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QUASILINEAR ELLIPTIC EQUATIONS WITH NATURAL GROWTH TERMS: THE REGULARIZING EFFECT OF THE LOWER ORDER TERMS

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Abstract. This work contains a survey of some results on regularizing properties of the lower order terms in nonlinear Dirichlet and a contribution on the same subjet (not previously published), presented in my lecture at "Recent Advances in Partial Differential Equations", Messina, December 2005.

"Noi navigammo dentro lo Stretto, gemendo:
da una parte c'era Scilla
e dall'altra la divina Cariddi"

1. A survey on the regularizing effect of the lower order terms.

We begin with a quick survey of some results on regularizing properties of the lower order terms in nonlinear Dirichlet problems in Ω, bounded, open subset of $\mathbb{R}^N$, with $N > 2$.

1.1. **Semilinear equations.** The solution u of the simple semilinear boundary value problem in $L^2(\Omega)$

$$\begin{cases} -\Delta u + |u|^{r-1}u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^m(\Omega)$, $m \geq 2$ and $r > 1$, belongs to $L^{mr}(\Omega)$. Thus the summability of the solutions increases as the power of the lower order term increases: for r very large u is "almost bounded". Furthermore, we repeat here a result of [15], written in the following model case. Consider the Dirichlet problems

$$u_r \in W_0^{1,2}(\Omega): \ -\Delta u_r + |u_r|^{r-1}u_r = f(x) \in L^2(\Omega).$$

As $r \to +\infty$, the sequence $\{u_r\}$ converges in $W_0^{1,2}(\Omega)$ to a bounded function u, solution of the bilateral problem with operator $-\Delta$ and datum f on the convex set

$$K = \{v \in W_0^{1,2}(\Omega) : |v| \leq 1 \ \text{ in } \ \Omega\}.$$

In L^1, the study of semilinear elliptic problems was initiated by H. Brezis and W. Strauss ([21], see also [3], [30]).

Then, in [13], is proved (for general elliptic operators with nonlinear principal part) that if $f \in L^1(\Omega)$, then the solution u of (1.1) belongs to $L^r(\Omega)$ and ∇u belongs to $L^q(\Omega)$, $q < \frac{2r}{r+1}$. Remark that $\frac{2r}{r+1} < 2$ and $\frac{2r}{r+1} \to 2$ (as $r \to +\infty$) and that also the summability of ∇u depends (increasing) on r and, in [23], is proved that if $f \in L^m(\Omega)$, $1 + \frac{1}{p} < m < \frac{2N}{N+2}$, then the solution u of (1.1) have finite energy.

We recall the results of [7] in the following simple framework

$$(1.2) \qquad \begin{cases} -\Delta u + h(u) = f(x) & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \leq f \in L^1(\Omega)$ and $h(s) : [0, \sigma) \mapsto \mathbb{R}^+$ is a continuous, increasing real function such that (vertical asymptote)

$$\lim_{s \to \sigma^-} h(s) = +\infty.$$

There exists a bounded, weak solution u of of the above boundary value problem. Moreover $|u| \leq \sigma$, and the measure of the set $\{x : u(x) = \sigma\}$ is zero.

The common point of the above existence results is: more growth in the lower order term gives more summability (until boundedness) of the solutions, even if the right hand side only belongs to $L^1(\Omega)$. Other contributions to the study of elliptic equations with vertical asymptotes in the nonlinear term can be found in [24], [26].

The case where the right hand side is a measure turns out different than one might expect. It was observed by Ph. Bénilan and H. Brezis (see [2]) that, if $N \geq 3$,

$$\begin{cases} -\Delta u + |u|^{r-1} u = \mu & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution when $r \geq \frac{N}{N-2}$ and $\mu = \delta_a$ (with $a \in \Omega$) and has a solution for any measure μ, if $r < \frac{N}{N-2}$.

In [7] it is proved also a nonexistence theorem for the problem (1.2) if $f = \delta_a$.

1.2. **Quasilinear equations.** Simple examples of functionals defined by multiple integrals in the Calculus of Variations as

$$I(v) = \frac{1}{2} \int_\Omega a(x, v) |\nabla v|^2 - \int_\Omega f(x) v(x), \quad v \in W_0^{1,2}(\Omega),$$

with $0 < \alpha \leq a(x, s) \leq \beta$, $|a'_s(x, s)| \leq \nu$, give models of Dirichlet problems with lower order terms with natural growth (that is, lower order terms with quadratic dependence with respect to the gradient) since

$$\begin{cases} \langle I'(u), \phi \rangle = \displaystyle\int_\Omega a(x, u) \nabla u \nabla \phi + \frac{1}{2} \int_\Omega a'_s(x, u) |\nabla u|^2 \phi - \int_\Omega f \phi, \\ \qquad\qquad \forall \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases}$$

Moreover the differential operator $I'(v)$ is not pseudomonotone in the sense of H. Brezis ([18]), since the lower order term maps $W_0^{1,2}(\Omega)$ only to $L^1(\Omega)$ and not to the dual of $W_0^{1,2}(\Omega)$. For existence of weak solutions of Dirichlet problems (1.3) with lower order terms with quadratic dependence with respect to the gradient, also not variational as (1.3), see [17] (bounded solutions), [6] (unbounded solutions) and the references therein.

Consider the boundary value problem

$$(1.3) \qquad \begin{cases} -\operatorname{div}(a(x, u) \nabla u) + g(x, u) |\nabla u|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the assumptions: there exist $\alpha \in \mathbb{R}^+$, $d(x) \in L^1(\Omega)$ and $\beta(s)$, $\nu(s)$ continuous and increasing real functions such that

$$f \in L^2(\Omega), \tag{1.4}$$

$$0 < \alpha \leq a(x,s) \leq \beta(s), \tag{1.5}$$

$$|g(x,s)| \leq d(x)\nu(s) \tag{1.6}$$

and

$$g(x,s)s \geq 0. \tag{1.7}$$

An example is given by the Euler-Lagrange equation of the functional

$$\frac{1}{2}\int_\Omega (1+|v|^m)|\nabla v|^2 - \int_\Omega fv.$$

The assumption (1.7), introduced in [19] for semilinear problems (that is, the lower order term does not depend on the gradient) and in [16] for quasilinear problems, is the key point in the study of the existence of unbounded solutions in $W_0^{1,2}(\Omega)$ (see [16], [4], [5], [6], [28]).

The theorem proved in [10] shows that the existence of finite energy weak solutions of (1.3) can be proved only with the weaker assumption $f \in L^1(\Omega)$, thanks to the presence of the order term with quadratic dependence with respect to the gradient and (1.7). This result is somewhat surprising because it is not true in the linear case!

In [12] is studied the boundary value problem

$$\begin{cases} -\Delta u + u|\nabla u|^2 = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

There exists a solution u in $W_0^{1,2}(\Omega)$ for the previous problem if and only the measure μ does not charge the sets of capacity zero in Ω. Moreover if we consider a sequence $\{u_n\}$ of solutions with $L^\infty(\Omega)$ data μ_n converging to a nonzero measure which is singular with respect to the capacity (such as, for example, a Dirac mass), then u_n converges to zero as n tends to infinity. This nonexistence result is closely related to the work of H. Brezis and L. Nirenberg (see [20]), where (as a particular case of more general results) it is proved that if μ is a bounded $L^\infty(\Omega)$ function, and u is a smooth solution in $\Omega \setminus K$, with K a closed set of zero capacity, then u is smooth in the whole of Ω; that is to say, u cannot be singular on sets of zero capacity.

Other results concerning the regularizing effect of lower order terms can be found in [26] (bounded solutions) and in [27] (extra summability of solutions).

Once more, the case where the right hand side is a general measure is different than the case where the right hand side is a summable function (or a measure absolutely continuous with repect to the capacity) since there may even be nonexistence of solutions instead of a regularizing effect (see [12], [7]).

2. New results

2.1. **Degenerate coercivity: existence and nonexistence.** In the papers [9], [8], [1] [25] existence and regularity results for the following elliptic problem (with degenerate coercivity) are studied:

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $M(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function (that is, measurable with respect to x for every $s \in \mathbb{R}$, and continuous with respect to s for almost every $x \in \Omega$) satisfying the following condition:

$$\frac{\alpha}{(1+|s|)^\theta} \le M(x,s) \le \beta\,, \tag{2.2}$$

for some real number θ such that

$$0 \le \theta \le 1\,, \tag{2.3}$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, where α and β are positive constants. The datum f belongs to $L^m(\Omega)$, for some $m \ge 1$.

The main difficulty in dealing with problem (2.1) is the fact that, because of assumption (2.2), the differential operator $A(v) = -\operatorname{div}(a(x,v)\nabla v)$, even if it is well defined between $W_0^{1,2}(\Omega)$ and its dual $W^{-1,2}(\Omega)$, is not coercive on $W_0^{1,2}(\Omega)$, since when v is large, $1/(1+|v|)^\theta$ goes to zero.

In [1] also a nonexistence theorem is proved. Here we repeat his statement in a particular case.

Consider the example u_λ of the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u_\lambda}{(1+|u_\lambda|)^\gamma}\right) = \lambda \in \mathbb{R}^+ & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Defining (u_λ is positive),

$$z_\lambda = \frac{1-(1+u_\lambda)^{1-\gamma}}{\gamma - 1}\,,$$

one has that z_λ is a solution of

$$\begin{cases} -\Delta z_\lambda = \lambda & \text{in } \Omega, \\ z_\lambda = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the Laplacian is linear, then $z_\lambda = \lambda\, z_1$, where z_1 is the unique solution of the problem

$$\begin{cases} -\Delta z_1 = 1 & \text{in } \Omega, \\ z_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since z_1 is radially symmetric, it can be explicitly calculated, and one has

$$z_1(\rho) = \frac{1}{2N}(1-\rho^2)\,, \qquad (\rho = |x|)$$

so that $z_\lambda(\rho) = \lambda\, z_1(\rho) = \frac{\lambda}{2N}(1-\rho^2)$. By definition, $z_\lambda \le \frac{1}{\gamma-1}$, so that one can recover an "actual" solution u_λ starting from z_λ if and only if the maximum of z_λ is strictly smaller than $\frac{1}{\gamma-1}$. Since

$$\max_{B(0,1)} z_\lambda(\rho) = z_\lambda(0) = \frac{\lambda}{2N}\,,$$

this can be done if and only if $\lambda < \frac{2N}{\gamma-1}$. Define $\lambda^* = \frac{2N}{\gamma-1}$. For $\lambda = \lambda^*$ one has

$$z_{\lambda^*}(\rho) = \frac{1}{\gamma-1}\,(1-\rho^2)\,,$$

which implies

$$u_{\lambda^*}(\rho) = \frac{1}{\rho^{\frac{2}{\gamma-1}}} - 1\,.$$

Note that u_{λ^*} is not in $L^\infty(\Omega)$, and that it belongs to $W_0^{1,2}(\Omega)$ if and only if $\gamma > \frac{N+2}{N-2}$. Moreover, a rather "bizarre" fact happens: the regularity of u_{λ^*} increases as γ increases, and this is contradiction with the properties of the solutions in the case $\gamma < 1$.

Observe also that if we consider as solutions of (2.4) the solutions given starting from z_λ also in the case $\lambda > \lambda^*$, one has

$$u_\lambda(\rho) = \begin{cases} \Big(1 - \dfrac{\lambda(\gamma-1)}{2N}(1-\rho^2)\Big)^{\frac{1}{1-\gamma}} - 1, & \text{if } \rho_\lambda < \rho \le 1 \\ +\infty, & \text{if } 0 \le \rho \le \rho_\lambda \end{cases}$$

where

$$\rho_\lambda = \sqrt{\frac{\lambda(\gamma-1) - 2N}{\lambda(\gamma-1)}}\,,$$

so that u_λ is equal to $+\infty$ on a set of positive Lebesgue measure.

Hence the degenerate coercivity of the differential operator gives solutions even less regular (or no solutions at all) than solutions of Dirichlet problems with singular right hand side.

2.2. **Degenerate coercivity: the regularizing effect of the lower order terms depending on the gradient.** In this section, we present a theorem (obtained in collaboration with Tommaso Leonori and Francesco Petitta) concerning existence of weak solutions for the Dirichlet problem

$$u \in W_0^{1,2}(\Omega):\ -\operatorname{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f(x). \tag{2.5}$$

Here $A(v) = -\operatorname{div}(a(x,v,\nabla v))$ is a differential operator from $W_0^{1,2}(\Omega)$ into $W^{-1,2}(\Omega)$ and $g(x,v,\nabla v)$ is a nonlinearity with natural growth and which satisfies a sign condition.

More in details we assume that $a(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ and $g(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are Caratheodory functions. Moreover we assume that there exist $d(x) \in L^1(\Omega)$, α, β, γ, σ, $\mu > 0$, and $b(s)$, continuous and increasing real function, such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$a(x,s,\xi)\cdot\xi \ge \frac{\alpha}{(1+|s|)^\gamma}\,|\xi|^2\,, \tag{2.6}$$

$$|a(x,s,\xi)| \le \beta\,|\xi| \tag{2.7}$$

and (see [10])

$$\begin{cases} 0 \le g(x,s,\xi)s, \\ \mu|\xi|^2 \le |g(x,s,\xi)|, \quad \forall\,|s| \ge \sigma, \\ |g(x,s,\xi)| \le b(|s|)d(x)|\xi|^2. \end{cases} \tag{2.8}$$

Finally, on the datum, we assume

$$f \in L^1(\Omega). \tag{2.9}$$

It is possible to prove the following existence theorem:

Theorem 2.1. *Under the assumptions (2.6), (2.7), (2.8), (2.9), there exists at least one solution of (2.5) in the following weak sense*

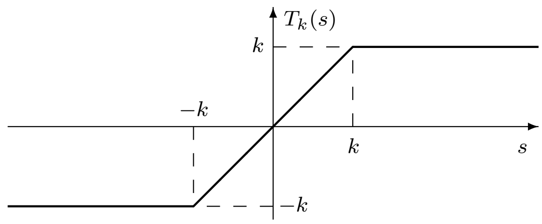
$$\begin{cases} u \in W_0^{1,2}(\Omega),\ g(x,u,\nabla u) \in L^1(\Omega): \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla\phi + \int_\Omega g(x,u,\nabla u)\phi = \int_\Omega f\phi, \\ \forall \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{2.10}$$

Sketch of the proof. Consider the sequence of approximate equations

$$u_n \in W_0^{1,2}(\Omega):\ -\operatorname{div} a(x, T_n(u_n), \nabla u_n) + g(x, u_n, \nabla u_n) = f_n, \tag{2.11}$$

where f_n is a sequence of smooth functions which converges strongly to f in $L^1(\Omega)$ and such that $\|f_n\|_1 \le \|f\|_{L^1(\Omega)}$, $T_k(v)$, $k \in \mathbb{R}^+$, is the usual truncation in $W_0^{1,2}(\Omega)$ defined by

$$T_k(s) = \begin{cases} -k & \text{if} \quad s \le -k, \\ s & \text{if} \quad -k < s < k, \\ k & \text{if} \quad s \ge k \end{cases}$$



and $G_k(s) = s - T_k(s)$.

Since $g(x,s,\xi)s \ge 0$, there exists at least one solution u_n ([16], [4]) and u_n belongs to $L^\infty(\Omega)$ ([29]).

We will follow the approach of [10] and we will use some techniques of [11], [16], [4].

In (2.11), the use of the test function $T_k(u_n)$ yields for any $k > 0$

$$\alpha \int_\Omega \frac{\nabla u_n}{(1+|u_n|)^\gamma} \nabla T_k(u_n) + \int_\Omega g(x,u_n,\nabla u_n) T_k(u_n) \le \int_\Omega f_n T_k(u_n)$$

which implies, for $k \ge \sigma$,

$$\frac{\alpha}{(1+k)^\gamma} \int_\Omega |\nabla T_k(u_n)|^2 + \mu\sigma \int_{\{|u_n|>k\}} |\nabla u_n|^2 \le k\|f\|_{L^1(\Omega)}$$

and so

$$\int_\Omega |\nabla u_n|^2 \le C_{\alpha,\sigma} \|f\|_{L^1(\Omega)}.$$

Therefore there exist $u \in W_0^{1,2}(\Omega)$ and a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$, and almost everywhere.

So we know that for any fixed $k \in \mathbb{R}^+$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,2}(\Omega).$$

We shall use in (2.11) $\varphi[T_k(u_n) - T_k(u)]$ as test function, with $\varphi(s) = (e^{\lambda|s|} - 1)\mathrm{sgn}(s)$ The use of the test function $\varphi(u_n)$ is one of the main tools in the existence proof of [16], [17].

Since $\varphi[T_k(u_n) - T_k(u)]$ converges to zero weakly in $W_0^{1,2}(\Omega)$ and $*$-weakly in $L^\infty(\Omega)$ we have

$$\int_\Omega f_n \varphi[T_k(u_n) - T_k(u)] \to 0. \tag{2.12}$$

Thus we get

$$\langle A(u_n), \varphi[T_k(u_n) - T_k(u)]\rangle + \int_\Omega g(x, u_n, \nabla u_n)\varphi[T_k(u_n) - T_k(u)] \to 0. \tag{2.13}$$

The main step of the proof is that (2.13) implies

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } W_0^{1,2}(\Omega). \tag{2.14}$$

The strong convergence (2.14) yields

$$\nabla u_n \to \nabla u \quad \text{in measure.} \tag{2.15}$$

Now we do not use the classical method (see [16], [4], [10]) of using the Vitali Theorem, in order to prove that $g(x, u_n, \nabla u_n)$ converges in $L^1(\Omega)$. We follow an other Measure Theory approach due to F. Cavalletti ([22]).

Fix $\epsilon > 0$. Let $k_0 \in \mathbb{R}^+$ such that $\int_{\{|u_n|>k_0\}} |f_n(x)| < \epsilon$ (uniformly with respect to n); let $n_1 \in \mathbb{N}$ such that, for $n > n_1$,

$$\int_\Omega |g(x, T_{k_0}(u_n), \nabla T_{k_0}(u_n)) - g(x, T_{k_0}(u), \nabla T_{k_0}(u))| < \epsilon$$

Moreover, for $\delta > 0$, the use of $\psi_{\delta,k}(u_n)$, where

$$\psi_{\delta,k}(s) = \begin{cases} 0 & \text{if } \ 0 \le s < k \\ \dfrac{s-k}{\delta} & \text{if } \ k \le s < k+\delta, \\ 1 & \text{if } \ s \ge k+\delta \\ \psi_{\delta,k}(-s) = -\psi_{\delta,k}(s) \end{cases}$$

as test function in (2.11) yields

$$\int_{\Omega} g(x,u_n,\nabla u_n)\psi_{\delta,k}(u_n) \le \int_{\Omega} f_n\psi_{\delta,k}(u_n)$$

which implies, as $\delta \to 0$,

$$\int_{\{|u_n|\ge k\}} |g(x,u_n,\nabla u_n)| \le \int_{\{|u_n|\ge k\}} |f_n|$$

and, for $k \ge \sigma$

$$\mu \int_{\{|u_n|\ge\sigma\}} |\nabla u_n|^2 \le \int_{\{|u_n|\ge\sigma\}} |f_n|.$$

Set $g_n = g(x,u_n,\nabla u_n)$ and $g_0 = g(x,u,\nabla u)$. Then, for $n > n_1$, we have

$$\begin{aligned}
&\int_{\Omega} |g(x,u_n,\nabla u_n) - g(x,u,\nabla u)| \\
&= \int_{\{|g_n-g_0|\le\epsilon\}} |g_n - g_0| + \int_{\{|g_n-g_0|>\epsilon\}} |g_n| + \int_{\{|g_n-g_0|>\epsilon\}} |g_0| \\
&\le \epsilon|\Omega| + \int_{\{|g_n-g_0|>\epsilon\}\cap\{|u_n|\le k_0\}} |g_n| + \int_{\{|u_n|>k_0\}} |g_n| + \int_{\{|g_n-g_0|>\epsilon\}} |g_0| \\
&\le \epsilon|\Omega| + \int_{\{|g_n-g_0|>\epsilon\}} |g(x,T_{k_0}(u_n),\nabla T_{k_0}(u_n))| + \int_{\{|u_n|>k_0\}} |f_n| + \int_{\{|g_n-g_0|>\epsilon\}} |g_0| \\
&\le \epsilon|\Omega| + \int_{\Omega} |g(x,T_{k_0}(u_n),\nabla T_{k_0}(u_n)) - g(x,T_{k_0}(u),\nabla T_{k_0}(u))| \\
&\quad + \int_{\{|g_n-g_0|>\epsilon\}} |g(x,T_{k_0}(u),\nabla T_{k_0}(u))| + \int_{\{|u_n|>k_0\}} |f_n| + \int_{\{|g_n-g_0|>\epsilon\}} |g_0| \\
&\le \epsilon|\Omega| + \epsilon + \int_{\{|g_n-g_0|>\epsilon\}} |g(x,T_{k_0}(u),\nabla T_{k_0}(u))| + \epsilon + \int_{\{|g_n-g_0|>\epsilon\}} |g_0|
\end{aligned}$$

Since g_n converges to g_0 in measure, $\text{meas}\{x \in \Omega : |g_n - g_0| > \epsilon\} \to 0$. Moreover g_0 and $g(x,T_{k_0}(u),\nabla T_{k_0}(u))$ belong to $L^1(\Omega)$: the absolute continuity of the Lebesgue integral yields

$$\lim_n \int_{\{|g_n-g_0|>\epsilon\}} |g(x,T_{k_0}(u),\nabla T_{k_0}(u))| = 0, \quad \lim_n \int_{\{|g_n-g_0|>\epsilon\}} |g_0| = 0$$

Thus we have

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega). \tag{2.16}$$

Again fix $\epsilon > 0$ and let $k_0 \in \mathbb{R}^+$ such that $\int_{\{|u_n|>k_0\}} |f_n(x)| < \epsilon$ (uniformly with respect to n); let $n_2 \in \mathbb{N}$ such that, for $n > n_2$

$$\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^2 \leq \epsilon.$$

Using that $\nabla u \in L^2(\Omega)$ and the absolute continuity of the Lebesgue integral, then ($k > k_0$, $k > \sigma$, $n > n_1$, $n > n_2$)

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u|^2 &\leq 2\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^2 + 4\int_\Omega |\nabla G_k(u_n)|^2 + 4\int_\Omega |\nabla G_k(u)|^2 \\ &\leq 2\epsilon + 4\frac{\epsilon}{\mu} + 4\epsilon\end{aligned}$$

which proves that

$$u_n \to u \quad \text{strongly in } W_0^{1,2}(\Omega). \tag{2.17}$$

This convergence result and (2.7) imply

$$a(x, T_n(u_n), \nabla u_n) \to a(x, u, \nabla u) \quad \text{strongly in } L^2(\Omega). \tag{2.18}$$

Use now ϕ as test function in (2.11)

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\nabla\phi + \int_\Omega g(x, u_n, \nabla u_n)\phi = \int_\Omega f_n\phi$$

and recall (2.18), (2.16). Then it is possible to pass to the limit in and we obtain that u is a solution of (2.5) in the sense of (2.10). □

Now we present two examples concerning the problem (2.5).

Example 2.2. Consider the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)}\right) + \operatorname{arctg}(u)|\nabla u|^2 = f(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Since $u \geq 0$, the change $z = \log(1+u)$ leads to

$$\begin{cases} -\Delta z + \operatorname{arctg}(e^z - 1)e^{2z}|\nabla z|^2 = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Existence and properties of z can be deduced by the results of [6].

Example 2.3. Consider the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^2}\right) + \operatorname{arctg}(u)|\nabla u|^2 = f(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Since $u \geq 0$, the change $w = [1 - (1+u)^{-1}]$ leads to

$$\begin{cases} -\Delta w + \operatorname{arctg}\Big(\dfrac{w}{1-w}\Big)\dfrac{|\nabla w|^2}{(1-w)^4} = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

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