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EXISTENCE OF CONTINUOUS SOLUTIONS TO TIME-DEPENDENT VARIATIONAL INEQUALITIES

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Abstract. The author considers equilibrium problems with data depending explicitly on time and studies under which assumptions the continuity of the solution with respect to the time can be guaranteed. We show continuity results for solutions to time-dependent variational inequalities for a general class of convex sets, and then we apply these results to dynamic traffic equilibrium problems. Taking into account the regularity results presented, we adapt extragradient method to solve a evolutionary variational inequality and we report the results of a numerical approximation.

1. Introduction

The aim of this paper is to present continuity results for the solutions to evolutionary variational inequalities associated to linear and nonlinear strongly monotone operators and to linear degenerate operators. Our results are related to the nonempty, closed and convex sets $\mathbf{K}(t)$, $t \in [0,T]$, which fulfil the Mosco's convergence property. In particular, the continuity result obtained in [1] in the core of linear strongly monotone operators for the set of constraints related to dynamic traffic equilibrium problems will be generalized for a general class of convex sets. Since, the set of constraints related to dynamic traffic equilibrium problems and many other equilibrium problems fulfils this condition, then for these general results it follows the continuity of equilibrium solutions.

The paper is organized as follows. In Sec. 2, we introduce the time-dependent variational inequality which models the time-dependent traffic equilibrium problem. In Sec. 3, we generalize Theorem 3.2 in [1], and we show that the solutions to time-dependent variational inequalities associated to linear degenerate and nonlinear strongly monotone operators are continuous mappings from the time interval $[0,T]$ to the Euclidian space $\mathbb{R}^m_+$ (see Theorems 3.3 and 3.4). At last, in Sec. 4, we apply the shown result to the traffic equilibrium problem and the associated variational inequality. In order to calculate an approximated solution of a dynamic traffic network we use a discretization procedure and then we compute, by means of the extragradient method, the solution of the finite-dimensional variational inequalities obtained using the discretization. Finally, we construct an approximation solution interpolating the static equilibrium solutions found.

2. The dynamic model

Let us consider a dynamic traffic network equilibrium problem. A traffic network is represented by a graph $G = [N, L]$, where N is the set of nodes (i.e. cross-roads,

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airports, railway stations) and L is the set of directed links between the nodes. Let r be a path consisting of a sequence of links which connect an Origin-Destination (O/D) pair of nodes. Let m be the number of the paths in the network. Let $\mathcal{W}$ denote the set of the O/D pairs with typical O/D pair w_j, $|\mathcal{W}| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. The topology of the network is described by the pair-link incidence matrix $\Phi = (\varphi_{j,r})$, where $\varphi_{j,r}$ is 1 if path R_r connects the pair w_j and 0 otherwise. Since the feasible flows have to satisfy time-dependent capacity constraints and demand requirements, also the flow vector is a time-dependent flow vector $F(t) \in \mathbb{R}^m_+$, where t varies in the fixed time interval $[0,T]$, while the topology remains fixed. Each component $F_r(t)$ of $F(t)$ gives the flow trajectory $F : [0,T] \to \mathbb{R}^m_+$ which has to satisfy almost everywhere on $[0,T]$ the capacity constraints

$$\lambda(t) \leq F(t) \leq \mu(t)$$

and the traffic conservation law

$$\Phi F(t) = \rho(t),$$

where the bounds $\lambda < \mu$ and the demand $\rho = (\rho_j)_{w_j \in \mathcal{W}}$ are given. We assume that λ and μ belong to $L^2([0,T], \mathbb{R}^m_+)$ and that ρ lies in $L^2([0,T], \mathbb{R}^m_+)$. Assuming in addition that

$$\Phi\lambda(t) \leq \rho(t) \leq \Phi\mu(t) \ \text{ a.e. in } [0,T],$$

we obtain that the set of feasible flows

$$\mathbf{K} = \Big\{ F \in L^2([0,T], \mathbb{R}^m_+) : \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t), \text{ a.e. in } [0,T] \Big\}$$

is nonempty, as it is shown in [7]. We remark that this kind of feasible set includes the constraint set related to dynamic market, evolutionary financial equilibrium problems, electric power supply chain networks with known demands and human migration problems. Clearly $\mathbf{K}$ is a convex, closed, bounded subset of $L^2([0,T], \mathbb{R}^m_+)$. Furthermore, we give the cost trajectory C, which becomes a function of the time $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$. The equilibrium condition is given by a generalized version of Wardrop's condition (see [5, 6]), namely:

Definition 2.1. A flow $H \in \mathbf{K}$ is a user traffic equilibrium flow if $\forall w_j \in \mathcal{W}$, $\forall q, s \in \mathcal{R}_j$ and a.e. in $[0,T]$ it results:

$$C_q(t, H(t)) > C_s(t, H(t)) \Longrightarrow H_q(t) = \lambda_q(t) \quad \textit{or} \quad H_s(t) = \mu_s(t). \tag{1}$$

The overall flow pattern obtained according with condition (1) fits very well in the framework of the theory of variational inequalities. In fact, in [5] and [6] the following result has been proved:

Theorem 2.1. *A flow $H \in \mathbf{K}$ is an equilibrium pattern if and only if it satisfies the following evolutionary variational inequality:*

$$\int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle dt \geq 0, \quad \forall F \in \mathbf{K}. \tag{2}$$

In order to give some results of existence of equilibria, we shall recall some definitions.

Definition 2.2. $C : [0,T] \times \mathbf{K} \to L^2([0,T],\mathbb{R}^m)$ is said to be:

- *strongly monotone* if for all $F, H \in \mathbf{K}$ there exists $\nu > 0$ such that
$$\int_0^T \langle C(t,F(t)) - C(t,H(t)), F(t) - H(t)\rangle dt \geq \nu \|F - H\|^2_{L^2([0,T],\mathbb{R}^m)};$$
- *strictly monotone* if for all $F, H \in \mathbf{K}$, $F \neq H$,
$$\int_0^T \langle C(t,F(t)) - C(t,H(t)), F(t) - H(t)\rangle dt > 0;$$
- *pseudomonotone* if for all $F, H \in \mathbf{K}$
$$\int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle dt \geq 0 \Longrightarrow \int_0^T \langle C(t,F(t)), H(t) - F(t)\rangle dt \geq 0;$$
- *upper hemicontinuous* if for all $F \in \mathbf{K}$ the function
$$H \to \int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle dt$$
is upper semicontinuous on $\mathbf{K}$;
- *upper hemicontinuous along line segments* if for all $H, F \in \mathbf{K}$ the function
$$G \to \int_0^T \langle C(t,G(t)), F(t) - H(t)\rangle dt$$
is upper semicontinuous on the line segment $[H,F]$.

The following general result holds:

Theorem 2.2. ([6]) *Let $C : [0,T] \times \mathbf{K} \to L^2([0,T],\mathbb{R}^m)$ and $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty and convex set. Assume that:*

(i) *there exist $A \subseteq \mathbf{K}$ nonempty, compact and $B \subseteq \mathbf{K}$ compact, convex such that, for every $H \in \mathbf{K} \setminus A$, there exists $\widehat{H} \in B$ with*
$$\int_0^T \langle C(t,H(t)), \widehat{H}(t) - H(t)\rangle dt < 0;$$

either (ii) or (iii) below holds:

(ii) *C is upper hemicontinuous;*
(iii) *C is pseudomonotone and upper hemicontinuous along line segments.*

Then, there exists $H \in A$ such that
$$\int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle dt \geq 0,$$
for all $F \in \mathbf{K}$.

It is well known that if C is in addition strictly monotone, then the solution to the evolutionary variational inequality is unique.

From Theorem 2.2 it is possible to derive the following existence theorem, which gives a sufficient condition in terms of the operator $C(t,F)$ (see [12]).

Theorem 2.3. *Let* $C(t,F) : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ *be a vector-function measurable in* t*, continuous in* F *and such that*

$$\|C(t,F)\|_m \leq A(t)\|F\|_m + B(t), \quad \text{a.e. in } [0,T], \tag{3}$$

with $B \in L^2([0,T],\mathbb{R}_+)$ *and* $A \in L^\infty([0,T],\mathbb{R}_+)$*, and for each* $F, H \in \mathbf{K}$ *it results*

$$\int_0^T \langle C(t,F(t)) - C(t,H(t)), F(t) - H(t)\rangle dt \geq 0.$$

Let $\lambda, \mu \in L^2([0,T],\mathbb{R}^m_+)$ *and let* $\rho \in L^2([0,T],\mathbb{R}^l_+)$ *be vector-functions. Then, the variational inequality* (2) *admits solutions.*

It is well known that if C is in addition strongly monotone, then the solution to the evolutionary variational inequality is unique.

We recall that problem (2) (see [13]) is also equivalent to the following one:

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{4}$$

where

$$\mathbf{K}(t) = \left\{ F(t) \in \mathbb{R}^m_+ : \ \lambda(t) \leq F(t) \leq \mu(t), \ \ \Phi F(t) = \rho(t) \right\},$$

for a.e. $t \in [0,T]$.

Now, we present some result related to time-dependent variational inequalities when the path cost vector is linear with respect to the path flow vector, i.e. $C(t,H(t)) = A(t)H(t) + B(t)$, where $A, B : [0,T] \to \mathbb{R}^{m\times m}_+$, that is

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \tag{5}$$

In this case the following results hold.

Theorem 2.4 ([9])**.** *Let* $A \in L^2([0,T],\mathbb{R}^m_+)$ *be a bounded positive definite matrix-function, that is,*

$$\exists M > 0 : \ \|A(t)\|_{m\times m} = \Big(\sum_{r,s=1}^m A^2_{rs}(t)\Big)^{\frac{1}{2}} \leq M, \quad \text{a.e. in } [0,T], \tag{6}$$

$$\exists \nu > 0 : \ \langle A(t)F(t), F(t)\rangle \geq \nu\|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{7}$$

and $B \in L^2([0,T],\mathbb{R}^m_+)$*, then there exists a unique solution to the time-dependent variational inequality* (5)*.*

Theorem 2.5 ([2])**.** *Let* $A \in L^2([0,T],\mathbb{R}^m_+)$ *be a bounded degenerate matrix-function, namely satisfies conditions* (6) *and*

$$\langle A(t)F(t), F(t)\rangle \geq \nu(t)\|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{8}$$

where $\nu \in L^\infty([0,T],\mathbb{R}^+_0)$ *is such that*

$$\nexists I \subseteq [0,T], \ \mu(I) > 0 : \ \nu(t) = 0, \quad \text{a.e. in } I,$$

and let $B \in L^2([0,T],\mathbb{R}^m_+)$ *be a matrix-function. Then the degenerate time-dependent variational inequality* (5) *admits a unique solution.*

3. Continuity results for time-dependent variational inequalities

In this section, we will generalize the theorem of continuity for solutions to evolutionary variational inequalities associated to linear strongly monotone operator proved in [1] for a general class of convex sets $\mathbf{K}(t)$, $t \in [0,T]$, and we will present analogous results for linear degenerate and nonlinear operators.

In the following, we will make use the important concept of the sets convergence in Mosco's sense (see [14]).

Definition 3.1. Let $(V, \|\cdot\|)$ be an Hilbert space and $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$, as $n \to +\infty$, in Mosco's sense, if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$,

(M2) for any subsequence $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all $n \in \mathbb{N}$, then the weak limit H belongs to $\mathbf{K}$.

Definition 3.2. A sequence of operators $A_n : \mathbf{K}_n \to V'$ converges to an operator $A : \mathbf{K} \to V'$ if

$$\|A_nH_n - A_nF_n\|_* \le M\|H_n - F_n\|, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{9}$$

$$\langle A_nH_n - A_nF_n, H_n - F_n\rangle \ge \nu\|H_n - F_n\|^2, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{10}$$

hold with fixed constants $M, \nu > 0$ and

(M3) the sequence $\{A_nH_n\}_{n\in\mathbb{N}}$ strongly converges to AH in V', for any sequence $\{H_n\}_{n\in\mathbb{N}}$, such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$, strongly converging to $H \in \mathbf{K}$.

In (9) $\|\cdot\|_*$ is the norm in the dual space of V.

It results that the set as in (4) fulfils the conditions of Definition 3.1.

Lemma 3.1. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, let $\rho \in C([0,T], \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets*

$$\mathbf{K}(t_n) = \Big\{F(t_n) \in \mathbb{R}^m : \lambda(t_n) \le F(t_n) \le \mu(t_n),\ \Phi F(t_n) = \rho(t_n)\Big\},$$

$\forall n \in \mathbb{N}$, converges to

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m : \lambda(t) \le F(t) \le \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

as $n \to +\infty$, in Mosco's sense.

Proof. See proof of Theorem 3.2 in [1]. □

We recall an abstract stability result due to Mosco (see [15], Theorem 4.1):

Theorem 3.1. *Let $\mathbf{K}_n \to \mathbf{K}$ in Mosco's sense* (M1)–(M2) *, $A_n \to A$ in the sense of* (M3) *and $B_n \to B$ in V'. Then the unique solutions H_n of*

$$H_n \in \mathbf{K}_n : \quad \langle A_nH_n - B_n, F_n - H_n\rangle \ge 0, \quad \forall F_n \in \mathbf{K}_n \tag{11}$$

converge strongly to the solution H of the limit problem (2)*, i.e.,*

$$H_n \to H \quad in\ V.$$

In this section, we suppose that $\mathbf{K}(t)$, $t \in [0, T]$, satisfies the following assumption

(M) $\mathbf{K}(t)$, $t \in [0, T]$, is a family of nonempty convex, closed sets of $\mathbb{R}^m$ such that $\mathbf{K}(t_n)$ converges to $\mathbf{K}(t)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$, with $t_n \to t$, as $n \to +\infty$.

Now, we can generalize the continuity result proved in [1], Theorem 3.2, for a general set satisfying condition (M).

Theorem 3.2. *Let $A \in C([0, T], \mathbb{R}_+^{m\times m})$ be a positive definite matrix-function and let $B \in C([0, T], \mathbb{R}_+^m)$ be a vector function. Let $\mathbf{K}(t)$, $t \in [0, T]$, be a family of sets satisfying condition* (M)*. Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \textit{in } [0, T], \tag{12}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}_+^m)$.

Proof. By virtue of Theorem 2.4, we have that (12) admits a unique solution $H(t) \in \mathbf{K}(t)$, for $t \in [0, T]$.

Now, we prove the continuity of solution applying Theorem 3.1. Let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, with $t_n \to t$. From the assumption of continuity of the function A, one has

$$A(t_n) \to A(t) \quad \text{in } \mathbb{R}^{m\times m},$$

moreover, if $\{F(t_n)\}_{n\in\mathbb{N}}$ is a sequence, with $F(t_n) \in \mathbf{K}(t_n)$, such that $F(t_n) \to F(t)$ in $\mathbb{R}^m$, it results

$$A(t_n)F(t_n) \to A(t)F(t) \quad \text{in } \mathbb{R}^m.$$

Finally, for the continuity of the function B we have

$$B(t_n) \to B(t) \quad \text{in } \mathbb{R}^m.$$

Taking into account that the set $\mathbf{K}(t)$, $t \in [0, T]$, satisfies condition (M) and using the stability Theorem 3.1, we can conclude that the unique solution $H(t_n)$ of

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),$$

converge strongly to the solution $H(t)$ of the limit problem (12), i.e.,

$$H(t_n) \to H(t) \quad \text{in } \mathbb{R}^m,$$

namely $H \in C([0, T], \mathbb{R}_+^m)$. □

We recall that this result can be extended for the degenerate time-dependent variational inequalities by a perturbation procedure (see [2], Theorem 3.2).

Theorem 3.3. *Let $A \in C([0, T], \mathbb{R}_+^{m\times m})$ be a matrix-function satisfying condition* (8) *and let $B \in C([0, T], \mathbb{R}_+^m)$ be a vector-function. Let $\mathbf{K}(t)$, $t \in [0, T]$, be a uniformly bounded family of sets satisfying condition* (M)*. Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \textit{in } [0, T],$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}_+^m)$.

We present the analogous result for nonlinear strongly monotone time-dependent variational inequalities (see [3], Theorem 3.2).

Theorem 3.4. *Let $C \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$ be a strongly monotone operator satisfying condition* (3)*. Let $\mathbf{K}(t)$, $t \in [0,T]$, be a family of sets satisfying condition* (M)*. Then, the evolutionary variational inequality*

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ in \ [0,T].$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m_+)$.

Remark 3.1. Theorems 3.3 and 3.4 still hold true for the convex set

$$\mathbf{K}(t) = \left\{ F(t) \in \mathbb{R}^m_+ : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t) \right\},$$

for $t \in [0,T]$, supposing that $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ and $\rho \in C([0,T], \mathbb{R}^l_+)$, under these assumptions the family of sets satisfies condition (M) (see Lemma 3.1).

4. Application to a dynamic traffic network

Now, we introduce a method to solve evolutionary variational inequalities related to a linear degenerate operator.

We consider the following evolutionary variational inequality

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ \text{a.e. in } [0,T], \tag{13}$$

with A satisfying the following condition

$$\langle A(t)F(t), F(t) \rangle \geq \nu(t) \|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{a.e. in } [0,T], \tag{14}$$

where $\nu \in L^\infty([0,T], \mathbb{R}^+_0)$ is such that $\nexists I \subseteq [0,T], \ \mu(I) > 0 : \ \nu(t) = 0$, a.e. in I, and

$$\mathbf{K} = \left\{ F(t) \in \mathbb{R}^m_+ : \ \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \rho(t) \right\},$$

for a.e. $t \in [0,T]$. We suppose that assumptions of Theorem 3.3 are satisfied and then the solution H belongs to $C([0,T], \mathbb{R}^m_+)$. As a consequence, (13) holds for each $t \in [0,T]$, namely

$$\langle C(H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall t \in [0,T].$$

A refined method to solve variational inequalities is the extragradient method, but it can be applied to evolutionary variational inequalities after that a discretization procedure has been made.

Let us consider a partition of $[0,T]$, such that:

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_N = T.$$

Then, for each value t_i, for $i = 0, 1, \ldots, N$, we obtain the static variational inequality

$$\langle C(H(t_i)), F(t_i) - H(t_i) \rangle \geq 0, \quad \forall F(t_i) \in \mathbf{K}(t_i), \tag{15}$$

where $C(H(t_i)) = A(t_i)H(t_i) + B(t_i)$ and

$$\mathbf{K}(t_i) = \left\{ F(t_i) \in \mathbb{R}^m_+ : \ \lambda(t_i) \leq F(t_i) \leq \mu(t_i), \quad \Phi F(t_i) = \rho(t_i) \right\}.$$

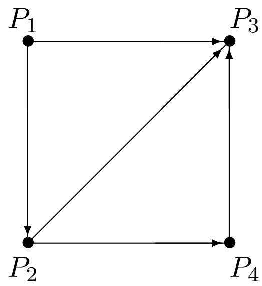

FIGURE 1. A network model.

We compute now the solution to the finite-dimensional variational inequality (15) using a modified version of the extragradient method introduced by Marcotte in [10].

At first, we present the extragradient method. The algorithm, as it is well known, starting from any $H^0(t_i) \in \mathbf{K}(t_i)$ and a number $\alpha > 0$ fixed, iteratively updates $H^{k+1}(t_i)$ from $H^k(t_i)$ according to the following projection formulas

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(\overline{H}^k(t_i))), \quad \overline{H}^k(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(H^k(t_i)))$$

for $k \in \mathbb{N}$, where $P_{\mathbf{K}(t_i)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_i)$.

In [4] and [17] the convergence of the extragradient method is proved under the following hypothesis: C is a monotone and Lipschitz continuous mapping and $\alpha \in (0, 1/L)$ where L is Lipschitz constant.

We remark that a drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all. Then, Khobotov in [8] introduced the idea to perform an adaptive choice of α, changing its value at each iteration. Now, we present a modification of Khoboton's algorithm obtained by Marcotte in [10].

The algorithm starting from any $H^0(t_i) \in \mathbf{K}(t_i)$ and a number $\alpha_0 > 0$ fixed, iteratively updates $H^{k+1}(t_i)$ from $H^k(t_i)$ according to the following projection formulas

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha_k C(\overline{H}^k(t_i))), \quad \overline{H}^k(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha_k C(H^k(t_i)))$$

for $k \in \mathbb{N}$, where α_k is chosen as following

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|H^k(t_i) - \overline{H}^k(t_i)\|_m}{\sqrt{2}\|C(H^k(t_i)) - C(\overline{H}^k(t_i))\|_m}\right\}.$$

If C is a monotone mapping, then, the convergence of the scheme is proved, hence the hypothesis on the Lipschitz continuity of C is removed. This method was improved by Tinti in [16].

After the iterative procedure, we can construct the dynamic equilibrium solution by means of a linear interpolation of the obtained static equilibrium solutions.

Let us consider a network as Figure 1. The network consists of four nodes and five links. The origin-destination pair is $w = (P_1, P_3)$, which is connected by the paths $R_1 = (P_1, P_3)$, $R_2 = (P_1, P_2) \cup (P_2, P_3)$ and $R_3 = (P_1, P_2) \cup (P_2, P_4) \cup (P_4, P_3)$.

We consider the cost operator on the path C defined by

$$C : L^2([0,2],\mathbb{R}^3_+) \to L^2([0,2],\mathbb{R}^3_+);$$

$$\begin{aligned} C_1(H(t)) &= (t+3)H_1(t) + 2t, \\ C_2(H(t)) &= (2t+4)H_2(t) + 1, \\ C_3(H(t)) &= 3tH_2(t) + (t+2)H_3(t) + t + 5. \end{aligned}$$

The set of feasible flows is given by

$$\mathbf{K} = \Big\{ F \in L^2([0,2],\mathbb{R}^3_+) : (2t,2t,0) \le (F_1(t),F_2(t),F_3(t)) \le (10t{+}5, 5t{+}3, 2t{+}1), \\ F_1(t)+F_2(t)+F_3(t) = 5t+3, \ \text{ in } [0,2] \Big\}.$$

It is easy to verify that the cost vector-function satisfies condition (14) and is continuous. Then, the theory of evolutionary variational inequalities states that the problem has a unique continuous equilibrium solution. To compute the solution, we apply the direct method (see [11]), obtaining the exact solution:

$$\begin{cases} H_1(t) = \dfrac{5t^3 + 23t^2 + 25t + 13}{t^2 + 7t + 13}, & \forall t \in [0,2], \\ H_2(t) = \dfrac{5t^3 + 30t^2 + 47t + 13}{2(t^2 + 7t + 13)}, & \forall t \in [0,2], \\ H_3(t) = \dfrac{-5t^3 + 75t + 39}{2(t^2 + 7t + 13)}, & \forall t \in [0,2]. \end{cases}$$

Now, we solve the numerical problem using Marcotte's version of the extragradient method. This methos is convergent for the property of C. Then, we can compute an approximate curve of equilibria, by selecting $t_i \in \big\{\frac{k}{10} : \ k \in \{0,1,\ldots,20\}\big\}$. Using a simple MatLab computation and choosing the initial point $H^0(t_i) = (2t_i+1, 2t_i+1, t_i+1)$ to start the iterative method, we obtain the static equilibrium solutions, as shows Table 1.

The interpolation of equilibria points yields the curves of equilibria, as shows Figure 2.

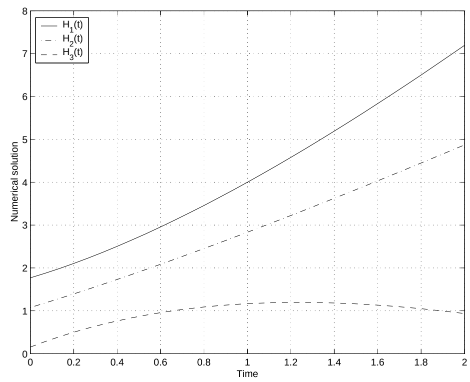

FIGURE 2. Network pattern of the numerical example

Table 1. Numerical results

t_i	$H_1(t_i)$	$H_2(t_i)$	$H_3(t_i)$
0	1.7692257	1.0769299	0.1538443
1/10	1.9288787	1.2332301	0.3378912
1/5	2.1052578	1.3947422	0.5
3/10	2.2972299	1.5610644	0.6417056
2/5	2.5037551	1.7318330	0.7644119
1/2	2.7238761	1.9067196	0.8694042
3/5	2.9567160	2.0854239	0.9578600
7/10	3.2014645	2.2676749	1.0308606
4/5	3.4573773	2.4532243	1.0893985
9/10	3.7237657	2.6418467	1.1343875
1	3.9999967	2.8333350	1.1666683
11/10	4.2854830	3.0275003	1.1870167
6/5	4.5796812	3.2241697	1.1961491
13/10	4.8820899	3.4231834	1.1947266
7/5	5.1922423	3.6243955	1.1833622
3/2	5.5097056	3.8276711	1.1626233
8/5	5.8340776	4.0328860	1.1330364
17/10	6.1649844	4.2399255	1.0950901
9/5	6.5020774	4.4486834	1.0492392
19/10	6.8450321	4.6590615	0.9959064
2	7.1935454	4.8709687	0.9354859

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