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DUALITY THEORY FOR A WALRASIAN EQUILIBRIUM PROBLEM

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Abstract. We consider a Walrasian equilibrium for a pure exchange economy. The equilibrium conditions are expressed in terms of a quasi-variational inequality for which, by recent results, the existence of solution is provided. Moreover, the Generalized Lagrangean and Duality theories are studied and, as an interesting consequence, we obtain the Lagrangean variables. This theory plays an extraordinary role, in fact it allows us to describe the behavior of the market. We also present an example and we find the equilibrium solution by means of the Lagrangean multipliers.

1. Introduction

In this paper we consider a competitive economic equilibrium problem with utility function, which is a particular case of a general economic equilibrium problem. It was Leon Walras [14] who in 1874 laid the fundamental ideas for the study of the general equilibrium theory, providing a succession of models, each taking into account more aspects of a real economy. The rigorous mathematical formulation of this problem, with possibly nonsmooth but convex data, was elaborated by Arrow and Debreu [1] in the 1954. There are several authors who applied themselves to the study of the equilibrium for a competitive economy [2, 3, 12, 13, 11]. In the articles [5] and [6] we showed that the Walrasian equilibrium of pure exchange economic market can be incorporated directly into a quasi-variational inequality model; furthermore we provided existence theorems by using variational approach. In this paper we characterize the equilibrium solution in terms of Lagrangean multipliers by applying the duality theory. It is worth remarking that the Lagrangean theory provides some interesting contributions, needed for a better understanding and handling of several equilibrium problems, for example the obstacle problem, the discrete and continuum traffic equilibrium problem, the spatial price problem, the financial problem ... (see e.g. [9] within its contained bibliography and [4, 7, 10]). In fact, the Lagrangean variables have an intrinsic meaning to the nature of the problems considered. Moreover it is essential to obtain the equivalence between the equilibrium conditions and the variational inequality. We also give an example where we are able to find the competitive equilibrium by means of Lagrangean multipliers.

2. Walrasian pure exchange model

We consider a marketplace consisting of l different goods indexed by $j = 1, \dots, l$ and n agents indexed by $a = 1, \dots, n$. Each agent $a = 1, 2, \dots, n$ has an initial endowment vector:

$$e_a = (e_a^1, e_a^2, \dots, e_a^l) \in \mathrm{R}_+^l.$$

We denote by x_a^j the consumption by agent a of goods j and represent with:

$$x_a = (x_a^1, x_a^2, \ldots, x_a^l) \in \mathrm{R}_+^l$$

the consumption choice vector and with:

$$x \equiv (x_1, x_2, \ldots, x_n)^T \in \mathrm{R}_+^{nl}$$

the consumption of market. In this economy there is only pure exchange, without production, that is the only activity that agents can perform is to consume and/or trade their commodities with each other. We presume that the "law of one price" is fulfilled, that is, traders scope out opportunities to the extent that each goods is sold and purchased at only one price. Each goods j, $j = 1, 2, \ldots, l$ associates with it a real positive number p^j representing its price and we denote by

$$p = (p^1, p^2, \ldots, p^l) \in \mathrm{R}_+^l$$

the price vector. We also presume a competitive behavior, that is, agents do not perceive that they can have any influence over these market prices. Competitive equilibrium price vector, which we denote by $\overline{p}$, is price at which every agent can simultaneously satisfy his desire to trade. As is standard in economic theory, the choice by the consumer from a given set of alternative consumption vectors is supposed to be made in accordance with a preference scale for which there is an utility function:

$$u_a : \mathrm{R}_+^l \to \mathrm{R}$$

$$\mathrm{R}_+^l \ni x_a \to u_a(x_a) \in \mathrm{R}.$$

In this market, the objective of each of the agents is to maximize their utility by performing pure exchanges of the given goods. There are natural constraints that the consumers must satisfy: the wealth of a consumer is his initial endowment, and the total amount of goods that a consumer can acquire or buy is at most equal to his initial wealth, i. e. the goods that the consumer sells off. This means that, for all $a = 1, \ldots, n$ and for all $p \in P$:

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(p)} u_a(x_a), \tag{1}$$

where

$$M_a(p) = \{x_a \in \mathrm{R}^l : \ x_a^j \geq 0 \ \forall j = 1, \ldots, l, \ \sum_{j=1}^{l} p^j (x_a^j - e_a^j) \leq 0\}, \quad \forall a = 1, \ldots, n,$$

and

$$p \in P = \Big\{p \in \mathrm{R}_+^l : \sum_{j=1}^{l} p^j = 1\Big\}.$$

For each $a = 1, \ldots, n$ and $p \in P$, $M_a(p)$ is a closed and convex set of R_+^l.

We define a particular aggregate excess demand function:

$$z^j : \mathrm{R}_+^{nl} \to \mathrm{R}, \quad j = 1, 2, \ldots, l$$

$$x \to z^j(x) = \sum_{a=1}^{n} (x_a^j - e_a^j)$$

where $x_a^j - e_a^j$ is the individual excess demand by agent a for goods j. Grouping this components in the vector we introduce:

$$z(x) = (z^1(x), z^2(x), \ldots, z^l(x)) \in \mathrm{R}^l.$$

Furthermore, for all $a = 1, \ldots, n$, we assume that:

(U_1) u_a is strictly concave,
(U_2) $u_a \in C^1(\mathrm{R}_+^l)$ in the usual sense,
(U_3) $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ and
$\forall x_a \in \partial M_a(p) : \dfrac{\partial u_a(x_a)}{\partial x_a^s} > 0$, when $x_a^s = 0, \quad \forall p \in P$,
(U_4) $\lim\limits_{\substack{||x_a|| \to +\infty, \\ x_a \in M_a(p)}} u_a(x_a) = -\infty$,
(U_5) Each agent is endowed at least of a positive quantity of goods:

$$\forall\, a = 1, \ldots, n \quad \exists j : \quad e_a^j > 0,$$

and for every goods j there exists at least an agent a such that $e_a^j > 0$.

In our assumptions, for all $a = 1, \ldots, n$, the maximization problem (1) has a unique solution for each $p \in P$, then it arises a function $\overline{x}_a(p)$ from P to R_+^l. So, we can define $z(x(p)) : P \to \mathrm{R}$ and in the following we will continue to denote with $z(p)$ the composite function $z(p) = z(x(p))$.

Then the competitive equilibrium condition of a pure exchange economic market takes the following form:

Definition 1. Let $\overline{p} \in P$ and $\overline{x}(\overline{p}) \in M(\overline{p}) = \Pi_{a=1}^n M_a(\overline{p})$. The pair $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ is a competitive equilibrium if and only if: for all $a = 1, \ldots, n$

$$u_a(\overline{x}_a(\overline{p})) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{2}$$

and for all $j = 1, 2, \ldots, l$:

$$z^j(\overline{x}(\overline{p})) = \sum_{a=1}^{n} (\overline{x}_a^j(\overline{p}) - e_a^j) \leq 0. \tag{3}$$

The vector $\overline{p}$ is the competitive equilibrium price.
For sake of brevity in the sequel we will write $\overline{x}$ instead of $\overline{x}(\overline{p})$.

In the work [5] we have proved that, in our assumptions, the market is regulated by Walras' law:

$$\sum_{j=1}^{l} p^j(\overline{x}_a^j(p) - e_a^j) = 0 \quad \forall p \in P, \;\; \forall a = 1, \ldots, n, \tag{4}$$

hence it is possible reformulate the equilibrium in the following way:

Definition 2. *A competitive equilibrium of a pure exchange economic market with utility function consists of a competitive equilibrium price vector $\overline{p} \in P$ and a consumption vector $\overline{x} \in \mathrm{R}_+^{nl}$ such that:*

a) *for all* $a = 1, \dots, n$, $\overline{x}_a$ *is a solution to maximization problem (2) and*

$$\sum_{j=1}^{l} \overline{p}^j (\overline{x}_a^j - e_a^j) = 0. \tag{5}$$

b) *For all* $j = 1, \dots, l$*:*

$$\sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \begin{cases} \leq 0 & \text{if } \overline{p}^j = 0 \\ = 0 & \text{if } \overline{p}^j > 0. \end{cases} \tag{6}$$

Problem (2) states that the consumption choice vector x_a of agent a must be such that his utility $u_a(x_a)$ is maximized, and the choice is subjected to the constraint that the amount that the agent a pays for acquiring the goods x_a, $\sum_{j=1}^{l} \overline{p}^j x_a^j$, is at must the amount that the agent receives for his initial endowment, $\sum_{j=1}^{l} \overline{p}^j e_a^j$. Condition (5) states that the amount that the agent a pays for acquiring the good that maximized his utility: $\sum_{j=1}^{l} \overline{p}^j \overline{x}_a^j$, is equal to the amount that the agent received for his initial endowment: $\sum_{j=1}^{l} \overline{p}^j e_a^j$. Condition (6) states that the market is usually considered to be in equilibrium when, for a goods, the supply equals the demand; but, there exists the possibility that at the zero price, the supply will exceed the demand. This is the classical case of the free goods.

In the work [5] we have proved that the competitive equilibrium of a pure exchange economic market is characterized as a solution to the quasi-variational inequality:

"Find $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *such that:*

$$\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \rangle_l + \sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_l \leq 0 \tag{7}$$

$$\forall\ (p, x) \in P \times M(\overline{p})\text{"},$$

in fact the following result holds:

Theorem 1. *The pair* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *is a competitive equilibrium of a pure exchange economic market with utility function if and only if is a solution to quasi-variational inequality (7).*

Proof. See e.g. [5]. □

In the work [6] we have proved the following existence theorem:

Theorem 2. *Let* $(-\nabla u_a(x_a))$ *be an operator such that:*

$$\langle -\nabla u_a(x_a) + \nabla u_a(y_a), x_a - y_a \rangle \geq \nu ||x_a - y_a||^2 \quad \forall x_a,\ y_a \in M_a(p).$$

Then there exists $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *solution to quasi variational inequality (7).*

Proof. See e.g. [6]. □

3. Lagrangean and Duality Theory

In this section our purpose is to give a characterization to the competitive equilibrium of a pure exchange economic market in terms of the Lagrangean multipliers, which, as it's well known, play a very important role in economic theory. To this end we can prove the following result:

Theorem 3. *$(\overline{p},\overline{x}) \in P \times M(\overline{p})$ is a competitive equilibrium of a pure exchange economic market if and only if there exist $\overline{\alpha} = (\overline{\alpha}_1,\dots,\overline{\alpha}_a,\dots,\overline{\alpha}_n)$, $\overline{\beta} = (\overline{\beta}_1,\dots,\overline{\beta}_a,\dots,\overline{\beta}_n)$, $\overline{\gamma} = (\overline{\gamma}^1,\dots,\overline{\gamma}^l)$ and $\overline{\delta}$ such that:*

i) $\overline{\alpha}_a \in \mathbb{R}^l_+$, $\quad \overline{\beta}_a \in \mathbb{R}_+ \setminus \{0\}$, $\quad$ *for all* $a = 1,\dots,n$
$\overline{\gamma} \in \mathbb{R}^l_+$, $\quad \overline{\delta} \in \mathbb{R}_+$

ii) *for all* $a = 1,\dots,n \quad \langle \overline{\alpha}_a, \overline{x}_a\rangle = 0, \quad \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a\rangle = 0; \quad \langle \overline{\gamma}, \overline{p}\rangle = 0;$

iii)
$$\begin{cases} \dfrac{\partial u_a(\overline{x}_a)}{\partial x_a^j} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}_a^j & \forall a = 1,\dots,n \quad \forall j = 1,\dots,l; \\ \displaystyle\sum_{a=1}^n (e_a^j - \overline{x}_a^j) = \overline{\gamma}^j & \forall j = 1,\dots,l \\ \overline{\delta} = 0 & \\ \overline{\beta}_a = \displaystyle\sum_{j=1}^l \Big(\frac{\partial u_a(\overline{x}_a)}{\partial x_a^j} + \overline{\alpha}_a^j \Big) & \forall a = 1,\dots,n. \end{cases}$$

Proof. Let $(\overline{p},\overline{x})$ a competitive equilibrium of a pure exchange economic market; then $(\overline{p},\overline{x})$ is a solution to quasi-variational inequality (7). We have that, for all $a = 1,\dots,n$, $\overline{x}_a$ is a solution to:

$$\langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a\rangle_l \geq 0 \quad \forall x_a \in M_a(\overline{p}) \tag{8}$$

and $\overline{p}$ is a solution to:

$$\langle \sum_{a=1}^n (e_a - \overline{x}_a), p - \overline{p}\rangle_l \geq 0 \quad \forall p \in P. \tag{9}$$

Being $\overline{x}_a \in M_a(\overline{p})$ a solution to variational inequality (8) for all $a = 1,\dots,n$, it results:

$$\min_{x_a \in M_a(\overline{p})} \Phi_{\overline{x}_a}(x_a) = \Phi_{\overline{x}_a}(\overline{x}_a) = 0$$

where

$$\Phi_{\overline{x}_a}(x_a) = \langle -\nabla u_a(\overline{x}_a), x_a - \overline{x}_a\rangle_l \quad \forall x_a \in \mathbb{R}^l.$$

For all $a = 1,\dots,n$, we associate to the variational inequality (8) the following Lagrange function:

$$\begin{aligned} \mathcal{L}_a^{(1)}(x_a,\alpha_a,\beta_a) &= \Phi_{\overline{x}_a}(x_a) - \langle \alpha_a, x_a\rangle - \beta_a \langle \overline{p}, e_a - x_a\rangle \\ &= -\sum_{j=1}^l \frac{\partial u_a(\overline{x}_a)}{\partial x_a^j}(x_a^j - \overline{x}_a^j) - \sum_{j=1}^l \alpha_a^j x_a^j - \beta_a \sum_{j=1}^l \overline{p}^j (e_a^j - x_a^j) \\ &\qquad \forall (x_a,\alpha_a,\beta_a) \in \mathbb{R}^l \times \mathbb{R}^l_+ \times \mathbb{R}_+ . \end{aligned}$$

From Theorem 5 of Appendix, there exists $(\overline{\alpha}_a, \overline{\beta}_a) \in \mathbb{R}^l_+ \times \mathbb{R}_+$ such that $(\overline{x}_a, \overline{\alpha}_a, \overline{\beta}_a)$ is a saddle point of $\mathcal{L}^{(1)}_a$ and furthermore:

$$0 = \mathcal{L}^{(1)}_a(\overline{x}_a, \overline{\alpha}_a, \overline{\beta}_a) = \Phi_{\overline{x}_a}(\overline{x}_a) - \langle \overline{\alpha}_a, \overline{x}_a \rangle - \overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle.$$

Being $\overline{x}_a \in M_a(\overline{p})$ and $\overline{\alpha}_a \in \mathbb{R}^l_+, \overline{\beta}_a \in \mathbb{R}_+$ it must be:

$$\langle \overline{\alpha}_a, \overline{x}_a \rangle = 0 \tag{10}$$

and

$$\overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0 \tag{11}$$

Taking into account of (24) it results:

$$\mathcal{L}^{(1)}_a(x_a, \overline{\alpha}_a, \overline{\beta}_a) \geq 0 = \mathcal{L}^{(1)}_a(\overline{x}_a, \overline{\alpha}_a, \overline{\beta}_a) \quad \forall x_a \in \mathbb{R}^l$$

namely $\mathcal{L}^{(1)}_a(\cdot\ , \overline{\alpha}_a, \overline{\beta}_a)$ assumes the minimal value in $\overline{x}_a$; then:

$$\frac{\partial \mathcal{L}^{(1)}_a}{\partial x^j_a}(\overline{x}_a \overline{\alpha}_a, \overline{\beta}_a) = -\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} - \overline{\alpha}^j_a + \overline{\beta}_a \overline{p}^j = 0 \quad \forall j = 1, \ldots, l,$$

so:

$$\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}^j_a \quad \forall j = 1, \ldots, l. \tag{12}$$

Summing (12) for all $j = 1, \ldots, l$, because $\sum_{j=1}^l \overline{p}^j = 1$, it results:

$$\overline{\beta}_a = \sum_{j=1}^l \Big(\frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} + \overline{\alpha}^j_a \Big) \quad \forall a = 1, \ldots, n.$$

We can observe that $\overline{\beta}_a \neq 0$ for all $a = 1, \ldots, n$; in fact, if it results $\overline{\beta}_a = 0$, from first of *iii*),

$$-\overline{\alpha}^j_a = \frac{\partial u_a(\overline{x}_a)}{\partial x^j_a} \leq 0, \tag{13}$$

for all $j = 1, \ldots, l$. From first of *ii*): $\dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} \cdot \overline{x}^j_a = 0$ for all $j = 1, \ldots, l$. If $\overline{x}^j_a > 0\ \forall j = 1, \ldots, l$, $\nabla u_a(\overline{x}_a) = 0$ in contradiction with (U_3); if there exists j such that $\overline{x}^j_a = 0$, from (U_3), $\dfrac{\partial u_a(\overline{x}_a)}{\partial x^j_a} > 0$ in contradiction with (13).

Being $\overline{p} \in P$ a solution to variational inequality (9) we have:

$$\min_{p \in P} \Psi_{\overline{p}}(p) = \Psi_{\overline{p}}(\overline{p}) = 0$$

where

$$\Psi_{\overline{p}}(p) = \langle \sum_{a=1}^n (e_a - \overline{x}_a), p - \overline{p} \rangle_l \quad \forall p \in \mathbb{R}^l.$$

We associate to the variational inequality (9) the following Lagrange function:

$$\mathcal{L}^{(2)}(p, \gamma, \delta) = \Psi_{\overline{p}}(p) - \langle \gamma, p \rangle - \delta \Big(\sum_{j=1}^l p^j - 1 \Big)$$

$$= \sum_{j=1}^{l} \Big(\sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) \Big) (p^j - \overline{p}^j) - \sum_{j=1}^{l} \gamma^j p^j - \delta \Big(\sum_{j=1}^{l} p^j - 1 \Big)$$
$$\forall (p, \gamma, \delta) \in \mathbb{R}^l \times \mathbb{R}_+^l \times \mathbb{R}_+ \,.$$

From Theorem 7 of Appendix, there exists $(\overline{\gamma}, \overline{\delta}) \in \mathbb{R}_+^l \times \mathbb{R}_+$ such that $(\overline{p}, \overline{\gamma}, \overline{\delta})$ is a saddle point of $\mathcal{L}^{(2)}$ and furthermore:

$$0 = \mathcal{L}^{(2)}(\overline{p}, \overline{\gamma}, \overline{\delta}) = \Psi_{\overline{p}}(\overline{p}) - \langle \overline{\gamma}, \overline{p} \rangle - \overline{\delta} \Big(\sum_{j=1}^{l} \overline{p}^j - 1 \Big).$$

Being $\overline{p} \in P$ and $\overline{\gamma} \in \mathbb{R}_+^l, \overline{\delta} \in \mathbb{R}_+$ it must be:

$$\langle \gamma, p \rangle = 0. \tag{14}$$

From condition (24) it results:

$$\mathcal{L}^{(2)}(p, \overline{\gamma}, \overline{\delta}) \geq 0 = \mathcal{L}^{(2)}(\overline{p}, \overline{\gamma}, \overline{\delta}) \quad \forall p \in \mathbb{R}^l \,,$$

namely $\mathcal{L}^{(2)}(\cdot\ , \overline{\gamma}, \overline{\delta})$ assumes the minimal value in $\overline{p}$; then:

$$\frac{\partial \mathcal{L}^{(2)}}{\partial p^j}(\overline{p}, \overline{\gamma}, \overline{\delta}) = \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) - \overline{\gamma}^j - \overline{\delta} = 0 \quad \forall j = 1, \ldots, l,$$

namely:

$$\overline{\gamma}^j = \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) - \overline{\delta}, \quad \forall j = 1, \ldots, l. \tag{15}$$

By (14) and (15) and by Walras' law we have:

$$0 = \langle \overline{\gamma}, \overline{p} \rangle = \sum_{a=1}^{n} \sum_{j=1}^{l} \overline{p}^j (e_a^j - \overline{x}_a^j) - \overline{\delta} \sum_{j=1}^{l} \overline{p}^j = \overline{\delta}.$$

Then:

$$\overline{\gamma}^j = \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j), \quad \forall j = 1, \ldots, l, \quad \overline{\delta} = 0 \,.$$

We observe that for all $a = 1, \ldots, n$ there exists j^* such that $\overline{\alpha}_a^{j^*} = 0$ and for $j = 1, \ldots, l$ there exists a^* such that $\overline{\alpha}_{a^*}^j = 0$. In fact: if $\overline{\alpha}_a^j > 0$ for all $j = 1, \ldots, l$, from (10) $\overline{x}_a^j = 0$ and from (U_3) we derive $\dfrac{\partial u_a x_a}{\partial x_a^j} > 0$ and from (12) $\overline{p}^j > 0$. From the third equality of $ii)$ it derives $\overline{\gamma}^j = 0$ and hence , from (15), $e_a^j = 0$ for all $j = 1, \ldots, l$, in contradiction with (U_5). Analogously, if $\overline{\alpha}_a^j > 0$ for all $a = 1, \ldots, n$, from (15), $e_a^j = 0$ for all $a = 1, \ldots, n$, namely the market is not endowed of good j in contradiction with (U_5).

Conversely, we suppose that there exist $\overline{x}, \overline{p}, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ that satisfy the conditions $i)$, $ii)$, $iii)$, we prove that $(\overline{x}, \overline{p})$ is a competitive equilibrium. Because $\overline{\beta}_a > 0$ for all $a = 1, \dots, n$, from $\overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0$ it follows $\overline{x}_a \in M_a(\overline{p})$. From *iii*) and *ii*), for all $a = 1, \dots, n$ and for all $x_a \in M_a(\overline{p})$, we have:

$$\langle \nabla u_a(\overline{x}_a), x - \overline{x}_a \rangle = \sum_{j=1}^{l} \overline{\beta}_a \overline{p}^j x_a^j - \sum_{j=1}^{l} \overline{\beta}_a \overline{p}^j \overline{x}_a^j - \overline{\alpha}^j x_a^j$$

Because $x_a \in M_a(\overline{p})$, from *ii*), it results:

$$\langle \nabla u_a(\overline{x}_a), x - \overline{x}_a \rangle \leq \sum_{j=1}^{l} \overline{\beta}_a \overline{p}^j (e_a^j - \overline{x}_a^j) - \langle \overline{\alpha}, \overline{x}_a \rangle = -\langle \overline{\alpha}, \overline{x}_a \rangle \leq 0;$$

namely $\overline{x}_a(\overline{p})$ is a solution to variational inequality (8), so, by well known results $\overline{x}_a$ satisfies the condition (2).

Now, by comparing first and last of *iii*), it follows that $\sum_{j=1}^{l} \overline{p}^j = 1$, that is $\overline{p} \in P$. Being:

$$\overline{\gamma}^j = \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) \geq 0, \quad \forall j = 1, \dots, l,$$

and $\langle \overline{\gamma}, \overline{p} \rangle = 0$, it results:

$$\langle \overline{\gamma}, \overline{p} \rangle = \sum_{j=1}^{l} \sum_{a=1}^{n} (e_a^j - \overline{x}_a^j) \overline{p}^j = 0;$$

then also the condition (6) holds. □

Remark. The importance of the multipliers α^*, γ^*, derives from the fact that they are able to describe the behavior of the market: if there exists a^* and j^* such that $\overline{\alpha}_{a^*}^{j^*} > 0$, from the first of *ii*), $\overline{x}_{a^*}^{j^*} = 0$. By (U_3), $\dfrac{\partial u_{a^*}(\overline{x}_{a^*})}{\partial x_{a^*}^{j^*}} > 0$, then, by (12), we have $\overline{p}^{j^*} > 0$. Analogously, if there exists j^* such that $\overline{\gamma}^{j^*} > 0$, from the second of *ii*), $\overline{p}^{j^*} = 0$. Being, from the first of *iii*), $-\overline{\alpha}_a^{j^*} = \dfrac{\partial u_a(\overline{x}_a)}{\partial x_a^{j^*}} \leq 0 \; \forall \, a = 1, \dots, n$, by (U_3) must be $\overline{x}_a^{j^*} > 0$ for all $a = 1, \dots, n$. Then, from the first of *ii*), $\overline{\alpha}_a^{j^*} = 0$ for all $a = 1, \dots, n$, hence $\dfrac{\partial u_a(\overline{x}_a)}{\partial x_a^{j^*}} = 0 \; \forall \, a = 1, \dots, n$.
By an economic point of view these mean that, if there exists an agent a^* and a goods j^* such that $\overline{\alpha}_{a^*}^{j^*} > 0$, then there is not consumption by agent a^* of goods j^*, the goods j^* has price greater than zero and the agent a^* is not satiated of goods j^* related to his budget constraint set $M_{a^*}(\overline{p})$. If there exists a goods j^* such that $\overline{\gamma}^{j^*} > 0$ then the goods j^* has price equal to zero, all agents consume goods j^* and are satiated of goods j^* related to their budget constraint set.

4. Example

We consider a pure exchange economy with two agent ($a = 1, 2$) and two goods ($j = 1, 2$). Each agent has an utility function:

$$u_1(x_1) = -\frac{1}{2}(x_1^1)^2 - \frac{1}{2}(x_1^2)^2 + 4x_1^1 + 8x_1^2, \quad u_2(x_2) = -\frac{1}{2}(x_2^1)^2 - \frac{1}{2}(x_2^2)^2 + 12x_2^1 + 5x_2^2,$$

and endowment:

$$e_1 = (3,4) \quad e_2 = (7,2)\,.$$

Our aim is to find a competitive equilibrium $(\overline{x}, \overline{p})$ by using sufficient conditions of Theorem 3. From

$$\frac{\partial u_a(\overline{x}_a)}{\partial x_a^j} = \overline{\beta}_a \overline{p}^j - \overline{\alpha}_a^j, \tag{16}$$

conditions $\sum_{a=1}^{n}(e_a^j - \overline{x}_a^j) = \overline{\gamma}^j$ become:

$$\begin{cases} (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^1 = \overline{\gamma}^1 + (\overline{\alpha}_1^1 + \overline{\alpha}_2^1) + 6 \\ (\overline{\beta}_1 + \overline{\beta}_2)\overline{p}^2 = \overline{\gamma}^2 + (\overline{\alpha}_1^2 + \overline{\alpha}_2^2) + 7 \end{cases} \tag{17}$$

From (17) it results $\overline{\beta}_1 + \overline{\beta}_2 > 0$, $\overline{p}^1 > 0$, $\overline{p}^2 > 0$. So $\overline{\gamma}^1 = \overline{\gamma}^2 = 0$, from $\langle \overline{\gamma}, \overline{p} \rangle = 0$, and

$$\begin{cases} \overline{p}^1 = \dfrac{1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) \\ \overline{p}^2 = \dfrac{1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) \end{cases} \tag{18}$$

From (18) and (16) we have:

$$\begin{cases} \overline{x}_1^1 = 4 - \dfrac{\overline{\beta}_1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) + \overline{\alpha}_1^1 \\ \overline{x}_1^2 = 8 - \dfrac{\overline{\beta}_1}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) + \overline{\alpha}_1^2 \\ \overline{x}_1^1 = 12 - \dfrac{\overline{\beta}_2}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^1 + \overline{\alpha}_2^1 + 6) + \overline{\alpha}_2^1 \\ \overline{x}_1^1 = 5 - \dfrac{\overline{\beta}_2}{(\overline{\beta}_1 + \overline{\beta}_2)}(\overline{\alpha}_1^2 + \overline{\alpha}_2^2 + 7) + \overline{\alpha}_2^2 \end{cases} \tag{19}$$

Considering the system $\overline{\beta}_a \langle \overline{p}, e_a - \overline{x}_a \rangle = 0$, $\forall a = 1,2$, we have that, it has solution if and only if $\overline{\alpha}_a^j = 0$ for all $a = 1,2,\ j = 1,2$ and it follows $\overline{\beta}_1 = \frac{34}{51}\overline{\beta}_2$. Moreover from last of iii): $\overline{\beta}_1 + \overline{\beta}_2 = 13$, so $\overline{\beta}_1 = \frac{26}{5}$, $\overline{\beta}_2 = \frac{39}{5}$. Hence the competitive equilibrium $(\overline{p}, \overline{x})$ is:

$$\overline{p} = \left(\frac{6}{13}, \frac{7}{13}\right),\ \overline{x}_1 = \left(\frac{9}{5}, \frac{26}{5}\right),\ \overline{x}_2 = \left(\frac{42}{5}, \frac{4}{5}\right).$$

5. Appendix of Lagrangean and duality theory

In this section for readers' convenience we report the well known results of the duality and Lagrangean theory and the generalized Lagrangean multiplier rule, that we used in the previous section.

First we consider a minimum problem with the present of constraints of the type $g(x) \in -C$. We suppose following assumptions:

$$
(20)\quad \begin{cases} \text{let } (X, ||\cdot||) \text{ be a real linear space;} \\ \text{let } (Y, ||\cdot||) \text{ be a partially ordered real normed space with the} \\ \text{ordering cone } C; \\ \text{let } \widehat{S} \text{ be a nonempty subset of } X; \\ \text{let } f : \widehat{S} \to \mathbb{R} \text{ be a given functional,} \\ \text{let } g : \widehat{S} \to Y \text{ be a given constraint mapping;} \\ \text{let the composite mapping } (f, g) : \widehat{S} \to \mathbb{R} \times Y \text{ be convex-like with} \\ \text{respect to the product cone } \mathbb{R}_+ \times C \text{ in } \mathbb{R} \times Y; \\ \text{let the constraint set be given as } S = \{x \in \widehat{S} : g(x) \in -C\} \neq \emptyset. \end{cases}
$$

Under these assumptions we consider the optimization problem, called *primal problem*:

$$
(21)\qquad \min_{x \in S} f(x).
$$

We associate to the primal problem the Lagrangean functional:

$$
\mathcal{L} : \widehat{S} \times C^* \to \mathbb{R}
$$

$$
\mathcal{L}(x, u) = f(x) + u(g(x)), \quad \forall x \in \widehat{S},\ u \in C^*,
$$

where C^* is the dual cone of Y.

Lemma 1 ([8])**.** *Let the assumptions (20) be satisfied and let the ordering cone C be closed. Then $\overline{x}$ is a minimal solution of the problem (21) if and only if $\overline{x}$ is a minimal solution of the problem:*

$$
(22)\qquad \min_{x \in \widehat{S}} \sup_{u \in C^*} \mathcal{L}(x, u)\,.
$$

In this case the extremal values of both problems are equal.

Theorem 4 ([8])**.** *Let the assumptions (20) be satisfied and let $intC$ be nonempty. If the primal problem (21) is solvable and the generalized Slater condition is satisfied, i.e. there is a vector $\widehat{x} \in \widehat{S}$ with $g(\widehat{x}) \in -intC$, then the dual problem*

$$
(23)\qquad \max_{u \in C^*} \inf_{x \in \widehat{S}} \mathcal{L}(x, u)
$$

is also solvable and the extremal values of the two problems are equal.

Now, we characterize a solution of minsup and maxinf problem as saddle point of Lagrangean functional.

Theorem 5 ([8])**.** *Let the assumptions (20) be satisfied and let the ordering cone C be closed. If $\overline{x} \in S$ is a minimal solution of the primal problem (21) and the generalized Slater condition is satisfied then there exists $\overline{u} \in C^*$ such that $(\overline{x}, \overline{u})$ is a saddle point of the Lagrange functional:*

$$
(24)\qquad \mathcal{L}(\overline{x}, u) \le \mathcal{L}(\overline{x}, \overline{u}) \le \mathcal{L}(x, \overline{u}), \quad \forall x \in \widehat{S},\ u \in C^*.
$$

Furthermore we have:

$$
\mathcal{L}(\overline{x}, \overline{u}) = \min_{x \in S} f(x) = f(\overline{x})\,.
$$

Now, we consider a minimum problem with the presence also of constraints of the kind $h(x) = 0$. First of all, we make the following assumptions:

$$(25)\quad \begin{cases} \text{let } (X, ||\cdot||) \text{ and } (Z, ||\cdot||) \text{ be real Banach spaces;} \\ \text{let } (Y, ||\cdot||) \text{ be partially ordered real normed space with the} \\ \ \text{ordering cone } C; \\ \text{let } \widehat{S} \text{ be a convex subset of } X; \\ \text{let } f : X \to \mathbb{R} \text{ be a given functional,} \\ \text{let } g : X \to Y,\ h : X \to Z \text{ be given mappings;} \\ \text{let the constraint set } S = \{x \in \widehat{S} : g(x) \in -C,\ h(x) = 0_Z\} \text{ be nonempty.} \end{cases}$$

Under these assumptions, we consider the optimization problem

$$\min_{x \in S} f(x)$$

and we associate to it the Lagrangean functional:

$$\mathcal{L} : \widehat{S} \times C^* \times Z^* \to \mathbb{R}$$

$$\mathcal{L}(x, u, v) = f(x) + u(g(x)) + v(h(x)), \quad \forall x \in \widehat{S},\ (u, v) \in C^* \times Z^*,$$

The following theorem presents the generalized Lagrangean multiplier rule.

Theorem 6 ([8]). *Let the assumptions (25) be satisfied and let $\overline{x}$ be a minimal point of f in S. Let the functional f and the mapping g be Fréchet differentiable at $\overline{x}$. Let the mapping h be Fréchet differentiable in a neighborhood of $\overline{x}$, let $h'(\cdot)$ be continuous at $\overline{x}$. Let the set*

$$\begin{pmatrix} g'(\overline{x}) \\ h'(\overline{x}) \end{pmatrix} cone(\widehat{S} - \{\overline{x}\}) + cone \begin{pmatrix} C + \{g(\overline{x})\} \\ \{0_Z\} \end{pmatrix} = Y \times Z.$$

Then there are continuous linear functionals $\overline{u} \in C^*$ and $\overline{v} \in Z^*$ such that

$$\frac{\partial \mathcal{L}(\overline{x})}{\partial x}(x - \overline{x}) \geq 0 \quad \forall x \in \widehat{S}, \quad \overline{u}(g(\overline{x})) = 0,$$

where $\dfrac{\partial \mathcal{L}(\overline{x})}{\partial x}$ is the Fréchet derivative of Lagrangean functional at $\overline{x}$.

Theorem 7 ([8]). *In addition to the assumptions of Theorem 6 let us assume that f, g, h are convex, then $\overline{x} \in S$ is a minimal point of the primal problem if and only if there exist $\overline{u} \in C^*$ and $\overline{v} \in Z^*$ such that $(\overline{x}, \overline{u}, \overline{v})$ is a saddle point of the Lagrangean functional, namely*

$$\mathcal{L}(\overline{x}, u, v) \leq \mathcal{L}(\overline{x}, \overline{u}, \overline{v}) \leq \mathcal{L}(x, \overline{u}, \overline{v}), \quad \forall x \in \widehat{S},\ u \in C^*, v \in Z^*$$

and, moreover, it result that

$$u(g(\overline{x})) = 0.$$

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