V} = \{(x_1, 0, 0)\}$ and $\mathbb{W} = \{(0, x_2, x_3)\}$; the map

$$L : \mathbb{V} \to \mathbb{W} \quad \text{defined as: } L(x, 0, 0) := (0, x, 0)$$

is an H-linear map $\mathbb{V} \to \mathbb{W}$ but it is not $\mathbb{H}^n$-linear because $\mathrm{graph}(L) = \{(x, x, x^2/2)\}$ is not a subgroup. Conversely, the map

$$L : \mathbb{V} \to \mathbb{W} \quad \text{defined as: } L(x, 0, 0) := (0, x, -x^2/2)$$

is $\mathbb{H}^n$-linear but it is not H-linear.

Proposition 4.3. *Assume* $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ *as in Definition 2.1.*

(i): If $L : \mathbb{V} \to \mathbb{W}$ *is* $\mathbb{H}^n$*-linear then* $\mathrm{graph}(L)$ *is a homogeneous subgroup of* $\mathbb{H}^n$ *and the map* Φ_L *defined as* $\Phi_L(v) := v \cdot L(v)$ *is a homogeneous homomorphism (i.e. a H-linear map)* $\mathbb{V} \to \mathrm{graph}(L)$.

(ii): If $L : \mathbb{W} \to \mathbb{V}$ *is* $\mathbb{H}^n$*-linear then* $\mathrm{graph}(L)$ *is a homogeneous subgroup of* $\mathbb{H}^n$ *and the map* $\Phi_L : \mathbb{W} \to \mathrm{graph}(L)$ *defined as* $\Phi_L(w) := w \cdot L(w)$ *satisfies*

$$\Phi_L(\delta_\lambda w) = \delta_\lambda(\Phi_L(w))$$
$$\Phi_L(w \cdot w') = \Phi_L(w) \cdot \Phi_L((Lw)^{-1} \cdot w' \cdot Lw).$$

Now if f acts between subgroups of $\mathbb{H}^n$, we define differentiability of f in the usual way in the points where f vanishes and we extend the definition everywhere making it invariant by graph translation. Precisely, if $f : \mathbb{W} \to \mathbb{V}$ (or $f : \mathbb{V} \to \mathbb{W}$) is such that $f(e) = e$ we say that f is $\mathbb{H}^n$*-differentiable in* e when there is a $\mathbb{H}^n$-linear map $df_e : \mathbb{W} \to \mathbb{V}$, such that

$$\left\| df_e(\xi)^{-1} \cdot f(\xi) \right\| = o(\|\xi\|) \quad \text{as } \|\xi\| \to 0,$$

and, setting $p := w \cdot f(w)$, we say that f is $\mathbb{H}^n$*-differentiable in* $w \in \mathbb{W}$ if $f_{p^{-1}}$ is $\mathbb{H}^n$-differentiable in e, that is, if there is a $\mathbb{H}^n$-linear map $df_w : \mathbb{W} \to \mathbb{V}$, such that

$$\left\| df_w(\xi)^{-1} \cdot f_{p^{-1}}(\xi) \right\| = o(\|\xi\|) \quad \text{as } \|\xi\| \to 0.$$

Finally, writing explicitly the expression of $f_{p^{-1}}$, we give the definition as follows

Definition 4.4. Assume $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ as in definition 2.1.

(i): let $f : \mathbb{W} \to \mathbb{V}$; we say that f is $\mathbb{H}$-differentiable in $w \in \mathbb{W}$ if there is a $\mathbb{H}$-linear map $df_w : \mathbb{W} \to \mathbb{V}$ such that

$$\left\| df_w(\xi)^{-1} \cdot f(w)^{-1} \cdot f(w \cdot f(w) \cdot \xi \cdot f(w)^{-1}) \right\| = o(\|\xi\|) \tag{15}$$

as $\|\xi\| \to 0$.

(ii): let $f : \mathbb{V} \to \mathbb{W}$; we say that f is $\mathbb{H}$-differentiable in $v \in \mathbb{V}$ if there is a $\mathbb{H}$-linear map $df_v : \mathbb{V} \to \mathbb{W}$ such that

$$\left\| df_v(\eta)^{-1} \cdot \eta^{-1} \cdot f(v)^{-1} \cdot \eta \cdot f(v \cdot \eta) \right\| = o(\|\eta\|) \tag{16}$$

as $\|\eta\| \to 0$.

We limit ourselves now in quoting a couple of elementary properties of $\mathbb{H}$-differentials.

Definition 4.5. Assume that $S := \{x \cdot f(x) : x \in A\}$, where A is an open neighborhood of e in $\mathbb{W}$. We say that a subgroup $\mathbb{T}$ of $\mathbb{H}^n$ is the *regular tangent group* of S at e if there is another subgroup $\mathbb{N}$, such that $\mathbb{T} \cap \mathbb{N} = \{e\}$ and $\mathbb{H}^n = \mathbb{T} \cdot \mathbb{N}$, and if, for all $\alpha > 0$, there is $\lambda > 0$ such that

$$C_{\mathbb{T},\mathbb{N}}(e, \alpha) \cap \delta_\lambda S \cap B(e, 1) = \{e\}.$$

More generally we say that $\mathbb{T}$ is the *regular tangent group* of S at $p \in S$ if $\mathbb{T}$ is the regular tangent plane of $\tau_{p^{-1}} S$ at e.

Proposition 4.6. *If $f : \mathbb{W} \to \mathbb{V}$ is $\mathbb{H}^n$-differentiable in x with differential df_x, then $\mathbb{T} := \mathrm{graph}(df_x)$ is the regular tangent group of S at $p = x \cdot f(x)$.*

Proposition 4.7. *Let $f : \mathbb{W} \to \mathbb{V}$. Assume that f is $\mathbb{H}^n$-differentiable and $Y \in \mathfrak{w} \cap \mathfrak{h}_1$, then the directional derivative $D_Y f(x)$ exists and*

$$D_Y f(x) = df_x(\exp Y). \tag{17}$$

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