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L^p ESTIMATES FOR THE ONE DIMENSIONAL WAVE OPERATOR AND APPLICATIONS TO DISPERSIVE ESTIMATES

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1. Introduction

Dispersive properties of evolution equations have become in recent years a crucial tool in the study of a variety of questions, including local and global existence for nonlinear equations, well posedness in Sobolev spaces of low order, scattering theory and many others. In particular, the free Schrödinger equation on $\mathbb{R}^n$

$$i\partial_t u - \Delta u = 0, \qquad u(0,x) = f(x) \tag{1.1}$$

and the free wave equation

$$\partial_{tt}^2 u - \Delta u = 0, \qquad u(0,x) = 0, \qquad \partial_t u(0,x) = g(x) \tag{1.2}$$

exhibit a rich set of dispersive and smoothing properties. The basic one is the the *dispersive estimate*; for Schrödinger, we have

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{n/2}} \|f\|_{L^1} \tag{1.3}$$

while for the wave equation we have

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{C}{t^{\frac{n-1}{2}}} \|g\|_{\dot{B}^{\frac{n-1}{2}}_{1,1}} \tag{1.4}$$

(the norm at the right hand side is a homogeneous Besov norm; see the next section and the references for precise definitions).

The dispersive estimates can be proved using the explicit expression of the fundamental solution or the stationary phase method. Then via interpolation with the conservation of energy one can obtain $L^p - L^q$ decay estimates and from these via suitable functional analytic arguments one can derive the Strichartz estimates. Standard references on the subject are [15] and [21]; see also [7].

A very interesting problem is to extend these methods to equations with variable coefficients. In general the above approach is not feasible since an explicit expression of the fundamental solution is not available, and more abstract methods are required. A great number of works have been devoted to the analysis of dispersive properties of equations with variable coefficients, in particular for the physically very relevant Schrödinger equation perturbed with a potential:

$$i\partial_t u - \Delta u + V(x)u = 0 \tag{1.5}$$

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(see, among others, [16], [20], [22] and the references therein). Of great interest is also the time-dependent case $V = V(t, x)$, which of course is much more delicate; almost all available results are of a perturbative nature, requiring some smallness of the potential $V(t, x)$ (see [16], [22], and, for small potentials of very low regularity, [23]; concerning the case of time-periodic potentials, see [32]).

For the wave equation there are just a few results (see in particular [5], [9], [14], [12], [10] and the papers by K.Yajima whose methods are discussed in detail in the next section).

In the following we shall focus on the one dimensional case only, for which is is possible to develop a complete theory covering in particular the Schrödinger, wave and Klein-Gordon equations with fully variable coefficients. This is obtained via a suitable extension of the method of K.Yajima, combined it with ideas from harmonic analysis. Note that in one space dimension, the general equations with variable coefficients can be reduced to the simplest case of a constant coefficient equation perturbed with a potential, via a simple change of variables.

2. The one dimensional case

The results listed in this section have been obtained in a joint work with L.Fanelli [11]; we refer to that paper for detailed proofs.

We start by setting our notations.

Let $H_0 = -d^2/dx^2$ be the one-dimensional Laplace operator on the line, and consider the perturbed operator $H = H_0 + V(x)$. For a potential $V(x) \in L^1(\mathbb{R})$, the operator H can be realized uniquely as a selfadjoint operator on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$. The absolutely continuous spectrum of H is $[0, +\infty[$, the singular continuous spectrum is absent, and the possible eigenvalues are all strictly negative. Moreover, the *wave operators*

$$W_{\pm} f = L^2 - \lim_{s \to \pm\infty} e^{isH} e^{-isH_0} f \tag{2.1}$$

exist and are unitary from $L^2(\mathbb{R})$ to the absolutely continuous space $L^2_{ac}(\mathbb{R})$ of H. A very useful feature of $W_{\pm}$ is the *intertwining property.* If we denote by P_{ac} the projection of L^2 onto $L^2_{ac}(\mathbb{R})$, the property can be stated as follows: for any Borel function f,

$$W_{\pm} f(H_0) W_{\pm}^* = f(H) P_{ac} \tag{2.2}$$

(see e.g. [13], [8]).

Thanks to (2.2), one can reduce the study of an operator $f(H)$, or more generally $f(t, H)$, to the study of $f(t, H_0)$ which has a much simpler structure. When applied to the operators e^{itH}, $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$, $\frac{\sin(t\sqrt{H+1})}{\sqrt{H+1}}$, this method can be used to prove decay estimates for the Schrödinger, wave and Klein-Gordon equations

$$iu_t - \Delta u + Vu = 0, \qquad u_{tt} - \Delta u + Vu = 0, \qquad u_{tt} - u_{xx} - \Delta u + u + Vu = 0,$$

provided one has some control on the L^p behaviour of $W_{\pm}$, $W_{\pm}^*$. Indeed, if the wave operators are bounded on L^p, the $L^q - L^{q'}$ estimates valid for the free operators extend immediately to the perturbed ones via the elementary argument

$$\|e^{itH}P_{ac}f\|_{L^q} \equiv \|W_+e^{itH_0}W_+^*f\|_{L^q}$$
$$\leq C\|e^{itH_0}W_+^*f\|_{L^q} \leq Ct^{-\alpha}\|W_+^*f\|_{L^{q'}} \leq Ct^{-\alpha}\|f\|_{L^{q'}}$$

Such a program was developed systematically by K.Yajima in a series of papers [28], [29], [30] where he obtained the L^p boundedness for all p of $W_\pm$, under suitable assumptions on the potential V, for space dimension $n \geq 2$. The analysis was completed in the one dimensional case in Artbazar-Yajma [3] and Weder [25]. We remark that in high dimension $n \geq 4$ the decay estimates obtained by this method are the best available from the point of view of the assumptions on the potential; only in low dimension $n \leq 3$ more precise results have been proved (see [16], [17], [22], [26], [31] and [12]). We also mention [18] for an interesting class of related counterexamples.

In order to explain the results in more detail we recall a few notions. The relevant potential classes are the spaces

$$L^1_\gamma(\mathbb{R}) \equiv \{f\colon (1+|x|)^\gamma f \in L^1(\mathbb{R})\}. \tag{2.3}$$

Moreover, given a potential $V(x)$, the *Jost functions* are the solutions $f_\pm(\lambda, x)$ of the equation $-f''+Vf = \lambda^2 f$ satisfying the asymptotic conditions $|f_\pm(\lambda, x)-e^{\pm i\lambda x}| \to 0$ as $x \to \pm\infty$. When $V(x) \in L^1_1$, the solutions $f_\pm$ are uniquely defined ([13]). Now consider the Wronskian

$$W(\lambda) = f_+(\lambda, 0)\partial_x f_-(\lambda, 0) - \partial_x f_+(\lambda, 0)f_-(\lambda, 0). \tag{2.4}$$

The function $W(\lambda)$ is always different from zero for $\lambda \in \mathbb{R} \setminus 0$, and hence for real λ it can only vanish at $\lambda = 0$. Then we say that 0 *is a resonance for H* when $W(0) = 0$, and that it is not a resonance when $W(0) \neq 0$. The first one is also called the *exceptional case.*

In [25] Weder proved that the wave operators are bounded on L^p for all $1 < p < \infty$, provided $V \in L^1_\gamma$ for $\gamma > 5/2$ (see also the following remark). The assumption can be relaxed to $\gamma > 3/2$ provided 0 is not a resonance. It is natural to conjecture that these conditions may be sharpened, also in view of the results Goldberg and Schlag [16] proved under the milder assumption $\gamma = 2$ in the general and $\gamma = 1$ in the nonresonant case.

Indeed, our main result is the following:

Theorem 2.1. *Assume $V \in L^1_1$ and 0 is not a resonance, or $V \in L^1_2$ in the general case. Then the wave operators $W_\pm, W_\pm^*$ can be extended to bounded operators on L^p for all $1 < p < \infty$. Moreover, in the endpoint L^∞ case we have the estimate*

$$\|W_\pm g\|_{L^\infty} \leq C\|g\|_{L^\infty} + C\|\mathcal{H}g\|_{L^\infty}, \tag{2.5}$$

for all $g \in L^\infty \cap L^p$ for some $p < \infty$ such that $\mathcal{H}g \in L^\infty$, where $\mathcal{H}$ is the Hilbert transform on $\mathbb{R}$; the conjugate operators $W_\pm^*$ satisfy the same estimate.

Remark 2.1. The appearence of the Hilbert trasform at the endpoint $p = \infty$ is not a surprise. Indeed, Weder proved that, under the assumptions $V \in L^1_\gamma$ for $\gamma > 5/2$ in the general case and $\gamma > 3/2$ in the nonresonant case, the wave operator involves explicitly the Hilbert transform. More precisely, let $\chi(x) \in C^{\infty}(\mathbb{R})$ be such that $\chi = 0$ for $x < 0$ and $\chi = 1$ for $x > 1$, then formula (1.12) in [25] states that

$$W_{\pm} = W_{\pm,r} \pm \chi(x) f_{+}(0,x)\mathcal{H}\Psi(D)(c_1 + c_2 P) \pm (1-\chi(x)) f_{-}(0,x)\mathcal{H}\Psi(D)(c_3 + c_4 P)$$

where the operators $W_{\pm,r}$ are bounded on L^1 and L^{∞}, $Pf(x) := f(-x)$, $\Psi(\xi) \in C_0^{\infty}$ is a suitable cutoff, and the constants c_j can be expressed in terms of the transmission and reflection coefficients. From this decomposition it is clear that the wave operator in general can not be bounded on L^{∞}, but only from L^{∞} to BMO. Notice also that the Hilbert transform terms vanish in the unperturbed case $V \equiv 0$.

At the opposite endpoint $p = 1$, we get an even weaker result by duality. Weder's decomposition suggests that the stronger bound

$$\|W_{\pm} g\|_{L^1} \leq C\|g\|_{L^1} + C\|\mathcal{H}g\|_{L^1} \tag{2.6}$$

should be true (and is indeed true under his assumptions on the potential). Notice that (2.6) is equivalent to say that $W_{\pm}$ are bounded operators from the Hardy space $\mathcal{H}_1$ to L^1, and by duality this would also imply that $W_{\pm}$ are bounded operators from L^{∞} to BMO.

Remark 2.2. Our proof is based on the improvement of some results of Deift and Trubowitz [13], combined with the stationary approach of Yajima [28], [3], and some precise Fourier analysis arguments. Quite inspirational have been the papers [16] and [27], both for showing there was room for improvement in the assumptions on the potential, and for the very effective harmonic analysis approach.

Remark 2.3. In the proof of Theorem 2.1 we split as usual the wave operator into high and low energy parts; the high energy part is known to be easier to handle since the resolvent is only singular at frequency $\lambda = 0$. Here we can prove that the high energy part is bounded on L^p for all p, including the cases $p = 1$ and $p = \infty$, under the weaker assumption $V \in L^1(\mathbb{R})$.

Remark 2.4. An essential step in the low energy estimate is a study of the Fourier properties of the Jost functions; this kind of analysis is classical (see [1]) and the fundamental estimates were obtained by Deift and Trubowitz in [13]. We can improve their results by showing that the L^1 norms of the Fourier transforms of the Jost functions satisfy a linear bound as $|x| \to +\infty$ instead of an exponential one as in [13]. In the resonant case we can prove a quadratic bound.

Remark 2.5. It is possible to continue the analysis and prove that the wave operators are bounded on Sobolev spaces $W^{k,p}$, under the additional assumption $V \in W^{k,1}$ (see also [25] where the boundedness from $W^{k,\infty}$ to BMO_k is proved under stronger assumptions on the potential), but we prefer not to pursue this question here.

Theorem 2.1 has several applications; here we shall focus on the dispersive estimates for the one dimensional Schrödinger and Klein-Gordon equations with variable rough coefficients.

Consider first the initial value problem

$$iu_t - a(x)u_{xx} + b(x)u_x + V(x)u = 0, \qquad u(0,x) = f(x). \tag{2.7}$$

Then we obtain the following decay result, where the notation $f \in L^2_1$ means $(1+|x|)f \in L^2$. Notice that the following result can also be obtained as a consequence of the dispersive $L^\infty - L^1$ estimate proved in [16] (and in [26] under stronger assumptions on the potential).

Proposition 2.1. *Assume $V \in L^1_2$, $a \in W^{2,1}(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R})$ with*

$$a(x) \geq c_0 > 0 \qquad a', b \in L^2_1, \qquad a'', b' \in L^1_2 \tag{2.8}$$

for some constant c_0. Then the solution of the initial value problem (2.7) *satisfies*

$$\|P_{ac}u(t,\cdot)\|_{L^q} \leq Ct^{\frac{1}{q}-\frac{1}{2}}\|f\|_{L^{q'}}, \qquad 2 \leq q < \infty, \qquad \frac{1}{q}+\frac{1}{q'} = 1. \tag{2.9}$$

The same result holds if $a = 1$, $b = 0$ and $V \in L^1_1$, provided 0 is not a resonance for H.

Proof. It is sufficient to perform the change of variables $u(t,x) = \sigma(x)w(t,c(x))$ with

$$c(x) = \int_0^x a(s)^{-1/2}ds, \qquad \sigma(x) = a(x)^{1/4}\exp\left(\int_0^x \frac{b(s)}{2a(s)}\,ds\right) \tag{2.10}$$

to reduce the problem to a Schrödinger equation with a potential perturbation $\widetilde{V}(y)$ defined by

$$\widetilde{V}(c(x)) = V(x) + \frac{1}{16a(x)}(2b(x)+a'(x))(2b(x)+3a'(x)) - \frac{1}{4}(2b(x)+a''(x)); \tag{2.11}$$

notice that $\widetilde{V}$ satisfies the assumptions of Theorem 2.1 provided (2.8) hold. Hence the solution of the transformed problem satisfies a dispersive estimate like (2.9), and coming back to the original variables we conclude the proof. $\square$

Remark 2.6. The range of indices allowed in (2.9) is sufficient to deduce the full set of Strichartz estimates, as it is well known. It is interesting to compare this with the result of Burq and Planchon [6] who proved the Strichartz estimates for the variable coefficient equation

$$iu_t - \partial_x(a(x)\partial_x u) = 0$$

assuming only that $a(x)$ is of BV class and bounded from below.

Remark 2.7. In view of the next application, we recall the definition of nonhomogeneous *Besov spaces.* Choose a Paley-Littlewood partition of unity, i.e., a sequence of smooth cutoffs $\phi_j \in C_0^\infty(\mathbb{R})$ with $\sum_{j\geq 0}\phi_j(\lambda) = 1$ and $\operatorname{supp}\phi_j = [2^{j-1}, 2^{j+1}]$ for $j \geq 1$, $\operatorname{supp}\phi_0 = [-2,2]$. Then the $B^s_{p,r}$ Besov norm is defined by

$$\|g\|^r_{B^s_{p,r}} \equiv \sum_{j\geq 0} 2^{jsr}\|\phi_j(\sqrt{H_0})g\|^r_{L^p}$$

with obvious modification for $r = \infty$. It is then natural to define the *perturbed Besov norm* corresponding to the selfadjoint operator $H = H_0 + V$ as

$$\|g\|^r_{B^s_{p,r}(V)} \equiv \sum_{j\geq 0} 2^{jsr}\|\phi_j(\sqrt{H})g\|^r_{L^p}.$$

Now, from the L^p boundedness of the wave operators and the intertwining property in the form

$$\phi_j(\sqrt{H})W_\pm = W_\pm \phi_j(\sqrt{H_0})$$

we obtain immediately the Besov space bounds

$$\|W_\pm f\|_{B^s_{p,r}(V)} \leq C\|f\|_{B^s_{p,r}}, \qquad \|W^*_\pm f\|_{B^s_{p,r}} \leq C\|f\|_{B^s_{p,r}(V)} \tag{2.12}$$

(in the second one we used the inequality $\|P_{ac}\phi(H)f\|_{L^p} \leq C\|\phi(H)f\|_{L^p}$ which is true since the eigenfunctions belong to $L^1 \cap L^\infty$).

We now consider the initial value problem for the one dimensional Klein-Gordon equation

$$u_{tt} - a(x)u_{xx} + u + b(x)u_x + V(x)u = 0, \qquad u(0,x) = 0, \quad u_t(0,x) = g(x). \tag{2.13}$$

Our second application is the following, proved in an identical way as Proposition 2.1:

Proposition 2.2. *Assume $a = 1$, $b = 0$ and $V \in L^1_2$, or $V \in L^1_1$ and 0 is not a resonance. Then the solution of the initial value problem* (2.13) *satisfies*

$$\|P_{ac}u(t,\cdot)\|_{L^q} \leq Ct^{\frac{1}{q}-\frac{1}{2}}\|g\|_{B^{\frac{1}{2}-\frac{3}{q}}_{q',q}(V)}, \qquad 2 \leq q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1. \tag{2.14}$$

The same decay rate is true for general coefficients a, b, V satisfying the assumptions of Proposition 2.1 (with the Besov norm replaced by a suitable norm of the initial data).

Proof. In the unperturbed case, (2.14) can be obtained as usual by interpolating the dispersive $L^\infty - B^{1/2}_{1,1}$ estimate with the conservation of the H^1 norm i.e. the energy. The perturbed case is handled by the change of variables (2.10) and an application of Theorem 2.1 as in the proof of Proposition 2.1. In the general case the Besov norm in (2.14) must be replaced by $\|h\|_{B^{\frac{1}{2}-\frac{3}{q}}_{q',q}(\widetilde{V})}$ with $\widetilde{V}$ as in (2.11) and $h = (g/\sigma)|_{c^{-1}(y)}$. □

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