



STABILITY OF LINEAR SEMI-COERCIVE VARIATIONAL INEQUALITIES IN HILBERT SPACES: APPLICATION TO THE SIGNORINI-FICHERA PROBLEM

SAMIR ADLY

Dedicated to the memory of Filippo Chiarenza.

ABSTRACT. In this paper we show how recent results concerning the stability of semi-coercive variational inequalities in reflexive Banach spaces, obtained in [2] and [3] can be applied to establish the stability of the semi-coercive Signorini-Fichera problem with respect to small perturbations.

1. INTRODUCTION AND POSITION OF THE PROBLEM

The theory of variational inequalities go back to the introduction of the calculus of variations, their development began in the sixties with the work of Hartman & Stampacchia [16], Stampacchia [22] and Fichera [14]. This theory was used as a tool for the study of partial differential equations with applications essentially drawn from mechanics (Signorini problem, obstacle problems in elasticity etc ...). The study of variational inequalities became an important mathematical tools and have been studied intensively, after the fundamental work of Lions & Stampacchia [20]. With the contributions of Brézis [7], [8], Duvaut-Lions [11], Browder [9], Kinderlehrer & Stampacchia [19] (among others), this field has known an increasing growth in both theory and applications. This branch of applied mathematics covers a large spectrum of problems and is a very attractive area in the calculus of variations, control theory, free boundary problems with a wide range of applications. Many classes of problems in unilateral mechanics or in plasticity theory, as well as in finance, economics, industry and engineering are modeled by variational inequalities.

Let X be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We shall consider the following variational inequality

$$\text{VI}(A, f, \Phi, K) \left\{ \begin{array}{l} \text{Find } u \in K \text{ such that:} \\ \langle Au - f, v - u \rangle + \Phi(v) - \Phi(u) \geq 0, \forall v \in K, \end{array} \right.$$

We suppose that the assumptions (\mathcal{H}) described below are satisfied:

- (1) $A : X \rightarrow X$ is a bounded, symmetric and linear operator such that: $\dim_{\mathbb{R}} \ker(A) < +\infty$. We suppose also that the operator A is *semi-coercive*

2000 *Mathematics Subject Classification.* 46N10, 47N10, 47J20, 49J40, 34A60.

Key words and phrases. Variational inequalities, recession analysis, convex analysis, semi-coercive operators, elastic contact with friction.

The research of the author has been supported by the “Fondation EADS” and the ANR project “Guidage”.

i.e.

$$(1) \quad \exists \kappa > 0 : \langle Au, u \rangle \geq \kappa \|Qu\|^2, \forall u \in X,$$

where $Q = I - P$ and $P : X \rightarrow \ker(A)$ is the orthogonal projection onto $\ker(A)$.

- (2) $K \subset X$ is a closed and nonempty convex set;
- (3) $f \in X$;
- (4) $\Phi \in \Gamma_0(X)$ i.e. $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, lower semi-continuous and proper. We note:

$$\text{Dom}(\Phi) = \{v \in X : \Phi(v) < +\infty\},$$

its effective domain.

Examples 1.1.

- (i) Recall that a linear and monotone operator such that $R(A)$ is closed, satisfies condition (1).
- (ii) Let $A : X \rightarrow X$ be a linear and monotone operator. The following conditions are equivalent
 - (a) there exists a strongly continuous operator $C : X \rightarrow X$ such that $A + C$ is coercive;
 - (b) A is semi-coercive and $\dim_{\mathbb{R}} \ker(A) < +\infty$;
- (iii) Let $(H, |\cdot|)$ be an other Hilbert space such that the embedding $X \hookrightarrow H$ is compact. If $A : X \rightarrow X$ is a linear, bounded and monotone operator satisfying the following Gding inequality

$$\exists \lambda > 0, \exists c > 0 \text{ tel que : } \langle Au, u \rangle + \lambda |u|^2 \geq c \|u\|^2, \forall u \in X,$$

then A is semi-coercive and $\dim_{\mathbb{R}} \ker(A) < +\infty$.

For a proof of these classical results see e.g. [15].

Several theoretical existence results for $VI(A, f, \Phi, K)$ (in general reflexive Banach spaces) are well known when the operator A is coercive. We can cite for instance the contributions of J.L. Lions [21], Brezis [7], [8], Browder [9] etc However, the variational formulation of many engineering problems leads generally to non-coercive variational inequalities. The theory of semi-coercive unilateral problems was studied first by Fichera [13] and Lions & Stampacchia [20], Duvaut & Lions [11] (for problems with frictional type functionals). Recently many mathematicians and engineers has focused their attention on non-coercive unilateral problems, using several different approaches such as the critical point theory, the Leray-Schauder degree theory, the recession analysis or the regularization method by approximating non-coercive problems by coercive ones (see e.g. [1], [4], [25], [5], [6], [24] and references cited therein). The main concern of these contributions is the obtainment of necessary or sufficient conditions for the solvability of such problems in a general setting by imposing some compactness conditions and some compatibility conditions on the right hand term f . More recently, S. Adly et al. [2], [3] has considered the situations in which the existence of the solution is stable with respect to small uniform perturbations of the data of the problem. More precisely, they characterized all data (A, f, Φ, K) for which there is some $\varepsilon > 0$ such that the variational inequality $VI(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ has solutions for every bounded and semi-coercive operator

A_ε , linear functional $f_\varepsilon \in X'$, proper lower semi-continuous convex function Φ_ε that is bounded from below, and closed convex set K_ε such that $K_\varepsilon \cap \text{Dom } \Phi_\varepsilon \neq \emptyset$, and satisfying the following conditions

$$(\mathcal{C}) \begin{cases} \|A(x) - A_\varepsilon(x)\|_* < \varepsilon, & \forall x \in X \\ \|f - f_\varepsilon\|_* < \varepsilon \\ K \subset K_\varepsilon + \varepsilon \mathbb{B}_X \text{ and } K_\varepsilon \subset K + \varepsilon \mathbb{B}_X, \\ \Phi(x) - \varepsilon \leq \Phi_\varepsilon(x) \leq \Phi(x) + \varepsilon, & \forall x \in X, \end{cases}$$

where \mathbb{B}_X is the open unit ball in X .

This type of perturbation could be applicable in finance or in engineering science where the data is only known with a certain precision (due e.g. to statistical measures) and it is desired that further refinement of the data of the problem should not cause the emptiness of the solutions set. We note that this kind of uniform stability with respect to small perturbations is taken in the sense of the existence of solutions and is completely different from Hadamard's stability which requires the continuous dependance with respect to the data of the problem. Note that in [17] was presented also a semi-coercive unilateral boundary value problem with perturbed (uncertain) data (with the restriction to the case of unique solution).

2. SOME STABILITY RESULT AND APPLICATION TO FRICTION PROBLEM

Let us also recall first some background results from convex analysis which will be used later.

Let K be a closed convex subset of X , the *recession cone* of K is the closed convex cone

$$K_\infty := \bigcap_{t>0} \left[\frac{K - x_0}{t} \right],$$

where x_0 is arbitrary chosen in K .

Let $\Phi \in \Gamma_0(X)$, the recession function Φ_∞ of Φ is defined by:

$$(2) \quad \Phi_\infty(x) := \lim_{\lambda \rightarrow +\infty} \frac{\Phi(x_0 + \lambda x) - \Phi(x_0)}{\lambda},$$

where $x_0 \in \text{Dom } \Phi$ is an arbitrary element. We set $\ker \Phi_\infty = \{x \in X : \Phi_\infty(x) = 0\}$, which is a closed convex cone in X .

The Fenchel conjugate $\Phi^* : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ of Φ is defined by:

$$\Phi^*(x^*) = \sup_{x \in X} \left\{ \langle x^*, x \rangle - \Phi(x) \right\}.$$

The indicator function to a convex set K is given by:

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K. \end{cases}$$

If K is a closed cone, its polar is defined by

$$K^\circ = \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \forall x \in K\}.$$

Let us now give a necessary condition for the existence of a solution of the variational inequality $\text{VI}(A, f, \Phi, K)$. The following proposition is in this sense.

Proposition 2.1. *Suppose that the assumptions (\mathcal{H}) hold. Then a necessary condition for the existence of a solution of $\text{VI}(A, f, \Phi, K)$ is that*

$$(3) \quad \langle f, w \rangle \leq \Phi_\infty(w), \quad \forall w \in \ker(A) \cap K_\infty.$$

Proof. We remark first that $\text{VI}(A, f, \Phi, K)$ is equivalent to the following variational inclusion: find $u \in K$ such that

$$f \in Au + \partial(\Phi + I_K)(u).$$

Hence,

$$f \in Au + \partial(\Phi + I_K)(u) \subset \bigcup_{u \in X} Au + \partial(\Phi + I_K)(u) \subset R(A) + R(\partial(\Phi + I_K)).$$

Therefore,

$$(4) \quad f \in \overline{R(A)} + \text{Dom}(\Phi + I_K)^*.$$

Let us introduce the following function $\Psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\Psi(u) = \kappa \|Qu\|^2 + \Phi(u) + I_K(u).$$

A classical result in convex analysis show us that

$$\text{Dom}(\Psi^*) = \text{Dom}(\|Q \cdot \|^2)^* + \text{Dom}(\Phi + I_K)^*.$$

A simple calculation of the Fenchel conjugate of the function $(\frac{1}{2}\|Q \cdot \|^2)$, give us

$$\left(\frac{1}{2}\|Q \cdot \|^2\right)^* = \frac{1}{2}\|\cdot\|^2 + I_{\ker(A)^\perp}.$$

Therefore,

$$(5) \quad \text{Dom}(\Psi^*) = \ker(A)^\perp + \text{Dom}(\Phi + I_K)^* = \overline{R(A)} + \text{Dom}(\Phi + I_K)^*.$$

Using (4), we have

$$f \in \text{Dom}(\Psi^*).$$

It is well known in convex analysis that

$$\overline{\text{Dom}(\Psi^*)} = \{g \in X : \langle g, w \rangle \leq \Psi_\infty(w), \quad \forall w \in X\}.$$

It can be easily checked that the recession function Ψ_∞ associated to Ψ is given by

$$\Psi_\infty(w) = I_{\ker(A)}(w) + \Phi_\infty(w) + I_{K_\infty}(w).$$

Consequently f must satisfies the following compatibility condition

$$\langle f, w \rangle \leq \Phi_\infty(w), \quad \forall w \in \ker(A) \cap K_\infty,$$

which completes the proof of the proposition. \square

In the sequel, we shall study the stability of the variational inequality $\text{VI}(A, f, \Phi, K)$ in the sense of (\mathcal{C}) i.e. we characterize the data (A, f, Φ, K) for which there is some $\varepsilon > 0$ such that $\text{Sol}(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon) \neq \emptyset$ for every $(A_\varepsilon, f_\varepsilon, \Phi_\varepsilon, K_\varepsilon)$ satisfying (\mathcal{C}) with A_ε bounded, linear symmetric and semi-coercive operator,

$\Phi_\varepsilon \in \Gamma_0(X)$ and K_ε a closed convex set. The following resolvent set will play an important role

$$\mathcal{R}(A, \Phi, K) = \{f \in X : \text{Sol}(A, f, \Phi, K) \neq \emptyset\}.$$

The stability of $\text{VI}(A, f, \Phi, K)$ is related to the characterization of the interior (with respect to the strong topology) of the resolvent set $\mathcal{R}(A, \Phi, K)$. Before starting our study, let us give some simple examples to motivate the stability of $\text{VI}(A, f, \Phi, K)$ with respect to small perturbations.

Examples 2.1.

- (i) We consider the following classical Neumann problem

$$\mathcal{N}(f) \begin{cases} -\Delta u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded domain of \mathbb{R}^n with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$.

It is well-known that $\mathcal{N}(f)$ has a solution if and only if $\int_\Omega f dx = 0$.

It is clear that if we replace f by $f_\varepsilon = f + \varepsilon$ with $\varepsilon > 0$, then the new problem $\mathcal{N}(f_\varepsilon)$ has no solution. Hence, the Neumann problem is unstable in the sense of (C).

- (ii) Consider now the obstacle problem without friction which consists to determine the equilibrium position of an elastic thin membrane $\Omega \subset \mathbb{R}^n$ submitted to loads $f \in L^2(\Omega)$ and required to stay on or above an obstacle Ψ . The classical formulation, assuming that linear elasticity applies, is to find the displacement u of the membrane such that

$$\mathcal{O}(f) \begin{cases} -\Delta u \geq f, & \text{in } \Omega \\ (-\Delta u - f)(u - \Psi) = 0, & \text{on } \Omega \\ u \geq \Psi & \text{on } \Omega \text{ and } \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega \end{cases}$$

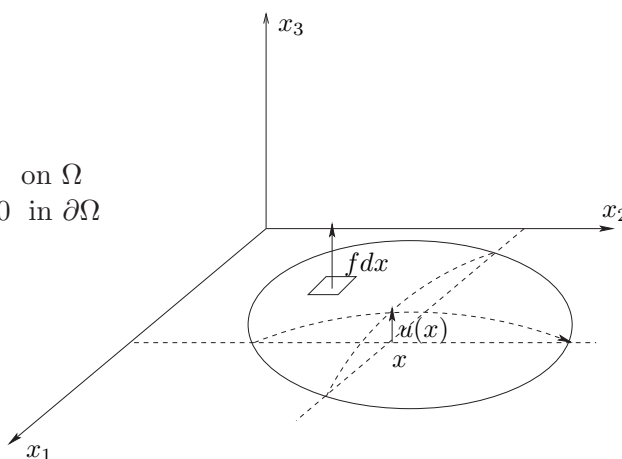


FIGURE 1. Frictionless obstacle problem

The weak formulation of problem $\mathcal{O}(f)$ is a variational inequality of the form $\text{VI}(A, f, K, 0)$ where $X = H^1(\Omega)$, $K = \{v \in H^1(\Omega) : v \geq \Psi \text{ a.e. in } \Omega\}$, the operator $A : X \rightarrow X'$ is defined by

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in X.$$

By Proposition 2.1, we have the following necessary condition for the existence of a solution to $VI(A, f, K, 0)$

$$(6) \quad f \in \left(\ker(A) \cap K_\infty \right)^\circ.$$

Since $\ker(A) = \mathbb{R}$ and $K_\infty = \{v \in H^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$, then $\ker(A) \cap K_\infty = \mathbb{R}^+$. Hence, condition (6) becomes

$$(7) \quad \int_{\Omega} f(x) dx \leq 0.$$

We can also show (see e.g. [1] or Theorem 2.1) that a sufficient condition for the existence of at least one solution of problem $VI(A, f, K, 0)$ is given by

$$(8) \quad f \in \text{Int} \left[\left(\ker(A) \cap K_\infty \right)^\circ \right],$$

or equivalently,

$$(9) \quad \int_{\Omega} f(x) dx < 0.$$

Consequently, the frictionless obstacle problem is stable with respect to small perturbation in the sense of (9) if and only if $f \in \text{Int} \left(\ker(A) \cap K_\infty \right)^\circ$.

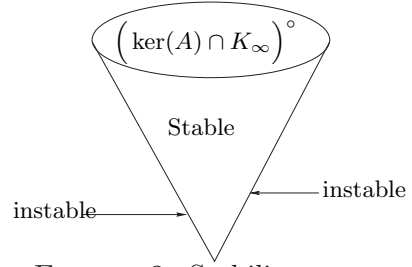


FIGURE 2. Stability cone of the frictionless obstacle problem.

- (iii) Set $X = \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $K_1 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and $\Phi \equiv 0$. In this case the resolvent set is given by

$$\mathcal{R}(A, 0, K_1) = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } y \leq 0\}.$$

Note that in this case $\text{Int } \mathcal{R}(A, 0, K_1) = \emptyset$ and hence problem $IV(A, f, 0, K_1)$ is instable.

Consider now, the new convex and closed subset K_2 given by

$$K_2 = \{(x, y) \in \mathbb{R}^2 : y \geq x^2\}.$$

In this case the resolvent set is given by

$$\mathcal{R}(A, 0, K_2) = \{(x, y) \in \mathbb{R}^2 : y < 0\} \cup \{(0, 0)\}.$$

Since $\text{Int } \mathcal{R}(A, 0, K_2) \neq \emptyset$, then problem $IV(A, f, 0, K_2)$ is stable.

We note that in this example the geometry of the convex and closed subset K plays an important role for the stability of problem $VI(A, f, K, \Phi)$ with respect to small perturbation. More generally, the notion of well-positioned

sets (introduced in [2]) and the coercivity of an associated energy-type functional to the problem play a crucial role for the stability the variational problem $VI(A, f, K, \Phi)$ with respect to small uniform perturbation (see Theorem 4.1 [2]).

We have the following the following existence and stability result related to the linear variational inequality $VI(A, f, \Phi, K)$ (for the proof we refer to Theorem 2.2 in [3]).

Theorem 2.1. *The linear variational inequality $VI(A, f, \Phi, K)$ is stable in the sense of (C) if and only if the following two conditions are satisfied*

- (i) $\ker(A) \cap K_\infty \cap \ker(\Phi_\infty)$ contains no lines;
- (ii) $\langle f, w \rangle < \Phi_\infty(w)$, $\forall w \in \ker(A) \cap K_\infty$, $w \neq 0$.

Example 2.1. Application to an elastic unilateral contact problem with friction.

Let an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with smooth boundary Γ . We suppose that $\Gamma = \Gamma_N \cup \Gamma_C$ where Γ_C is the part of Γ over which the body may come into contact with a rigid foundation S . We assume that the body Ω is subjected to body forces $f = (f_1, \dots, f_d)$ on Ω and surface traction $G = (G_1, \dots, G_d)$ acting on Γ_N . Let $u = (u_i)_{1 \leq i \leq d}$ be the displacement of the body and $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

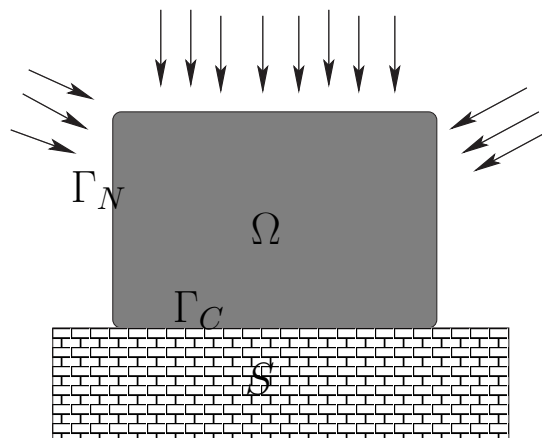


FIGURE 3. Elastic body in contact with a rigid foundation.

We denote by $\varepsilon_{ij}(u)$ and $\sigma_{ij}(u)$ the components of the strain and the stress tensor respectively. Assuming that linear elasticity holds, then we have as usual

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}(u) \text{ and } \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where the elasticity coefficients C_{ijkl} satisfy the usual conditions of boundness, symmetry and uniform ellipticity.

Let n denotes the unit outward normal vector to Γ . We set $u_N = u \cdot n$, $u_T = u - u_N n$ the normal and tangential displacement respectively, σ_N and σ_T the normal and tangential stress respectively.

The displacement u of the body Ω satisfies the following equations

$$(10) \quad -\sigma_{ij,j} = f_i \text{ in } \Omega, \quad i = 1, \dots, d$$

$$(11) \quad \sigma_{ij}(u)n_j = G_i \text{ on } \Gamma_N, \quad i = 1, \dots, d$$

The classical Signorini's unilateral contact conditions are expressed as follows

$$(12) \quad u_N \leq \psi, \quad \sigma_N \leq 0, \quad \sigma_N(u_N - \psi) = 0 \text{ on } \Gamma_C,$$

where $\psi \in L^\infty(\Gamma_C)$ is the initial gap between Γ_C and the foundation.

Assuming that frictional effects are governed by a modified version of Coulomb's law with a prescribed bound [11], the friction conditions are expressed as follows

$$(13) \quad |\sigma_T| \leq g \text{ on } \Gamma_C$$

$$(14) \quad |\sigma_T| < g \implies u_T = 0$$

$$(15) \quad |\sigma_T| = g \implies \text{there exists } \lambda \geq 0 : u_T = -\lambda \sigma_T$$

where $g \in L^\infty(\Gamma_C)$ is a non-negative function representing a friction bound.

The weak formulation of this problem is a variational inequality of the form $VI(A, l, \Phi, K)$ with

$$X = H^1(\Omega; \mathbb{R}^d), \quad K = \{v \in H^1(\Omega; \mathbb{R}^d) : v_N \leq \psi \text{ on } \Gamma_C\},$$

$$\langle Au, v \rangle = \int_{\Omega} C\varepsilon(u) : \varepsilon(v) dx, \quad \Phi(v) = \int_{\Gamma_C} g |v_T| d\sigma, \quad \langle l, v \rangle = \int_{\Omega} f \cdot v dx + \int_{\Gamma_N} G \cdot v d\sigma.$$

Note that the operator A is semi-coercive since Korn's inequality is not satisfied (no prescribed boundary displacements is imposed on any part of Γ). Let $\mathcal{R} = \{v \in H^1(\Omega; \mathbb{R}^d) : \varepsilon_{ij}(v) = 0\}$ be the space of rigid motions, e.g. for $d = 2$, we have

$$(16) \quad \mathcal{R} = \{v = (v_1, v_2) \in H^1(\Omega; \mathbb{R}^2) : v_1(x_1, x_2) = a_1 + bx_2 \text{ et } v_2(x_1, x_2) = a_2 - bx_1\},$$

where a_1, a_2 and b are arbitrary real constants.

Since the functional Φ is positively homogeneous, we have $\Phi_\infty = \Phi$.

Using Proposition 2.1, we have the following necessary condition for the existence of a solution of problem $VI(A, l, \Phi, K)$

$$(17) \quad \langle l, w \rangle \leq \Phi(w), \quad \forall w \in K_\infty \cap \mathcal{R},$$

where $K_\infty = \{v \in H^1(\Omega; \mathbb{R}^d) : v_N \leq 0 \text{ on } \Gamma_C\}$.

Using now Theorem 2.1, the Signorini-Fichera problem $VI(A, l, \Phi, K)$ is stable with respect to small uniform perturbations in the sense of (C) if and only if

- (i) $\mathcal{R} \cap K_\infty \cap \ker(\Phi)$ contains no line;
- (ii) $\langle l, w \rangle < \Phi(w), \quad \forall w \in \mathcal{R} \cap K_\infty, \quad w \neq 0$, i.e.

$$\int_{\Omega} f \cdot w dx + \int_{\Gamma_N} G \cdot w d\sigma < \int_{\Gamma_C} g |w_T| d\sigma, \quad \forall w \in \mathcal{R} \cap K_\infty, \quad w \neq 0.$$

We now discuss how to apply this stability result to some simple situation where the body Ω is a rectangle of \mathbb{R}^2 and the contact surface are as in Figures 4 and 5. We consider first the case of Figure 4. Using (16), it is clear that in this case the space $\mathcal{R} \cap K_\infty = \{0\}$. Hence, the Signorini-Fichera problem $VI(A, l, \Phi, K)$ is stable with respect to small uniform perturbations in the sense of (C) for every $l \in (H^1(\Omega; \mathbb{R}^d))'$.

In the case of Figure 5, we have $\mathcal{R} \cap K_\infty = \{0\} \times \mathbb{R}_+$. Hence, the Signorini-Fichera problem $VI(A, l, \Phi, K)$ is stable with respect to small uniform perturbations in the sense of (C) for every l such that $\langle l, e_2 \rangle < 0$ where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

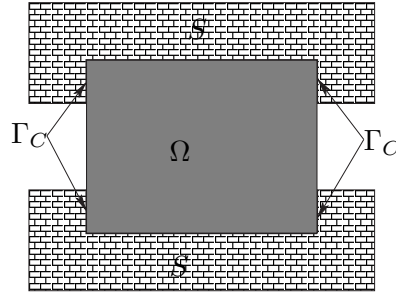


FIGURE 4. Contact surface 1

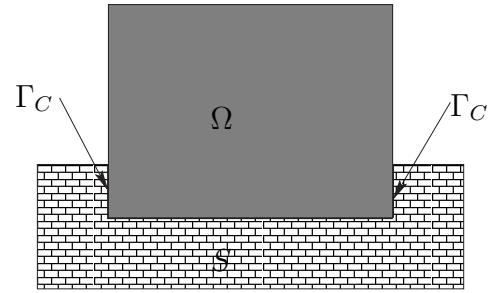


FIGURE 5. Contact surface 2

For a discussion about the solvability of the semi-coercive contact problem with Coulomb friction, we refer to [12] and references therein.

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Manuscript received ,
revised ,

SAMIR ADLY

Département Mathématiques-Informatique, Institut XLIM (UMR-CNRS 6172) Université de Limoges, 123, Avenue Albert Thomas, 87060 Limoges CEDEX, FRANCE.

E-mail address: samir.adly@unilim.fr