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REGULARITY FOR MINIMIZERS OF DEGENERATE ELLIPTIC FUNCTIONALS

MENITA CAROZZA AND ANTONIA PASSARELLI DI NAPOLI

Dedicated to the memory of Prof. Filippo Chiarenza

Abstract. We prove a regularity result for local minimizers of degenerate variational integrals. The degeneracy function $\mathcal{K}(x)$ is assumed to be exponentially integrable.

The right space of the gradient of a local minimizer u turns out to be the Zygmund class $L^p \log^{-1} L$. Our result states that if λ is sufficiently large, then Du belongs to $L^{p+\epsilon}$, some $\epsilon > 0$.

1. Introduction

Since the pioneering work by Chiarenza, Frasca and Longo ([4]) appeared, many papers have been devoted to the study of existence and regularity of solutions of PDE's, whose coefficients are discontinuous and even not bounded.

The papers by Filippo Chiarenza treated PDE's with VMO coefficients, i.e. the space of functions with vanishing mean oscillation, and motivated the subsequent works on equations with BMO and exponentially integrable coefficients.

Here, we present regularity results for local minimizers of variational integrals satisfying a degenerate ellipticity condition. More precisely, we treat the variational integrals of the form

$$I(\Omega, v) = \int_\Omega f(x, Dv)dx \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $v : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function.

We assume that there exits a function $1 \le k(x) < \infty$ such that

$$\frac{1}{k(x)}|\xi|^p \le f(x,\xi) \le k(x)|\xi|^p \tag{H1}$$

for $1 < p < \infty$, almost every $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$ and

$$\xi \to f(x,\xi) \text{ is strictly convex a. e. } x \in \Omega. \tag{H2}$$

Remark that, even in the case $p = 2$, we are dealing with genuine degenerate functionals since the ratio between the eigenvalues is unbounded.

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Definition 1. A function $u \in W^{1,1}_{loc}(\Omega)$ is a local minimizer of functional (1.1) if $I(\Omega', u) < \infty$ for every $\Omega' \subset\subset \Omega$ and

$$\int_{supp\varphi} f(x, Du)dx \leq \int_{supp\varphi} f(x, Du + D\varphi)dx \tag{1.2}$$

for every function $\varphi \in C^\infty_0(\Omega)$.

Our basic assumption is that the growth coefficient $k(x)$, which measures the degree of degeneracy of our variational integral, belongs to the Orlicz space $EXP(\Omega)$. Precisely, setting $\mathcal{K}(x) = \max\{k(x), k^{\frac{1}{p-1}}(x)\}$, we assume that

$$\int_\Omega e^{\lambda \mathcal{K}(x)} dx < \infty \tag{H3}$$

for some $\lambda > 0$.

Remark 1. By its definition, a minimizer u satisfy

$$I(\Omega', u) \; = \; \int_{\Omega'} f(x, Du)dx < \infty$$

for every $\Omega' \subset\subset \Omega$. Then assumptions (H1) and (H3) immediately imply, via Hölder's inequality, that

$$\int_{\Omega'} |Du|^p \log^{-1}(e + |Du|)dx$$

is finite, i.e. the gradient of u belongs to the Zygmund class $L^p \log^{-1} L_{loc}(\Omega)$.

Indeed, we have the following

Theorem 1. *([2]) Let u be a local minimizer of the variational integral (1.1) such that $Du \in L^p \log^{-1} L_{loc}(\Omega)$. Assume that $f(x, \xi)$ verifies assumptions (H1) and (H2). For every $\alpha > 0$ there exists $\lambda_\alpha = \lambda_\alpha(N)$ such that whenever $\mathcal{K}(x)$ satifies (H3) with $\lambda > \lambda_\alpha$ then $Du \in L^p \log^\alpha L_{loc}(\Omega)$ and*

$$||Du||_{L^p \log^\alpha L(\Omega'')} \leq C_\alpha(N, \Omega') \int_{\Omega'} f(x, Du)dx$$

for every domain $\Omega'' \subset \Omega' \subset\subset \Omega$.

Our main idea to get this result is to *approximate* the integrand f by more regular ones. More precisely, we shall approximate f by strictly convex and uniformly elliptic functions f_ε, $\varepsilon > 0$, growing as $|\xi|^p$ in the gradient. To every minimizer h_ε of such regular problems, according to the duality theory in the sense of Ekeland and Temam, we can associate a suitable divergence free vector field denoted by B_ε.

For the pair $(Dh_\varepsilon, B_\varepsilon)$ we are entitled to apply an a priori estimate proved in [9], which is the basic tool in our proof (see Theorem 4).

More important, these a priori estimates are preserved in passing to the limit and thus the minimizers of the regular integrals converge to a minimizer of our original problem.

Thanks to Theorem 1, we can improve the integrability of Du. Namely, Theorem 1 tells us, in particular, that there exists $\bar{\lambda} = \bar{\lambda}(N)$ such that if $\mathcal{K}(x)$ satifies (H3) with $\lambda > \bar{\lambda}$ then $Du \in L^p_{loc}(\Omega)$ and

$$||Du||_{L^p(\Omega'')} \leq C(N, \Omega') \int_{\Omega'} f(x, Du) dx$$

for every domain $\Omega'' \subset \Omega' \subset\subset \Omega$. Then we can use test functions proportional to the minimizers, thus arriving to the following

Theorem 2. *Let u be a local minimizer of the variational integral (1.1) such that $Du \in L^p \log^{-1} L_{loc}(\Omega)$. Assume that $f(x, \xi)$ verifies assumptions (H1) and (H2). There exists $\bar{\lambda} = \bar{\lambda}(N)$ such that whenever $\mathcal{K}(x)$ satifies (H3) with $\lambda > \bar{\lambda}$ then $Du \in L^{p+\varepsilon}_{loc}(\Omega)$ and*

$$||Du||_{L^{p+\varepsilon}}(\Omega'') \leq C(N, \Omega') \int_{\Omega'} f(x, Du) dx$$

for every domain $\Omega'' \subset \Omega' \subset\subset \Omega$, for some positive $\varepsilon = \varepsilon(p, N, \bar{\lambda})$.

We conclude noting that our results apply also when f is Gateaux differentiable and then they apply also to the solutions of the corresponding Euler - Lagrange equation, which degenerates too.

2. p-Harmonic couples

Let Ω be an open subset of $\mathbb{R}^N$. A pair $[B, E]$ of vector fields such that

$$\text{div} B(x) = 0 \qquad \text{curl} E(x) = 0$$

will be called a p-harmonic couple, $1 < p < \infty$, of distortion $\mathcal{K}(x) \geq 1$ if

$$\frac{|E|^p}{p} + \frac{|B|^q}{q} \leq \mathcal{K}(x) \langle B, E \rangle \quad \text{in} \quad \Omega \tag{2.1}$$

We will be interested in the distortion $\mathcal{K} = \mathcal{K}(x)$ of the exponential class $EXP(\Omega)$. Precisely, we shall assume that

$$\int_\Omega e^{\lambda \mathcal{K}(x)} dx < \infty \tag{2.2}$$

for some $\lambda > 0$.

It has to be noted (see [10]) that the distortion function $\mathcal{K}(x)$ admits a BMO-majorant. Precisely we can majorize $\mathcal{K}(x)$ point-wise by a function $K(x) \in BMO(\mathbb{R}^N)$ such that

$$e^{\lambda(K - K_0)} - 1 \in L^1(\mathbb{R}^N)$$

for some $K_0 \in L^\infty(\mathbb{R}^N)$, with $1 \leq K_0(x) \leq K(x)$. Moreover, we can also ensure a bound for the BMO-norm of $K(x)$ in terms of the exponent λ, i.e.

$$||K||_{BMO} \leq \frac{c(N)}{\lambda}. \tag{2.3}$$

The following quantity will be used in the sequel

$$[K] = ||K_0||_\infty + \frac{1}{\lambda}\int_{\mathbb{R}^N} (e^{\lambda(K-K_0)} - 1) \tag{2.4}$$

The right spaces for p-harmonic couples of exponential integrable distortion are the Orlicz-Zygmund spaces $L^s \log^\alpha L$, $1 \leq s < \infty$, $\alpha \in \mathbb{R}$, equipped by the Luxemburg norm ([13]).

Let us recall that for $\alpha \geq 0$ the non-linear functional

$$[f]_{s,\alpha} = \left[\int |f|^s \log^\alpha\left(e + \frac{|f|}{\|f\|_s}\right)\right]^{\frac{1}{s}}$$

is comparable with the Luxemburg norm in the sense that

$$||f||_{L^s \log^\alpha L} \leq [f]_{s,\alpha} \leq 2||f||_{L^s \log^\alpha L}, \tag{2.5}$$

see [9].

If Ω is a measurable subset of $\mathbb{R}^N$, with positive and finite measure, if $s > 1$ the following relations are straightforward

$$L^s \log^{-1} L(\Omega) \subset L^{s-\varepsilon}(\Omega) \qquad \forall \varepsilon > 0$$

and

$$f \in L^s \log^{-1} L(\Omega) \text{ implies } \lim_{\varepsilon \to 0} \varepsilon \int_\Omega |f|^{s-\varepsilon} dx = 0$$

Let us recall now some results about div-curl couples, that we shall use in the following. We begin with a result concerning the integrability properties of p-harmonic couples in $\mathbb{R}^N$. More precisely, we have the following basic a priori estimate was proved in [9].

Theorem 3. *Let $F = [B, E]$ be a p-harmonic couple of distortion $\mathcal{K}$ with $E \in L^p \log^\alpha L$ and $B \in L^q \log^\alpha L$, where $1 < p, q < \infty$, $p + q = pq$. For any $\alpha \geq 0$ there exists $\lambda_\alpha \geq 1$ such that, whenever $\mathcal{K}$ satisfies (2.2) with $\lambda \geq \lambda_\alpha$, we have*

$$\| \, |E|^p + |B|^q \|_{L \log^\alpha L(\mathbb{R}^N)} \leq c([K]) \| \, |E|^p + |B|^q \|_{L \log^{\alpha-1} L(\mathbb{R}^N)}.$$

A local version of previous theorem is also avalaible.

Theorem 4. *For any $\alpha \geq p+1$ there exists $\lambda_\alpha \geq 1$ such that, whenever $\mathcal{K}$ satisfies (2.2) with $\lambda \geq \lambda_\alpha$, we have*

$$\| \, |\varphi^q E|^p \|_{L \log^\alpha L} \leq c([K]) \| \, |D\varphi^q||E|^p \, \|_{L^{\frac{N}{N-1}}} \tag{2.8}$$

for any function $\varphi \in C_0^\infty(\Omega)$.

The proof of estimate (2.8) can be easily achieved using Theorem 3 and a suitable version of Sobolev-Poincaré inequality.

3. Proof of Theorem 1

The proof, given in [2], is reported here for sake of completeness.

Let us fix a ball Ω' compactly contained in Ω. We divide the proof in two steps. In the first one we construct a map $h : \Omega' \to R^N$, whose gradient lies in $L^p \log^\alpha L_{\text{loc}}(\Omega')$, for all $\alpha > 0$, while in the second one we show that the map h has the same energy of our minimizer u on the ball Ω'. Thus we conclude that $u = h$ a.e. using strict convexity of f.

More precisely, we have the following:

Step 1. The approximation.

Let ε denote a sequence of positive real numbers converging to zero. Define u_ε as the ε-mollification of u through φ_ε, where $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is a family of smooth mollifiers. Then $Du_\varepsilon \to Du$ in $L^p \log^{-1} L(\Omega')$

Then, for $\delta > 0$, we set

$$f_{\varepsilon,\delta}(x,\xi) \;=\; \frac{1+\varepsilon\delta}{1+\varepsilon\delta+\varepsilon k(x)} f(x,\xi) + \frac{\varepsilon k(x)}{1+\varepsilon\delta+\varepsilon k(x)} |\xi|^p$$

One can easily check that

$$\frac{1}{k(x)}|\xi|^p \le f_{\varepsilon,\delta}(x,\xi) \le k(x)|\xi|^p \tag{3.1}$$

$$\frac{\varepsilon}{1+\varepsilon+\varepsilon\delta}|\xi|^p \le f_{\varepsilon,\delta}(x,\xi) \le \frac{1+\varepsilon+\varepsilon\delta}{\varepsilon}|\xi|^p \tag{3.2}$$

Next, consider the variational integrals

$$I_{\varepsilon,\delta}[h] \;=\; \int_{\Omega'} f_{\varepsilon,\delta}(x, Dh)dx$$

and the problems

$$\inf_{h\in u_\varepsilon + W_0^{1,p}(\Omega')} \int_{\Omega'} f_{\varepsilon,\delta}(x, Dh)dx \tag{$\mathcal{P}_{\varepsilon,\delta}$}$$

that are well defined since $u_\varepsilon \in W^{1,p}(\Omega')$ for every $\varepsilon > 0$.

Since $f_{\varepsilon,\delta}$ is strictly convex by definition and it is coercive thanks to (3.2), each problem $\mathcal{P}_{\varepsilon,\delta}$ admits a unique solution $h_\varepsilon = h_{\varepsilon,\delta} \in W^{1,p}(\Omega')$. The duality theory of Ekeland and Temam tells us that the dual problem to $\mathcal{P}_\varepsilon$ is given by

$$\sup_{B\in L^q,\, divB=0} \left\{ \int_{\Omega'} \langle B, Du_\varepsilon\rangle dx - \int_{\Omega'} f^*_{\varepsilon,\delta}(x,B)dx \right\} \tag{$\mathcal{P}^*_{\varepsilon,\delta}$}$$

and has a solution denoted by $B_\varepsilon = B_{\varepsilon,\delta}$, which satisfies

$$\langle Dh_{\varepsilon,\delta}, B_{\varepsilon,\delta}\rangle = f_{\varepsilon,\delta}(x, Dh_{\varepsilon,\delta}) + f^*_{\varepsilon,\delta}(x, B_{\varepsilon,\delta}) \tag{3.3}$$

where f^* denotes the Young-Fenchel transform of f. The div-curl couple $(Dh_{\varepsilon,\delta}, B_{\varepsilon,\delta})$ is a p-harmonic couple in Ω' of distortion $\mathcal{K}(x)$, thanks to (3.1).

Moreover, (3.2) implies that, for every ε, $(Dh_{\varepsilon,\delta}, B_{\varepsilon,\delta})$ is also a p-harmonic couple of bounded distortion and then

$$|B_{\varepsilon,\delta}| \le C_{\varepsilon,\delta}|Dh_{\varepsilon,\delta}|^{p-1} \tag{3.4}$$

Classical higher integrability results for minimizers of strictly elliptic variational integrals tell us that h_ε belongs to $W^{1,r}_{\mathrm{loc}}(\Omega')$ for some $r > p$ and then $Dh_\varepsilon \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega')$, for every $\alpha > 0$ (see [6]).

By (3.4), $B_\varepsilon \in L^q \log^\alpha L_{\mathrm{loc}}(\Omega')$, for every $\alpha > 0$ where, as usual, q is the Hölder conjugate exponent of p.

Therefore, we are legitimate to apply Theorem 2.5 to find , for every Ω'' compactly contained in Ω', that

$$\begin{aligned}||Dh_\varepsilon||^p_{L^p \log^\alpha L(\Omega'')} &\leq C \int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon(x))dx \\ &\leq C \int_{\Omega'} f_{\varepsilon,\delta}(x, Du_\varepsilon(x))dx\end{aligned} \tag{3.5}$$

where we also used (3.1) and the minimality of h_ε. Here and in what follows C denotes a constant depending on α, N and Ω''.

Now, we choose δ related to ε via the equality

$$\delta = \delta_\varepsilon := \left(\int_{\Omega'} k(x)|Du_\varepsilon|^p dx\right)^2 \tag{3.6}$$

Using the definition of $f_{\varepsilon,\delta(\varepsilon)}$, Jensen inequality, dominated convergence Theorem and (3.6) we get

$$\begin{aligned}||Dh_\varepsilon||^p_{L^p \log^\alpha L(\Omega'')} &\leq C \int_{\Omega'} f_{\varepsilon,\delta(\varepsilon)}(x, Du_\varepsilon(x))dx \\ &\leq C \left[\int_{\Omega'} f(x, Du_\varepsilon(x))dx + \int_{\Omega'} \frac{\varepsilon k}{1+\varepsilon\delta+\varepsilon k}|Du_\varepsilon|^p dx\right] \\ &\leq C \left[\int_{\Omega'} f(x, Du(x))dx + o(\varepsilon)\right]\end{aligned}$$

A routine diagonal argument gives us the sequence $h_j = h_{\varepsilon_j}$ weakly converging. The gradient of limit mapping h, Dh, actually belongs to $L^p \log^\alpha L_{loc}(\Omega')$, by the lower semicontinuity of the norms.

Step 2. The uniqueness.

Thank to the strict convexity of f, to complete the proof we need to show that

$$\int_{\Omega'} f(x, Du) = \int_{\Omega'} f(x, Dh) \tag{3.7}$$

Fix an arbitrary compact subset Ω'' of Ω' and a number $0 < s < 1$. Observe that

$$\lim_{s\to 1} f^s(x, Dh_\epsilon(x)) = f(x, Dh_\epsilon(x))$$

and

$$f^s(x, Dh_\epsilon(x)) \leq k^s|Dh_\epsilon(x)|^{sp}$$

Since

$$\int_{\Omega'} k^s|Dh_\epsilon(x)|^{sp}dx \leq ||k||^s_{exp}||Dh_\epsilon||^{sp}_{L^p \log^m L} \leq C\left(\int_{\Omega'} f(x, Du)dx\right)^s$$

we are legitimate to apply dominated convergence Theorem and obtain

$$\lim_{s\to 1}\int_{\Omega'} f^s(x, Dh_\epsilon(x))dx = \int_{\Omega'} f(x, Dh_\epsilon(x))dx \tag{3.8}$$

The lower semicontinuity of the integral, the definition of $f_{\varepsilon,\delta}$ and (3.8) imply, up to a not relabelled subquence, that

$$\begin{aligned}
\int_{\Omega''} f(x, Dh)dx &\le \lim_{\varepsilon\to 0}\int_{\Omega'} f(x, Dh_\varepsilon)dx \\
&= \lim_{\varepsilon\to 0}\lim_{s\to 1}\int_{\Omega'} f(x, Dh_\varepsilon)^s dx \\
&\le \lim_{\varepsilon\to 0}\lim_{s\to 1}\int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon)^s \Big(\frac{1+\varepsilon\delta+\varepsilon k}{1+\varepsilon\delta}\Big)^s \\
&\le \lim_{\varepsilon\to 0}\lim_{s\to 1}\Big(\int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon)\Big)^s\Big[\int_{\Omega'}\Big(\frac{1+\varepsilon\delta+\varepsilon k}{1+\varepsilon\delta}\Big)^{\frac{s}{1-s}}\Big]^{1-s} \\
&\le \lim_{\varepsilon\to 0}\Big[\int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon)\lim_{s\to 1}\Big(|\Omega'|^{1-s}+\varepsilon||k||^s_{exp}\Big)\Big] \\
&= \lim_{\varepsilon\to 0}\int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon)
\end{aligned} \tag{3.9}$$

where we used Hölder inequality. Now, using in (3.9) the minimality of h_ε, choosing δ given by (3.6), again by Jensen inequality and dominated convergence Theorem we get

$$\begin{aligned}
\int_{\Omega''} f(x, Dh)dx &\le \lim_{\varepsilon\to 0}\int_{\Omega'} f_{\varepsilon,\delta}(x, Dh_\varepsilon) \\
&\le \lim_{\varepsilon\to 0}\int_{\Omega'} f_{\varepsilon,\delta}(x, Du_\varepsilon) \le \int_{\Omega'} f(x, Du)
\end{aligned}$$

Since Ω'' is an arbitrary subset of Ω' previous inequality yields that

$$\int_{\Omega'} f(x, Dh(x))dx \le \int_{\Omega'} f(x, Du(x))dx$$

The opposite inequality follows by the minimality of u, once we observe that the limit map h coincides with u on the boundary of Ω'.

4. Proof of Theorem 2

In this section we prove Theorem 2, which is a new result concerning the regularity of minimizers of degenerate functionals of the type (1.1). All previous results show that the scale of improved integrability of the gradient of finite energy minimizers is only logarithmic. Indeed, here we prove that the gain of regularity can also be obatined in terms of the growth exponent p.

Proof. Thanks to Theorem 1, for $\alpha = 1$ we know that there exists a $\bar\lambda = \bar\lambda(n)$ such that if $e^{\lambda K} \in L^1(\Omega)$ for some $\lambda > \bar\lambda$ then $Du \in L^p \log L_{loc}(\Omega)$. In particular, we have

$$\int_{B_{\frac{R}{2}}} \Big(|Du|^p + |A(x,Du)|^{\frac{p}{p-1}}\Big)dx \le c([K]) \int_{B_R} \langle A(x,Du), Du\rangle dx \tag{4.1}$$

for every ball B_R such that B_{2R} is well contained in Ω, where $A(x,Du)$ denotes the divergence free vector field which solves the dual problem. This yields that $u \in W^{1,p}_{loc}(\Omega)$ and then $\varphi = \eta(u - u_{B_{2R}})$ can be used as a test function in the equation . As usual η is a $C_0^\infty(B_{2R})$ cut-off function between B_R and B_{2R}, such that $|\nabla\varphi| \le \frac{c}{R}$. Then

$$\int_{B_{2R}} \langle A(x,Du), D\varphi\rangle dx = 0 \tag{4.2}$$

which implies

$$\begin{aligned} \int_{B_{2R}} \langle A(x,Du), Du\rangle \eta dx &= -\int_{B_{2R}} \langle A(x,Du), u - u_{B_{2R}}\rangle \nabla\eta \\ &\le \frac{c}{R}\int_{B_{2R}} |A(x,Du)||u - u_{B_{2R}}|dx \\ &\le \frac{c}{R}\Big(\int_{B_{2R}} |A(x,Du)|^{\frac{pn}{(p-1)(n+1)}}dx\Big)^{\frac{(p-1)(n+1)}{pn}} \Big(\int_{B_{2R}} |Du|^{\frac{pn}{n+1}}dx\Big)^{\frac{n+1}{pn}} \\ &\le \frac{c}{R}\Big(\int_{B_{2R}} H dx\Big)^{\frac{n+1}{n}} \end{aligned} \tag{4.3}$$

where we set $H = |A(x,Du)|^{\frac{pn}{(p-1)(n+1)}} + |Du|^{\frac{pn}{n+1}}$.

From inequality (4.3), we deduce that

$$⨍_{B_R} \langle A(x,Du), Du\rangle dx \le c\Big(⨍_{B_{2R}} H dx\Big)^{\frac{n+1}{n}} \tag{4.4}$$

Combining (4.1) with (4.4), we get

$$⨍_{B_{\frac{R}{2}}} \Big(|Du|^p + |A(x,Du)|^{\frac{p}{p-1}}\Big)dx \le c([K])\Big(⨍_{B_{2R}} H dx\Big)^{\frac{n+1}{n}} \tag{4.5}$$

and then, by the definition of H,

$$\Big(⨍_{B_{\frac{R}{2}}} H^{\frac{n+1}{n}} dx\Big)^{\frac{n}{n+1}} \le c([K])\Big(⨍_{B_{2R}} H dx\Big) \tag{4.6}$$

Inequality (4.6) and the use of Gehring's lemma concludes the proof. □

References

[1] M. Carozza, G. Moscariello, A. Passarelli Di Napoli *Nonlinear Equations with growth coefficients in BMO*, Houston J. of Math. 28, n.4 (2002), 917-929.

[2] M. Carozza, G. Moscariello, A. Passarelli Di Napoli *Regularity results via duality of degenerate functionals*, Asympt. Anal. 44 no.3-4, 221-235

[3] F. Chiarenza *L^p regularity for systems of PDEs, with coefficients in VMO*, Nonlinear analysis, function spaces and applications, Vol. 5 (Prague, 1994), 1-32, Prometheus, Prague, 1994.

[4] F. Chiarenza, M. Frasca, P. Longo *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. 336 (1993), no. 2, 841-853.

[5] I. Ekeland, R. Temam , *Analyse convexe et problémes variationelles*, Dunod-Gauthier-Villas (1974).
[6] M. Giaquinta, G. Modica *Regularity results for some classes of higher order nonlinear elliptic systems*, J. Reine Angew. Math. 311/312 (1979) 145-169.
[7] E. Giusti, *Metodi diretti nel Calcolo delle Variazioni*, Un. Mat. Ital. 1984.
[8] T. Iwaniec, P. Koskela, G. Martin, C. Sbordone, *Mappings of exponentially integrable distortion*, Journal of the London Math. Soc. 67 n.2 (2003), 123-136.
[9] T. Iwaniec, L. Migliaccio, G. Moscariello, A. Passarelli di Napoli, *A priori estimates for nonlinear elliptic complexes*, Advances in Differential Equations 8(5) (2003), 513-546.
[10] T. Iwaniec, C. Sbordone, *Quasiharmonic fields*, Ann. Inst. H. Poincaré AN 18,5 (2001), 519-572.
[11] N. Meyers, A. Elcrat, *Some results on regularity for solutions of nonlinear elliptic systems and quasiregular functions*, Duke Math. J. 42 (1) (1975), 121-136.
[12] G. Moscariello , *On the integrability of "finite energy" solutions for p-harmonic equations*, NoDEA 11 n.3 (2004), 393-406.
[13] M.M. Rao, Z.D. Ren , *Theory of Orlicz Spaces*, Marcel Dekker, Inc. new York (1991).

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Menita Carozza
Dipartimento Pe.Me.Is., Università degli Studi del Sannio, Piazza Arechi II - 82100, Benevento
E-mail address: `carozza@unisannio.it`

Antonia Passarelli di Napoli
Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", via Cintia - 80126, Napoli
E-mail address: `antonia.passarelli@unina.it`