Journal of Nonlinear and Convex Analysis
Volume 7, Number 3, 2006, 411–422

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

GLOBAL DIFFERENTIABILITY RESULTS FOR SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS WITH CONTROLLED GROWTHS

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Abstract. Let Ω be a bounded open subset of $\mathbb{R}^n$, $n > 2$. In Ω we deduce the global differentiability result

$$u \in H^2(\Omega, \mathbb{R}^N)$$

for the solutions $u \in H^1(\Omega, \mathbb{R}^n)$ of the Dirichlet problem:

$$\begin{gathered} u - g \in H_0^1(\Omega, \mathbb{R}^N) \\ -\sum_i D_i a^i(x, u, Du) = B_0(x, u, Du) \end{gathered} \tag{A}$$

with controlled growth and non linearity $q = 2$.

The result has been obtained in two steps:at first in a particular case of controlled growths and subsequently, in a more general case, for solutions $u \in H^1(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ of problem (A). In the later case we prove the local differentiability result to the interior of Ω. We then prove the differentiability result near the boundary and afterwards the global differentiability result, making use of the covering procedure.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$, $n > 2$ [(1)], for instance of class C^2 with points $x = (x_1, x_2, \ldots, x_n)$.

We denote by d_Ω the diameter of Ω. N is an integer > 1, $(\cdot|\cdot)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in $\mathbb{R}^k$, respectively. We will drop the subscript k when there is no fear of confusion.

If $u : \Omega \to \mathbb{R}^N$, we set $Du = (D_1 u, \ldots, D_n u)$ where, as usual, $D_i = \frac{\partial}{\partial x_i}$; clearly $Du \in \mathbb{R}^{nN}$ and we denote by $p = (p^1, \ldots, p^n)$, $p^j \in \mathbb{R}^N$, a typical vector of $\mathbb{R}^{nN}$. If

$$u \in H^1(\Omega, \mathbb{R}^N)^{(2)} \tag{1}$$

2000 *Mathematics Subject Classification.* 5J60, 35D10, 58B10.

Key words and phrases. global differentiability of weak solutions, elliptic problems, controlled growth, non linearity with $q = 2$..

(1) This argumentation is obviously modified if $n = 2$

(2) N is an integer > 1, $H^k = H^{k,2}, H_0^k = H_0^{k,2}$ (k integer ≥ 0) are the usual Sobolev spaces, $H^0(\Omega) = L^2(\Omega)$, $|u|_{k,\Omega}^2 = \int_\Omega \sum_{|\alpha|=k} \|D^\alpha u\|^2\, dx$, $\|u\|_{k,\Omega} = \left\{ \sum_{h=0}^k |u|_{h,\Omega}^2 \right\}^{\frac{1}{2}}$, $|u|_{0,\Omega} = \|u\|_{0,\Omega} = \{\int_\Omega \|u\|^2 dx\}^{\frac{1}{2}}$

we consider the following variational elliptic non linear system

$$-\sum_{i=1}^{n} D_i a^i(x,u,Du) = B_0(x,u,Du). \tag{2}$$

We suppose that:

(3) $a^i(x,u,p)$, $\forall i = 1,2,\dots,n$, are vectors of class $C^1(\overline{\Omega},\mathbb{R}^N,\mathbb{R}^{nN})$ such that $a^i(x,u,0) = 0$ $\forall x \in \Omega$, $\forall u \in \mathbb{R}^N$, $\forall i = 1,2,\dots,n$

$$\left\| \frac{\partial a^i(x,u,p)}{\partial u_k} \right\| + \left\| \frac{\partial a^i(x,u,p)}{\partial p_k^j} \right\| \le M,$$
$$\forall\, (x,u,p) \in \Lambda = \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$$
$$\forall\, i,j = 1,2,\dots,n \quad \forall\, k = 1,2,\dots,N$$

where M is a suitable positive constant;

$$\left\| a^i(x,u,p) \right\| + \left\| \frac{\partial a^i(x,u,p)}{\partial x_s} \right\| \le f(x) + c \left\{ \|u\|^\alpha + \sum_j \|p^j\| \right\}$$

$\forall i,s = 1,2,\dots,n \quad \forall\, (x,u,p) \in \Lambda$ with $\alpha \le \dfrac{n}{n-2}$ and $f \in L^2(\Omega)$;

(4) there exists a positive constant ν such that

$$\sum_{i,j} \sum_{h,k} \frac{\partial a_h^i(x,u,p)}{\partial p_k^j} \xi_h^i \xi_k^j \ge \nu \, \|\xi\|^2$$

$\forall\, (x,u,p) \in \Lambda$, $\forall\, \xi \in \mathbb{R}^{nN}$.

(5) the vector $B^0(x,u,p)$ defined in Λ, is measurable in x, continuous in (u,p) and
satisfies the following condition
$\forall (x,u,p) \in \Lambda$

$$\|B^0(x,u,p)\| \le f_0(x) + c \left\{ \|u\|^\alpha + \sum_j \|p^j\|^\gamma \right\}$$

where $f_0 \in L^2(\Omega)$, $\alpha \le \dfrac{n}{n-2}$, $\gamma \le \dfrac{n+2}{n}$.

From (3) it easily follows that $\forall\, (x,u,p) \in \Lambda$, $\forall\, i = 1,2,\dots,n$

$$\|a^i(x,u,p)\| \le M\|p\|. \tag{6}$$

A solution of system (2) in Ω is a vector $u \in H^1(\Omega,\mathbb{R}^N)$ such that

$$\int_\Omega \sum_i (a^i(x,u,Du) \mid D_i\varphi) dx = \int_\Omega (B^0 \mid \varphi) dx \tag{7}$$

$\forall \varphi \in H_0^1(\Omega,\mathbb{R}^N)$.

S. Campanato in [4] investigated the problem of the differentiability of the solution $u \in H^1(\Omega, \mathbb{R}^N)$ of basic systems

$$\sum_i D_i a^i(Du) = 0$$

both in the interior and near the boundary, achieving results of the same type in both cases.

Moreover the problem of achieving global differentiability results has been investigated by Campanato in [2], regarding the solutions $u \in H^k(\Omega, \mathbb{R}^N)$ of linear elliptic systems there by proving that the solutions of the Dirichlet problem with zero boundary data belong to $H^{k+2}(\Omega, \mathbb{R}^N)$.

In the first part of this paper we investigate the problem of the global differentiability of the solutions $u \in H^1(\Omega, \mathbb{R}^N)$ of the Dirichlet problem with non zero boundary data for non-linear elliptic systems (2) in a particular case ($\gamma = 1$) of controlled growth (3), (5) and we prove the following:

Theorem 1. *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the Dirichlet problem:*

$$u - g \in H^1_0(\Omega, \mathbb{R}^N) \tag{8}$$

$$-\sum_i D_i a^i(x, u, Du) = B^0(x, u, Du)$$

if Ω is of class C^2, $g \in H^2(\Omega, \mathbb{R}^N)$ and if a^i, B_0 satisfy conditions $(3)-(5)$ with $\gamma = 1$ then it results

$$u \in H^2(\Omega, \mathbb{R}^N)$$

and we have

$$\int_\Omega \sum_{ij} \|D_{ij}u\|^2 \, dx \le c(\nu, M) \int_\Omega \Big[1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2 + \|Dg\|^2 + \|D^2 g\|^2\Big] \, dx. \tag{9}$$

In order to prove this result, we must first recall a local differentiability result due to S. Campanato [3] and from this we immediately obtain an interior differentiability result:

Theorem 2. *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the system (2) where we suppose that the conditions $(3)-(5)$ with $\gamma = 1$ are verified. Then for every open set $\Omega^* \subset\subset \Omega$ we have that $u \in H^2(\Omega^*, \mathbb{R}^N)$ and*

$$|u|^2_{2,\Omega^*} \le c \int_\Omega [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \sum_i \|D_i u\|^2] dx$$

where the constant c depends also on $d = dist(\overline{\Omega}^*, \partial\Omega)$.

We can extend this theorem to the solution of the system

$$-\sum_{i=1}^n D_i a^i(x, u + g, Du + Dg) = B_0(x, u + g, Du + Dg) \tag{10}$$

with $g \in H^2(\Omega, \mathbb{R}^N)$, we obtain immediately the following

Theorem 3. *Let $u\in H^1(\Omega,\mathbb{R}^N)$ be a solution of the system (10) with $g\in H^2(\Omega,\mathbb{R}^N)$, where we suppose that the conditions (3)−(5) with $\gamma = 1$ hold. Then for every open set $\Omega^* \subset\subset \Omega$ it results that $u \in H^2(\Omega^*,\mathbb{R}^N)$ and*

$$|u|^2_{2,\Omega^*} \leq c \int_\Omega [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|g\|^{2^*} + \|Du\|^2 + \|Dg\|^2 + \|D^2 g\|^2]dx$$

where the constant c depends also on $d = dist(\overline{\Omega}^*, \partial\Omega)$.

Afterwards we prove the differentiability result near the boundary.

We define

$$B(x^0,\sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\};$$

moreover, if $x^0_n = 0$

$$B^+(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n > 0\}$$
$$\Gamma(x^0,\sigma) = \{x \in B(x^0,\sigma) : x_n = 0\}$$

We will simply write $B^+(\sigma)$, $\Gamma(\sigma)$ and Γ instead of $B^+(0,\sigma),\Gamma(0,\sigma)$ and $\Gamma(0,1)$ respectively.

Let us consider $t \in (0,1)$, $h \in \mathbb{R}$, such that $|h| < (1-t)\sigma$ and $x \in B(t\sigma) \subset \Omega$. If $u : B(\sigma) \to \mathbb{R}^N$ we define

$$\tau_{s,h}\, u(x) = u(x + h\, e^s) - u(x) \quad s = 1, 2, \ldots, n$$

where $\{e^s\}_{s=1,2,\ldots,n}$ is the standard base of R^n.

In the hemisphere $B^+(1)$ let us consider the problem

$$\begin{aligned} & u \in H^1(B^+(1),\mathbb{R}^N) \\ & u = 0 \text{ on } \Gamma \\ & -\sum_{i=1}^n D_i a^i(x,u,Du) = B^0(x,u,Du) \end{aligned} \tag{11}$$

The last equality means that

$$\int_{B^+(1)} \sum_i (a^i(x,u,Du)|D_i\varphi)dx = \int_{B^+(1)} (B^0(x,u,Du)|\varphi)dx \tag{12}$$

$\forall\, \varphi \in H^1_0(B^+(1),\mathbb{R}^N)$.

Then the following differentiability theorem near the boundary holds:

Theorem 4. *If $u \in H^1(B^+(1),\mathbb{R}^N)$ is a solution of the problem (11), under the conditions (3)–(5), with $\gamma = 1$, for every $\sigma < 1$*

$$u \in H^2(B^+(\sigma),\mathbb{R}^N) \tag{13}$$

and

$$|Du|_{1,B^+(\sigma)} \leq \frac{c(\nu,M)}{(1-\sigma)} \left\{ \int_{B^+(1)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2]\, dx \right\}^{\frac{1}{2}}. \tag{14}$$

Proof. See e.g. [8]. □

Now let us consider for $g \in H^1(B^+(1), \mathbb{R}^N)$ the following problem

$$u = 0 \quad \text{on } \Gamma$$

$$(15) \qquad -\sum_{i=1}^{n} D_i a^i(x, u+g, Du+Dg) = B^0(x, u+g, Du+Dg)$$

and let us assume that conditions (3)–(5)with $\gamma = 1$ are verified with Ω replaced by $B^+(1)$.

Then the following result holds:

Theorem 5. *Let $u \in H^1(B^+(1), |R^N)$ be a solution of the problem (15) under the conditions (3)–(5),with $\gamma = 1$.Let us assume that $g \in H^2(B^+(1), \mathbb{R}^N)$. Then for every $0 < R < 1$, $D\,u$ belongs to $H^1(B^+(R), \mathbb{R}^N)$ and it results*

$$|Du|_{1,B^+(\mathbb{R})} \leq \frac{c(\nu, M)}{(1-R)} \left\{ \int_{B^+(1)} [1 + |f|^2 + |f_0|^2 + \right.$$
$$\left. + \|u\|^{2^*} + \|Du\|^2 + \|Dg\|^2 + \|D^2 g\|^2] dx \right\}^{\frac{1}{2}}.$$

Proof. See e.g. [8]. □

Using this local differentiability result near the boundary, together with the interior differentiability result, we can prove, by means of the usual covering argument, the global differentiability result (Theorem 1)[(3)].

Afterwards, in order to obtain a more general differentiability result, we prove the following local differentiability result assuming that $\gamma \leq \dfrac{n+2}{n}$.

Theorem 6. *If $u \in H^1(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ $(0 < \lambda < 1)$ is a solution of the system (2), if the conditions (3),(5) are verified, then $\forall\, B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$ it result*

$$(16) \qquad u \in H^{1+\vartheta}(B(\sigma), \mathbb{R}^N)^{(4)} \quad \forall \vartheta \in (0, \frac{\lambda}{2}),$$

and

$$(17) \quad |Du|_{\vartheta,B(\sigma)} \leq \frac{c(\nu, k, U, \lambda, \vartheta, n)}{(1-\sigma)^2} \int_{B(3\sigma)} \left[1 + |f|^2 + |f_0| + \|u\|^{2^*} + \|Du\|^2\right] dx$$

where we pose $k = \sup_{\Omega} u$ and $U = [u]_{\lambda,\overline{\Omega}} = \sup_{x,y \in \overline{\Omega}\ x \neq y} \dfrac{\||u(x) - u(y)\|}{\|x-y\|}$

---

(3)This result has been submitted for publication to the Czechoslovak Mathematical Journal, see [8].

(4)$H^\theta(\Omega, \mathbb{R}^N), 0 < \theta < 1$, is a space of the vectors $u \in L^2(\Omega, \mathbb{R}^N)$ such that

$$|u|_{\theta,\Omega} = \int_\Omega dx \int_\Omega \frac{\|u(x) - u(y)\|^2}{\|x-y\|^{n+2\theta}} dy < +\infty$$

and $H^{1+\theta}(\Omega, \mathbb{R}^N)$is subspace of $H^1(\Omega, \mathbb{R}^N)$ of vectors $u \in H^1(\Omega, \mathbb{R}^N)$ such that $D_i u \in H^\theta(\Omega, \mathbb{R}^N)$, $\quad i = 1, 2, \ldots, n$. $|Du|^2_{\vartheta,B(\sigma)} = \sum_{i=1}^n |D_i u|^2_{\vartheta,B(\sigma)}$

Proof. Let us choose $\sigma < 1$ such that $B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$ and the function $\psi \in C_0^\infty(\mathbb{R}^n)$ fulfilling these properties

$$(18)\qquad 0 \le \psi \le 1,\ \psi = 1 \text{ in} B(\sigma),\ \ \psi = 0 \text{ in } \mathbb{R}^n \backslash B(2\sigma),\ |D_i\psi| \le \frac{c}{1-\sigma}$$

If $\rho < \frac{\sigma}{2}$ we assume in (2) as a test function

$$\varphi = \tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)$$

Then we have

$$(19)\qquad \int_\Omega \sum_i (\tau_{r,\rho} a^i(x,u,Du) | D_i(\psi^2 \tau_{r,\rho} u) dx) = \int_\Omega (B^0 | \tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)) dx$$

Arguing as in the proof of theorem (3.1) of [8], we obtain, for each integer i, $1 \le i \le n$,

$$\begin{aligned}(20)\ \ A = \frac{\nu}{2} \int_{B(2\sigma)} \psi^2 \|\tau_{r,\rho} Du\|^2 dx \\ \le \frac{c(M,\nu)}{(1-\sigma)^2} |\rho|^2 \int_{B(3\sigma)} \{1 + |f|^2 + \|u\|^{2^*} + \|Du\|^2\} dx \\ + \int_{B(2\sigma)} \big\{ |f_0(x)| + c(k) \big[\|u\|^\alpha + \sum_j \|p^j\|^\gamma \big] \|\tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)\| \big\} dx = B + C.\end{aligned}$$

Taking into account that $u \in C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$ we have :

$$\|\tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)\| \le 2[u]_{\lambda,\overline{\Omega}}\ |\rho|^\lambda \quad \forall x \in B(3\sigma)$$

and then from (20) we have

$$\begin{aligned}(21)\quad \int_{B(2\sigma)} \psi^2 \|\tau_{r,\rho} Du\|^2 dx \le \frac{c(M,\nu)}{(1-\sigma)^2} |\rho|^2 \int_{B(3\sigma)} \{1 + |f|^2 + \|u\|^{2^*} + \|Du\|^2\} dx \\ + c(\nu,k,U,\sigma)|\rho|^\lambda \int_{B(3\sigma)} (|f_0| + \|u\|^\alpha + \|Du\|^\gamma) dx;\end{aligned}$$

therefore, $\forall \rho < 2\sigma$

$$\begin{aligned}(22)\quad \sum_{r=1}^n \int_{B(\sigma)} \|\tau_{r,\rho} Du\|^2 dx \\ \le c(M,\nu,k,U,n,\lambda) \frac{|\rho|^\lambda}{(1-\sigma)^2} \int \{1 + |f_0| + |f|^2 + \|u\|^\alpha + \|u\|^{2^*} \\ + \|Du\|^2 + \|Du\|^\gamma\} dx.\end{aligned}$$

Then, given that $\alpha < 2^*$ and $\gamma \le 2$, if $0 < \vartheta < \frac{\lambda}{2}$ we have

$$\begin{aligned}(23)\quad \sum_{r=1}^n \int_{-2\sigma}^{2\sigma} \frac{d\rho}{|\rho|^{1+2\vartheta}} \int_{B(\sigma)} \|\tau_{r,\rho} Du\|^2 dx \\ \le \frac{c(M,n,\nu,k,U,\lambda)}{(1-\sigma)^2} \int_{B(3\sigma)} \{1 + |f_0| + |f|^2 + \|u\|^{2^*} + \|Du\|^2\} dx\end{aligned}$$

and by a well known theorem[5] we have (16) and (17). □

We must underline that the proof of this theorem is analogous to that of theorem 2, except for the estimation of the integral

$$\int_\Omega \|B^0(x,u,Du)\|\ \|\tau_{r,-\rho}(\theta^2\tau_{r,\rho}u)\|dx, \tag{24}$$

where we need the assumption $u \in C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$. By means of this assumption we get

$$\|\tau_{r,-\rho}(\theta^2\tau_{r,\rho}u)\| \le 2[u]_{\lambda,\overline{\Omega}}|\rho|^\lambda$$

and hence the assertion.

Afterwards, taking into account this results, we obtain a better local regularity result.

Theorem 7. *If $u \in H^{1+\theta}(\Omega,\mathbb{R}^N)\cap C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$, $0<\lambda<1$, $0<\theta<1$ is a solution of the system (2) and if the assumptions (3)–(5) hold, then $\forall\, B(3\sigma) = B(x^0,3\sigma) \subset\subset \Omega$ it result*

$$u \in H^{1+\theta_1}(B(\sigma),\mathbb{R}^N) \quad \forall\theta_1 \in (0,\theta+\frac{\lambda}{2}(1-\theta)) \tag{25}$$

and

$$\begin{aligned} |Du|_{\theta_1,B(\sigma)} \le{}& \frac{c(\nu,k,U,\theta,\theta_1,\lambda,n)}{(1-\sigma)^2} \\ &\cdot\Big\{\int_{B(3\sigma)}[1+|f|^2+|f_0|^2+\|u\|^{2^*}+\|Du\|^2]dx + |Du|^2_{\theta,B(3\sigma)}\Big\}. \end{aligned} \tag{26}$$

Proof. We set $\sigma<1$ such that $B(3\sigma)=B(x^0,3\sigma)\subset\subset\Omega$ and $\psi(x)\in C_0^\infty(\mathbb{R}^n)$ a real function defined in (18).

If we set ρ such that $|\rho| < \frac{\sigma}{2}$, arguing as in the proof of the theorem 6, we obtain,for each integer r, $1\le r\le n$, and for each $\epsilon>0$ (20).

Let us consider the last integral that appears at the right hand side of (20). The hypothesis that $u \in H^{1+\theta}(\Omega,\mathbb{R}^N)\cap C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$, $0<\lambda<1$, $0<\theta<1$ and a well known interpolation theorem [6], means that

$$u \in H^{1,p}(\Omega,\mathbb{R}^N) \quad \forall 2<p<q=\frac{2(1+\vartheta)n}{n-2\vartheta\lambda} \tag{27}$$

and $\forall B(\rho)=B(x^0,\rho)\subset\subset\Omega$

$$\sum_{i=1}^n\int_{B(\rho)}\|D_iu-(D_iu)_{B(\rho)}\|^p \le c(\vartheta,\lambda,n,p)(misB(\rho))^{1-\frac{p}{q}}\left[u\right]^{\frac{p\vartheta}{1+\vartheta}}_{\lambda,\overline{\Omega}}|Du|^{\frac{p}{1+\vartheta}}_{\vartheta,B(\rho)}. \tag{28}$$

Arguing as in the proof of the theorem 3.II of [6], we obtain

$$|u|^{2(1+\vartheta)}_{1,p,B(\rho)} \le c(k,U,\vartheta,\lambda,\rho,n,p)\{1+|Du|^2_{\vartheta,B(\rho)}\} \tag{29}$$

(5) see [6], lemma 2.3

(6) See [5], theorem 2.1

In particular (27) hold $\forall p$ such that $2 < p < 4 \wedge q$, and then, $\forall \varepsilon > 0$

(30)

$$
\begin{aligned}
C \leq & \Big\{ \int_{B(2\sigma)} |\rho|^{p-2}[|f_0(x)| + c(k)\big(\|u\|^\alpha + \|Du\|^\gamma\big)]^{\frac{p}{2}} \cdot \|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^{2-\frac{2}{p}} dx \Big\}^{\frac{2}{p}} \\
& \cdot \Big\{ \int_{B(2\sigma)} |\rho|^{-2}\|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^2 dx \Big\}^{1-\frac{2}{p}} \\
& \leq \frac{\varepsilon}{2}|\rho|^{-2}\int_{B(2\sigma)} \|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^2 dx + c(k,\varepsilon,p)|\rho|^{p-2}\int_{B(2\sigma)} \|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^{2-\frac{p}{2}} \\
& \cdot \big(|f_0(x)| + \|u\|^\alpha + \|Du\|^\gamma\big)^{\frac{p}{2}} dx
\end{aligned}
$$

Moreover by lemma 2.II of [5], we have

$$
\begin{aligned}
(31) \qquad & \frac{\varepsilon}{2}|\rho|^{-2}\int_{B(2\sigma)} \|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^2 dx \leq \frac{\varepsilon}{2}\int_{B(\frac{5}{2}\sigma)} \|D(\psi^2\tau_{r,\rho}u)\|^2 dx \\
& \leq \varepsilon \int_{B(2\sigma)} \psi^4\|\tau_{r,\rho}Du\|^2 dx + c(\sigma,\varepsilon)\int_{B(\frac{5}{2}\sigma)} \psi^2\|\tau_{r,\rho}u\|^2 dx \\
& \leq \varepsilon \int_{B(2\sigma)} \psi^2\|\tau_{r,\rho}Du\|^2 dx + c(\sigma,\varepsilon)|\rho|^2\int_{B(3\sigma)} \|Du\|^2 dx
\end{aligned}
$$

and, taking into account that $u \in C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$

$$
\begin{aligned}
& c(\varepsilon,k,p)|\rho|^{p-2}\int_{B(2\sigma)} [f_0(x) + \|u\|^\alpha + \|Du\|^\gamma]^{\frac{p}{2}} \cdot \|\tau_{r,-\rho}(\psi^2\tau_{r,\rho}u)\|^{2-\frac{p}{2}} dx \\
& \leq c(\varepsilon,k,p,n,U,\sigma)|\rho|^{p-2+\lambda(2-\frac{p}{2})}\int_{B(2\sigma)} [1 + |f_0(x)|^2 + \|u\|^{2\alpha} + \|Du\|^p] dx \leq \\
& \leq c(\varepsilon,k,p,n,U,\sigma)|\rho|^{p-2+\lambda(2-\frac{p}{2})}\int_{B(3\sigma)} [1 + |f_0(x)|^2 + \|u\|^{2^*} + \|Du\|^p] dx
\end{aligned}
$$

From (30)–(32), if we set $\varepsilon = \dfrac{\nu}{4}$ and $p = 2(1+\vartheta)$ we have, by (29) :

$$
\begin{aligned}
(32) \quad C \leq & \frac{\nu}{4}\int_{B(2\sigma)} \psi^2\|\tau_{r,\rho}Du\|^2 dx + c(\nu,k,\lambda,U,\vartheta,\sigma,n)|\rho|^{2\vartheta+\lambda(1-\vartheta)} \\
& \cdot \int_{B(3\sigma)} [1 + |f_0(x)|^2 + |f|^2 + \|u\|^{2^*} + \|Du\|^2] dx + |Du|^2_{\vartheta,B(3\sigma)}.
\end{aligned}
$$

By (20) and (33) we obtain

$$
\begin{aligned}
(33) \quad \sum_{r=1}^{n}\int_{B(\sigma)} \|\tau_{r,\rho}Du\|^2 dx \leq & \frac{c(\nu,k,\lambda,U,\vartheta,\sigma,n)}{(1-\sigma)^2}|\rho|^{2\vartheta+\lambda(1-\vartheta)} \\
& \cdot \int_{B(3\sigma)} [1 + |f_0|^2 + |f|^2 + \|u\|^{2^*} + \|Du\|^2] dx + |Du|^2_{\vartheta,B(3\sigma)}
\end{aligned}
$$

taking into account (17) and then, arguing as in the proof of theorem 6, we obtain (25) and (26). $\square$

Subsequently, with iterative procedure and an interpolation theorem[(7)] we obtain a still better local regularity result:

Theorem 8. *If* $u \in H^1(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$ *is a solution of the system (2) and if the conditions (3)–(5), hold then* $\forall\, B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$

$$u \in H^{1+\vartheta}(B(\sigma), \mathbb{R}^N) \quad \forall \vartheta \in (0,1) \tag{34}$$

and

$$|Du|^2_{\vartheta, B(\sigma)} \leq \frac{c(\nu, k, U, \vartheta, \lambda, n)}{(1-\sigma)^2} \Big\{ \int_{B(3\sigma)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx \Big\}. \tag{35}$$

Proof. Set $\vartheta \in (0,1)$; if we assume $\vartheta_0 = \dfrac{\lambda}{4}$,then $i \in N$, $i = i(\vartheta, \lambda)$ exists, such that

$$\vartheta_i = \vartheta_0 \sum_{r=0}^{i} (1 - \vartheta_0)^r \in (0,1)$$

then, by theorem 6 we deduce

$$\forall B(3\rho) \subset\subset \Omega, \quad u \in H^{1+\vartheta_0}(B(\rho), \mathbb{R}^N) \cap C^{0,\lambda}(\overline{B(\rho)}, \mathbb{R}^N)$$

and

$$|Du|^2_{\vartheta_0, B(\rho)} \leq \frac{c(\nu, k, \sigma, \vartheta, \lambda, n)}{(1-\rho)^2} \int_{B(3\rho)} [1 + |f|^2 + |f_0| + \|u\|^{2^*} + \|Du\|^2] dx.$$

By theorem 7

$$u \in H^{1+\vartheta_1}(B(3^{-2}\rho), \mathbb{R}^N) \cap C^{0,\lambda}(\overline{B(3^{-2}\rho)}, \mathbb{R}^N)$$

and

$$|Du|^2_{\vartheta_1, B(\rho)} \leq \frac{c(\nu, k, U, \vartheta, \lambda, n)}{(1-\rho)^2} \int_{B(3\rho)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx$$

with $\vartheta_1 = \vartheta_0 + \vartheta_0(1 - \vartheta_0) = \vartheta_0 \sum_{r=0}^{1} (1 - \vartheta_0)^r < \vartheta_0 + \dfrac{\lambda}{2}(1 - \vartheta)$.
If we repeat this process we deduce

$$u \in H^{1+\vartheta_i}(B(3^{-2i}\rho), \mathbb{R}^N) \cap C^{0,\lambda}(\overline{B(3^{-2i}\rho)}, \mathbb{R}^N) \tag{36}$$

and

$$|Du|^2_{\vartheta_i, B(3^{-2i}\rho)} \leq \frac{c(\nu, k, U, \vartheta, \lambda, n)}{(1-\rho)^2} \int_{B(3\rho)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx$$

and by lemma 1.III of [6] we have $\forall B(\sigma) \subset\subset B(\sigma_0) \subset\subset \Omega$

$$|Du|^2_{\vartheta_i, B(\sigma)} \leq \frac{c(\nu, k, U, \vartheta, \lambda, n)}{(1-\sigma)^2} \int_{B(\sigma_0)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx \tag{37}$$

and then we have the thesis. □

Finally we prove the searched local differentiability result :

(7) See [7] lemma 2.IV

Theorem 9. *If* $u \in H^1(\Omega, \mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega}, \mathbb{R}^N)$, $0 < \lambda < 1$ *is a solution of the system (2) and if the assumptions (3)–(5) hold, then* $\forall\, B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$ *it results*

$$u \in H^2(B(\sigma), \mathbb{R}^N) \tag{38}$$

and the following estimate holds

$$|u|^2_{2,B(\sigma)} \le \frac{c(\nu, k, U, \lambda, n)}{(1-\sigma)^2} \cdot \Big\{ \int_{B(3\sigma)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx \Big\}. \tag{39}$$

Proof. Fixed $B(3\sigma) = B(x^0, 3\sigma) \subset\subset \Omega$, $h \in \mathbb{R}$ such that $|h| < \dfrac{\sigma}{2}$, let $\psi(x)$ be a real function $\in C_0^\infty(\mathbb{R}^n)$ fulfilling (18). Arguing as in the proof of theorem 3.I of [8], we obtain (20).
Let us consider the last integral that appears at the right side of (20). By Corollario 2.I of [9], with $\Omega = B(\frac{5}{2}\sigma)$ and $\vartheta = 1 - \dfrac{\lambda}{2}$, we have that

$$u(x,t) \in W^{1,p}(B(\frac{5}{2}\sigma), \mathbb{R}^N)$$

and

$$\|u\|_{1,p,B(\frac{5}{2}\sigma)} \le c(\lambda, \sigma, n) \|u\|^{\frac{1}{2}}_{2-\frac{\lambda}{2}, B(\frac{5}{2}\sigma)} \cdot \|u\|^{\frac{1}{2}}_{C^{0,\lambda}(B(\frac{5}{2}\sigma), \mathbb{R}^N)} \tag{40}$$

where $p = 4 + \dfrac{4\lambda}{n - \lambda}$.

Now, since $p > 4$, $H^{1,p}(B(\frac{5}{2}\sigma), \mathbb{R}^N) \subset H^{1,4}(B(\frac{5}{2}\sigma), \mathbb{R}^N)$ and then $u \in H^{1,4}(B(\frac{5}{2}\sigma), \mathbb{R}^N)$ and by (28)

$$\begin{aligned} \|u\|^4_{1,4,B(\frac{5}{2}\sigma)} &\le c(k, U, \lambda, \sigma, n) \{1 + |u|^2_{1,B(\frac{5}{2}\sigma)} + |Du|^2_{1-\frac{\lambda}{2}, B(\frac{5}{2}\sigma)}\} \\ &\le \frac{c(\sigma, k, U, \lambda, n)}{(1-\sigma)^2} \Big\{ \int_{B(3\sigma)} [1 + |f|^2 + |f_0|^2 + \|u\|^{2^*} + \|Du\|^2] dx \Big\} \end{aligned} \tag{41}$$

Now we observe that, taking into account (42), $\forall \varepsilon > 0$ we have

$$\int_{B(2\sigma)} \{|f_0(x)| + c(k)[\|u\|^\alpha + \sum_j \|p^j\|^\gamma]\} \|\tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)\| dx \tag{42}$$

$$\le \Big(\int_{B(2\sigma)} |\rho|^2 \{|f_0(x)| + c(k)[\|u\|^\alpha + \|Du\|^\gamma]\}^2 \Big)^{\frac{1}{2}} \cdot \Big(\int_{B(2\sigma)} |\rho|^{-2} \|\tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)\|^2 dx \Big)^{\frac{1}{2}}$$

$$\le \frac{\varepsilon}{2} |\rho|^{-2} \int_{B(2\sigma)} \|\tau_{r,-\rho}(\psi^2 \tau_{r,\rho} u)\|^2 dx + c(k, \varepsilon) |\rho|^2 \int_{B(2\sigma)} \{|f_0(x)| + \|u\|^\alpha + \|Du\|^\gamma\}^2 dx$$

$$\le \varepsilon \int_{B(2\sigma)} \psi^2 \|\tau_{r,\rho} Du)\|^2 dx + c(k, \sigma, \varepsilon) |\rho|^2$$

$$\cdot \int_{B(2\sigma)} \{1 + |f_0(x)|^2 + \|u\|^{2\alpha} + \|Du\|^2 + \|Du\|^{2\gamma}\}^2 dx^{(8)}$$

$$\le \varepsilon \int_{B(2\sigma)} \psi^2 \|\tau_{r,\rho} Du)\|^2 dx + c(k, \sigma, \varepsilon) |\rho|^2 \Big[\int_{B(3\sigma)} \{1 + |f_0(x)|^2 + \|u\|^{2^*} + \|Du\|^2\}^2 dx$$

$$+ c(\sigma)\int_{B(2\sigma)} \|Du\|^4 dx\Big] \le \varepsilon \int_{B(2\sigma)} \psi^2 \|\tau_{r,\rho} Du)\|^2 dx + \frac{c(\sigma,k,U,\lambda,n)}{(1-\sigma)^2}|\rho|^2$$

$$\cdot \int_{B(3\sigma)} [1 + |f_0(x)|^2 + |f(x)|^2 + \|u\|^{2^*} + \|Du\|^2] dx.$$

Finally by (20) and (43), applying usually a well known theorem[9] we obtain (39), (40). □

From Theorem 9 we can immediately deduce the interior differentiability result and afterwards, arguing as in the first part of this paper, the differentiability result near the boundary and finally the following global differentiability result :

Theorem 10. *Let* $u \in H^1(\Omega,\mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$ $(0 < \lambda < 1)$ *be a solution of the Dirichlet problem*

$$u - g \in H^1_0(\Omega,\mathbb{R}^N)$$

$$-\sum_i D_i a^i(x,u,Du) = B^0(x,u,Du)$$

if Ω *is of class* C^2 *and* $g \in H^2(\Omega,\mathbb{R}^N) \cap C^{0,\lambda}(\overline{\Omega},\mathbb{R}^N)$, *if the conditions (3),(4),(5) are verified, then it results*

$$u \in H^2(\Omega,\mathbb{R}^N)$$

and we have

$$\int_\Omega \sum_{ij} \|D_{ij}u\|^2 dx \le c(\nu,U,k,\lambda,n)\int_\Omega \big[1 + |f|^2$$

$$+ |f_0|^2 + \|u\|^{2^*} + \|g\|^{2^*} + \|Du\|^2 + \|Dg\|^2 + \|D^2 g\|^2\big] dx$$

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(8) by (31)

(9) See theorem 3.X cap.1 of [2]

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Manuscript received March 21, 2006
revised November 3, 2006

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