Journal of Nonlinear and Convex Analysis
Volume 7, Number 2, 2006, 299–313

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CONTINUOUS SELECTIONS FOR MULTI-VALUED MAPS WITHOUT CONVEXITY

YOUCEF ASKOURA AND CHRISTIANE GODET-THOBIE

Abstract. In this paper, we prove two new existence results of continuous selections without convexity. Our results are improvements, in a large class of topological spaces, of Yannelis and Prabhakar's selection theorem [24] and its generalizations obtained by Wu [23] and by Ding et *al*. [7]. Furthermore, we apply our selection results to deduce some results of fixed point, coincidence and equilibrium for abstract economy.

1. Introduction and preliminaries

Over the last few years, many results in nonlinear analysis which require convexity (existence of continuous selections and fixed points, coincidence results, inequalities,...etc.) have been generalized with important extensions of the notion of convexity.

We can find in the literature various generalizations of convexity : the topological convex structures (Van de Vel [21]), the H-convexity (Horvath [12, 13], Bardaro and Ceppitelli [3]), G-convexity (Park and Kim [16], Park [17]), $L-G$-convexity (Park [18]), $G-H$-convexity (Verma [22]), simplicial convexity (Bielawski [6]), L-convexity and B'-simplicial convexity (Ben-El-Mechaiekh et al. [4]),....and others. These notions are defined and investigated, in order to generalize results (of nonlinear analysis) needing classical convexity. By these developments, many fixed point results, selection results, coincidence results, inequality results and various applications in game theory have been obtained in topological spaces without linear structure [21, 12, 13, 3, 16, 17, 22, 6, 4, 5, 19, 20]

When reading some of these papers (in particular that of Horvath [12, 13], we are inspired to do this work. The idea is the following : Consider a multi-valued map $T : X \to Y$. A process to obtain a selection for T, without the use of the convexity of its values (see for example Horvath [12], theorem 2, section 3), is to construct it as a composition of two functions: $f : X \to W$, and $g : W \to Y$, where W is a geometric realization of a nerve of some covering of X. We remarked that if we slightly change this composition and look for the two functions as follows : $f : X \to X \times W$ and $g : X \times W \to Y$; where W stands for a similar set, and do the necessary adaptation of the proof, we can transform the statement of the convexity involved in the whole space Y to local needs of some properties of the classical convexity. We give in the end of this paper an example proving that our conditions are not a rewriting of the H-convexity.

2000 *Mathematics Subject Classification.* 54C60; 54C65.
Key words and phrases. Non convex set-valued maps; Continuous selections; Contractible spaces.

The selection results of this paper are improvements, in a large class of topological spaces of Yannelis and Prabhakar's selection theorem [24] and its generalizations obtained by Wu and Shen [23] and by Ding et *al*. [7]. These results have been generalized in another way, by the use of generalized convexities, by Horvath [12, 13], Mao [15], Tarafdar [20], Park [17], Ding and Park [8] and Yu and Lin [25]. The no requirement of any form of convexity in our work can be an advantage. It is, in fact, clear that in practice, the definition (or the construction) of a convenient convexity in order to study a given map can be difficult.

One of our results, Theorem 2 below, is also a generalization of a selection theorem obtained in [1].

If not specified, we denote by X and Y two separated topological spaces such that the following condition (A1) is fulfilled:

(A1) X is paracompact and Y is an absolute extensor for $X \times \Delta_N$, for every geometric simplex Δ_N.

Note that the condition $A1$ is satisfied if X is metrizable and Y is a convex subset of a locally convex topological vector space [9] or an *mc*-space [13].

Let $T : X \to 2^Y$ be a multi-valued map (2^Y denotes the set of all subsets of Y) and $f : X \to Y$, be a single valued map, $Dom(f)$ denotes the definition domain of f. For $A \subset X, Gr_A T = GrT \cap A \times Y$, GrT specifies the graph of T. $\overline{A}$ denotes the adherence of A and $int(A)$ designates the interior. If K is a subset of a given vector space, $co(K)$ denotes the convex hull of K. The abbreviation *u.s.c.* means : upper semi-continuous and Tx will be used to denote $T(x)$.

2. Existence of continuous selections

Let us begin by defining the so called local intersection property. We say that a multi-valued map $T : X \to 2^Y$ satisfy the *local intersection property* if : $\forall x \in X, \exists U_x$ a neighborhood of x such that $\bigcap_{x' \in U_x} Tx' \neq \emptyset$. This condition can also be found, in the literature, under this equivalent formulation : $X = \cup\{int(T^{-1}y), y \in Y\}$.

It is well known that the local intersection property guarantees the existence of continuous selections for multi-valued maps with convex values (see [23, 24]). In the following theorem, we prove the existence of continuous selections provided, some local intersections of the values of the considered multi-valued map are nonempty and contractible. Then, we use here the local intersection property and instead of the convexity of the values, we need only one of the properties of convex sets : *Every non empty intersection of convex sets is convex, then contractible.*

Theorem 1. *Let $C : X \to 2^Y$ be a multi-valued map with non empty values. Suppose that the following condition is satisfied:*

(*c*) *$\forall x \in X$, there exists a neighborhood U_x of x such that, for every open set $V \subset U_x$, $\bigcap_{x' \in V} Cx'$ is nonempty and contractible.*

Then, C has a continuous selection.

Proof. Let $\mathcal{W}$ be an open covering of X by open sets U_x satisfying the condition (*c*) of Theorem 1. Let $\mathcal{U} = \{U_i, i \in I\}$, be a locally finite open refinement of W, and let $\mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$ (the geometric realization of the abstract nerve of

$\mathcal{U}$ endowed with its natural topology). In order to avoid confusions, we denote, for all $i \in I$, the vertex of $\mathcal{N}(\mathcal{U})$ corresponding to U_i by minuscule letter u_i. Take for every $i \in I$, an $y_i \in \bigcap_{x \in U_i} Cx$, which exists by the condition (c) of Theorem 1. The simplex of $\mathcal{N}(\mathcal{U})$ which have the points $u_i, i \in J$ for vertices is denoted by Δ_J. $\partial\Delta_J$ designates the boundary of Δ_J. For every finite $J \subset I$, let $O_J = \bigcap_{i \in J} U_i$.

We consider a continuous partition of unity $\{\Psi_i, i \in I\}$ subordinated to the cover $\mathcal{U}$. Then we have, $\forall i \in I$, $\operatorname{supp}(\Psi_i) \subset U_i$, where $\operatorname{supp}(\Psi_i)$ denotes the support of the function Ψ_i. For every $L \subset I$, L finite, we denote $S_L = \bigcap_{i \in L} \operatorname{supp}(\Psi_i)$ and we remark that $S_L \subset O_L$.

Lemma 1. *Let F be a closed subset of X, M a contractible subset of Y and $f : F \times \partial\Delta_K \to M$ (Δ_K is a simplex of $\mathcal{N}(\mathcal{U})$) a continuous function.*

Then f has a continuous extension defined from $F \times \Delta_K$ to M.

Proof. (One part of the construction of this proof can be found in [15]). Without loss of generalities, we can denote $K = \{1, 2, 3, ..., n\}$.

Let $\overline{z} = \sum_{i=1}^{n} \frac{1}{n} u_i$. Denote $z \in \Delta_K$ by $\sum_{i=1}^{n} z_i u_i$. For all $z \in \Delta_K \backslash \{\overline{z}\}$, let

$$k(z) = (1 - n \min_{i=\overline{1,n}} \{z_i\})^{-1}. \tag{1}$$

$k(z)$ is defined on all $\Delta_K \backslash \{\overline{z}\}$; because $z \neq \overline{z}$, $\exists z_{i_0} \neq \frac{1}{n}$ and $\exists z_{i_1} < \frac{1}{n}$.

k is continuous (as composition of continuous functions) and $k(z) \geq 1$, $\forall z \in \Delta_K \backslash \{\overline{z}\}$. Let

$$r(z) = k(z)z + (1 - k(z))\overline{z} \quad \text{and} \quad t(z) = \frac{1}{k(z)}. \tag{2}$$

We have: $t(z) \in \,]0, 1]$ and $r(z) \in \partial\Delta_K$ (if i_* is the index where the minimum is reached in (1), $r(z)_{i_*} = 0$).

$\lim_{z \to \overline{z}} t(z) = 0$, then $t(.)$ can be extended by continuity to $\widetilde{t}(.)$ defined on Δ_K as follows:

$$\widetilde{t}(z) = \begin{cases} t(z), & \text{if } z \neq \overline{z} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\chi : M \times [0, 1] \to M$ be the function which contracts M to a given point $\overline{y}$.

Define $\widetilde{f} : F \times \Delta_K \to Y$ by :

$$\widetilde{f}(x, z) = \begin{cases} \chi(f(x, r(z)), \widetilde{t}(z)) & \text{if } z \neq \overline{z}, x \in F \\ \overline{y} & \text{if } z = \overline{z}, x \in F \end{cases}$$

$\widetilde{f}$ is continuous on $F \times \Delta_K \backslash \{\overline{z}\}$ (by the continuity of functions $r, \widetilde{t}$, f and χ). The continuity of $\widetilde{f}$ at $(\overline{x}, \overline{z}), \overline{x} \in F$ remains to be solved.

Supposing that the multivalued map $\Gamma : F \times \Delta_K \to Y, (x, z) \longmapsto \chi(f(x, \partial\Delta_K), \widetilde{t}(z))$ is $u.s.c.$ at $(\overline{x}, \overline{z})$, we easily obtain the continuity of $\widetilde{f}$ as following: We have: $\Gamma(\overline{x}, \overline{z}) = \{\overline{y}\} = \{\widetilde{f}(\overline{x}, \overline{z})\}$. Let O be a neighborhood of $\Gamma(\overline{x}, \overline{z})$. There exists a neighborhood Q of $(\overline{x}, \overline{z})$, such that $\forall (x, z) \in Q$, $\widetilde{f}(x, z) \in \Gamma(x, z) \subset O$. Then $\widetilde{f}$ is continuous at the point $(\overline{x},\overline{z})$. Let us prove that Γ is *u.s.c.* at the point $(\overline{x},\overline{z})$. Let the function

$$\begin{array}{rcl} h: F\times\Delta_K\times\partial\Delta_K & \to & Y, \\ (x,z,p) & \mapsto & \chi(f(x,p),\widetilde{t}(z)) \end{array}$$

We have: $\Gamma(x,z)=h(x,z,\partial\Delta_K)$.

Let O be a neighborhood of $\overline{y}=h(\overline{x},\overline{z},p),\forall p\in\partial\Delta_K$. From the continuity of h, for all $p\in\partial\Delta_K,\exists Q^p_{\overline{x}}$ an open neighborhood of $\overline{x}$ in F, $\exists Q^p_{\overline{z}}$ an open neighborhood of $\overline{z}$ in Δ_K and $\exists Q_p$ an open neighborhood of p in $\partial\Delta_K$, such that,

$$(*)\qquad \forall(x,z,p')\in Q^p_{\overline{x}}\times Q^p_{\overline{z}}\times Q_p, h(x,z,p')\in O$$

when p varies in $\partial\Delta_K$, we obtain a cover of $\partial\Delta_K$ by open sets Q_p satisfying (*) (with the neighborhoods of $\overline{x}$ and of $\overline{z}$ associated in (*)). Since $\partial\Delta_K$ is compact, it can be covered by a finite number of these open sets, let $Q_{p_i}, i\in\{1,...,m\}$. Let $Q_{\overline{x}}=\bigcap\limits_{i=1}^{m}Q^{p_i}_{\overline{x}}$ and $Q_{\overline{z}}=\bigcap\limits_{i=1}^{m}Q^{p_i}_{\overline{z}}$ ($Q^{p_i}_{\overline{x}}$ and $Q^{p_i}_{\overline{z}}$ are the neighborhoods of $\overline{x}$ and of $\overline{z}$ corresponding to Q_{p_i} in the formula (*)). $\forall(x,z,p)\in Q_{\overline{x}}\times Q_{\overline{z}}\times\partial\Delta_K,\exists i\in\{1,...,m\}$ such that $(x,z,p)\in Q^{p_i}_{\overline{x}}\times Q^{p_i}_{\overline{z}}\times Q_{p_i}$, and then $h(x,z,p)\in O$. Thereafter, $\forall(x,z)\in Q_{\overline{x}}\times Q_{\overline{z}}$, $h(x,z,\partial\Delta_K)\subset O$. Let $Q=Q_{\overline{x}}\times Q_{\overline{z}}$, $\forall(x,z)\in Q,\Gamma(x,z)\subset O$, which means that Γ is *u.s.c.* at the point $(\overline{x},\overline{z})$.

Therefore, $\widetilde{f}$ is continuous at $(\overline{x},\overline{z})$. It is finally easy to see that $\widetilde{f}\mid_{F\times\partial\Delta_K}=f$. □

Lemma 2. *Let $f: X\times\partial\Delta_K\to Y$ (Δ_K is a simplex of $\mathcal{N}(\mathcal{U})$) be a continuous function.*

Suppose that

$$(3)\qquad \forall L\subsetneq K,\forall x\in S_L, f(x,\Delta_L)\subset\bigcap_{z\in O_L}Cz$$

Then, there exists $\widetilde{f}$ a continuous extension of f defined on $X\times\Delta_K$, such that

$$(4)\qquad \forall L\subset K,\forall x\in S_L,\widetilde{f}(x,\Delta_L)\subset\bigcap_{z\in O_L}Cz$$

Proof. Denote $g=f\mid_{S_K\times\partial\Delta_K}$. We have: $\bigcap\limits_{z\in O_K}Cz\supset\bigcap\limits_{z\in O_L}Cz,\forall L\subset K$ (because $O_K\subset O_L$, if $L\subset K$). Then the image of $S_K\times\partial\Delta_K$ by g is a subset of $\bigcap\limits_{z\in O_K}Cz$ $\left(g(S_K\times\partial\Delta_K)\subset\bigcap\limits_{z\in O_K}Cz\right)$. Indeed, $\forall(x,p)\in S_K\times\partial\Delta_K$, $(x,p)\in S_L\times\Delta_L$, for some $L\subsetneq K$. Considering (3), $g(x,p)\in\bigcap\limits_{z\in O_L}Cz\subset\bigcap\limits_{z\in O_K}Cz$, *i.e.* $g: S_K\times\partial\Delta_K\to\bigcap\limits_{z\in O_K}Cz$. From (c), $\bigcap\limits_{z\in O_K}Cz$ is contractible ($O_K\subset U_i,\forall i\in K$), then by Lemma 1, we conclude that g has a continuous extension $\widetilde{g}$ defined on $S_K\times\Delta_K$ to $\bigcap\limits_{z\in O_K}Cz$.

Define

$$f_1=\begin{cases} f & \text{on } X\times\partial\Delta_K\\ \widetilde{g} & \text{on } S_K\times\Delta_K\end{cases}$$

From the above, $\forall x\in S_K, f_1(x,\Delta_K)=\widetilde{g}(x,\Delta_K)\subset\bigcap\limits_{z\in O_K}Cz$. Then we remark that any continuous extension of f_1 on $X\times\Delta_K$ satisfies (4) □

Lemma 3. *There exists a function* $f : X \times \mathcal{N}(\mathcal{U}) \to Y$, *such that:*

(5) *for every simplex* Δ_J *of* $\mathcal{N}(\mathcal{U}), f$ *is continuous on* $X \times \Delta_J$, *and:*

$$\forall x \in S_J, f(x, \Delta_J) \subset \bigcap_{z \in O_J} Cz$$

Proof. Let

$$\mathcal{H} = \left\{ \begin{array}{c} (f, \mathcal{L}),\ \mathcal{L} \text{ is a subcomplex of } \mathcal{N}(\mathcal{U}), \\ f : X \times \mathcal{L} \to Y \text{ satisfies (5) with } \mathcal{L} \text{ in the place of } \mathcal{N}(\mathcal{U}) \end{array} \right\}$$

Define in $\mathcal{H}$ the ordering:

$$(f, \mathcal{L}) \leq (g, \mathcal{L}') \Leftrightarrow \mathcal{L} \subseteq \mathcal{L}' \text{ and } g\mid_{X \times \mathcal{L}} = f.$$

Prove that $\mathcal{H}$ is nonempty and inductive.

- $\mathcal{H}$ is nonempty:
  We denote $h^i(x, u_i) = y_i, \forall x \in X, \forall i \in I$. We see that $(h^i, \{u_i\}) \in H, \forall i \in I$, i.e. $\mathcal{H}$ is nonempty.
- $\mathcal{H}$ is inductive:
  Let $\{(f^\lambda, \mathcal{L}_\Lambda)\}_{\lambda \in \Lambda}$ be a chain in $\mathcal{H}$, where Λ is an arbitrary set of indices. Consider the function f defined on $X \times \mathcal{L}$ (where $\mathcal{L} = \bigcup_{\lambda \in \Lambda} \mathcal{L}_\lambda$), as follows: $f\mid_{X \times \mathcal{L}_\lambda} = f^\lambda$. f is well defined, because $\{(f^\lambda, \mathcal{L}_\Lambda)\}_{\lambda \in \Lambda}$ is a chain of $\mathcal{H}$. For every simplex Δ_K of $\mathcal{L}$, $\exists \mathcal{L}_{\lambda_0} \supset \Delta_K$. Consequently, $(f, \mathcal{L}) \in \mathcal{H}$, i.e. $\mathcal{H}$ is inductive.

By the Zorn's lemma, $\mathcal{H}$ has a maximal element $(f, \mathcal{L})$. Prove that $\mathcal{L} = \mathcal{N}(\mathcal{U})$. Suppose the contrary.

Then, there exists a $k-$skeleton $\mathcal{N}(\mathcal{U})^k$ of $\mathcal{N}(\mathcal{U})$ such that f is not defined on $X \times \mathcal{N}(\mathcal{U})^k$, i.e. $\mathcal{N}(\mathcal{U})^k \nsubseteq \mathcal{L}$. Let k_0 the minimal integer such that $\mathcal{N}(\mathcal{U})^{k_0} \nsubseteq \mathcal{L}$.

If $k_0 = 0$, there exists $u_{i_0} \in \mathcal{N}(\mathcal{U}) \backslash \mathcal{L}$. Define $\widetilde{f}(x, u_{i_0}) = y_{i_0}, \forall x \in X$, and $\widetilde{f} = f$ on $Dom(f)$, we obtain a continuous extension of f defined on $X \times \mathcal{L} \cup \{u_{i_0}\}$ such that $(\widetilde{f}, \mathcal{L} \cup \{u_{i_0}\}) \in \mathcal{H}$ which contradicts the maximality of $(f, \mathcal{L})$ in $\mathcal{H}$.

Else, $k_0 > 0$. This means that there exists a simplex Δ_K of $\mathcal{N}(\mathcal{U})^{k_0}$, with strictly positive dimension such that $\partial\Delta_K \subset \mathcal{L}$ and $\Delta_K \nsubseteq \mathcal{L}$. In other words, f is defined on $X \times \partial\Delta_K$ and satisfies (3), but it is not defined on $X \times \Delta_K$. We infer, using Lemma 2, an extension $\widetilde{f}$ of $f_{|X \times \partial\Delta_K}$ defined on $X \times \Delta_K$ so that (4) is satisfied. Put $\overline{f} = f$ on $Dom(f)$ and $\overline{f} = \widetilde{f}$ on $X \times \Delta_K$. As a result an extension $\overline{f}$ of f defined on $X \times \mathcal{L}'$, where $\mathcal{L}' = \mathcal{L} \cup \Delta_K$ which is a subcomplex of $\mathcal{N}(\mathcal{U})$, such that $(\overline{f}, \mathcal{L}') \in \mathcal{H}$. This contradicts also the maximality of $\mathcal{L}$.

Finally, we can claim that $\mathcal{L} = \mathcal{N}(\mathcal{U})$. □

Now, it remains to end the proof of Theorem 1.

Let for this reason a function $g : X \times \mathcal{N}(\mathcal{U}) \to Y$ satisfying (5) and $\Psi(x) = \sum_{i \in I} \Psi_i(x) u_i$, the canonical application from X to $\mathcal{N}(\mathcal{U})$. Let, in a last time, the function f defined as follows :

$$\begin{array}{rccc} f: & X & \longrightarrow & Y \\ & x & \longmapsto & g(x, \Psi(x)) \end{array}$$

Prove that f is continuous.

Let $x_0 \in X$. There exists O_{x_0} an open neighborhood of x_0 which intersects only finite number of sets of $\mathcal{U}$. Let $I_0 = \{i \in I, O_{x_0} \cap U_i \neq \emptyset\}$. Then, there exists a finite set of indices I_1, so that $\Psi(O_{x_0}) \subset (\underset{j\in I_1}{\cup} \Delta_{J_j})$, where $J_j \subset I_0, \forall j \in I_1$. Since g is continuous on $X \times \Delta_{J_j}, \forall j \in I_1$, g is continuous on $X \times (\underset{j\in I_1}{\cup} \Delta_{J_j})$, consequently, $f(.) = g(., \Psi(.))$ is continuous on O_{x_0}, and therefore on X.

Verify that f is a selection of C. Let $x \in X$ and $J_x = \{i \in I, \Psi_i(x) \neq 0\}$. We have $x \in S_{J_x} \subset O_{J_x}$ and then $f(x) = g(x, \Psi(x)) \in g(x, \Delta_{J_x}) \subset \underset{z\in O_{J_x}}{\cap} Cz \subset Cx$ □

Now, we show a result which generalizes the principal result of [1]. In order to state it, we need the following definition:

Definition 1. We say that the set-valued map $T : X \to 2^Y$ is locally continuously contractible, if $\forall x \in X, \exists U_x$ a neighborhood of x such that: for every continuous selection σ of T over U_x, there exists a continuous function $M : Gr_{U_x}T \times [0, 1] \to Y$ satisfying the following conditions:

1) $\forall (x', z) \in Gr_{U_x}T$, $M(x', z, 1) = z$ and $M(x', z, 0) = \sigma(x')$.
2) $\forall x' \in U_x$, $\forall z \in Tx'$, $M(x', z, [0, 1]) \subset Tx'$.

Remark 1. It is clear that, if Y is a topological vector space and Tx is convex, $\forall x \in X$, then, $M(x', z, t) = \sigma(x') + t(z - \sigma(x'))$ satisfies the previous conditions.

Theorem 2. *Let $T : X \to 2^Y$, be a set-valued map satisfying the following conditions :*

(i) *$\forall x \in X, \exists U_x$ a neighborhood of x, on which T possesses a continuous selection,*
(ii) *T is locally continuously contractible,*

Then, T has a continuous selection.

Remark 2. The condition (i) of the previous theorem is weaker than the local intersection property.

Proof of Theorem 2. Let $\mathcal{W}$ be an open cover of X by open sets U_x such that : U_x is an open set contained in the intersection of two neighborhoods of x such that : the first one satisfies (i) and the second is the neighborhood of x which is mentioned in the definition 1. Let $\mathcal{U} = \{U_i, i \in I\}$, be a locally finite open refinement of $\mathcal{W}$, and let $\mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$ (the geometric realization of the abstract nerve of $\mathcal{U}$ endowed with its natural topology). For every $i \in I$, let σ_i be a continuous selection of T on U_i. We resume in this proof the notations of the proof of Theorem 1 relating to the simplexes of $\mathcal{N}(\mathcal{U})$. We consider, as in the proof of Theorem 1, a continuous partition of unity $\{\Psi_i, i \in I\}$ subordinated to the cover $\mathcal{U}$. We resume also the notation $S_L = \underset{i\in L}{\cap} \text{supp}(\Psi_i), \forall L \subset I, L$ finite.

Lemma 4. *Let $f : X \times \partial\Delta_K \to Y$ (Δ_K is a simplex of $\mathcal{N}(\mathcal{U})$) be a continuous function such that,*

$$\forall L \subsetneq K, \forall x \in S_L, f(x, \Delta_L) \subset Tx \tag{6}$$

Then, $\exists g : X \times \Delta_K \to Y$, a continuous extension of f satisfying :

$$\forall L \subset K, \forall x \in S_L, g(x, \Delta_L) \subset Tx \tag{7}$$

Proof. Without loss of generalities, we take $K = \{1,...,n\}$. If S_K is empty, any extension of f satisfies (7). In this case, since $X \times \partial\Delta_K$ is closed in $X \times \Delta_K$, f possesses a continuous extension to $X \times \Delta_K$.

Suppose that S_K is nonempty. $\forall (x,p) \in S_K \times \partial\Delta_K$, $(x,p) \in S_L \times \Delta_L$ for some $L \subset K$, then $f(x,p) \in Tx$. We define, as in Lemma 1, $\overline{z} = \sum_{i=1}^{n} \frac{1}{n} u_i$ and we consider the functions k, r, t and $\widetilde{t}$ constructed in the same lemma. Let σ be a continuous selection of T on S_K and $M : Gr_{S_K}T \times [0,1] \to Y$ satisfying the conditions of Definition 1 with this selection σ. Define the function $\widetilde{f} : S_K \times \Delta_K \to Y$, as follows:

$$\widetilde{f}(x,z) = \begin{cases} M(x, f(x, r(z)), \widetilde{t}(z)) & \text{if } z \neq \overline{z}, x \in S_K \\ \sigma(x) & \text{if } z = \overline{z}, x \in S_K \end{cases}$$

The continuity of $\widetilde{f}$ can be obtained by the same way as in Lemma 1. The function h of Lemma 1 is taken here as follows:

$$\begin{array}{ccc} h : S_K \times \Delta_K \times \partial\Delta_K & \longrightarrow & Y, \\ (x,z,p) & \longmapsto & M(x, f(x,p), \widetilde{t}(z)) \end{array}$$

Let

$$g' = \begin{cases} f & \text{on } X \times \partial\Delta_K \\ \widetilde{f} & \text{on } S_K \times \Delta_K \end{cases},$$

Let finally g be any continuous extension of g'(g exists because $Dom(g')$ is closed), g satisfies (7). □

Lemma 5. *There exists a function $f : X \times \mathcal{N}(\mathcal{U}) \to Y$, such that:*

(8) *For every simplex Δ_J of $\mathcal{N}(\mathcal{U}), f$ is continuous on $X \times \Delta_J$, and:*

$$\forall x \in S_J, f(x, \Delta_J) \subset Tx$$

Proof. We consider as in the proof of Lemma 3,

$$\mathcal{H} = \left\{ \begin{array}{c} (f, \mathcal{L}),\ \mathcal{L} \text{ is a subcomplex of } \mathcal{N}(\mathcal{U}), \\ f : X \times \mathcal{L} \to Y \text{ satisfies (8) with } \mathcal{L} \text{ in the place of } \mathcal{N}(\mathcal{U}) \end{array} \right\}.$$

To show the non emptiness of H, we define for every $i \in I$, $h^i(., u_i)$ a continuous extension of $\sigma_i \mid_{S_{\{i\}}}$. We see that $(h^i, \{u_i\}) \in \mathcal{H}$, $\forall i \in I$, i.e. $\mathcal{H}$ is not empty.

The end of the proof is analoguous[1] with the similar part of Lemma 3 □

The end of the proof of Theorem 2 is identical with that of Theorem 1. Let a continuous function $g : X \times \mathcal{N}(\mathcal{U}) \to Y$ satisfying (8). Let $\Psi : X \to \mathcal{N}(\mathcal{U})$ the canonical function defined by $\Psi(x) = \sum_{i \in I} \Psi_i(x) u_i, \forall x \in X$. Define, in a last time, the function $f : X \to Y$, by $f(x) = g(x, \Psi(x)), \forall x \in X$. The continuity of f can be obtained as in the proof of Theorem 1.

Verify that f is a selection of T.

Indeed, let $x \in X$ and $J_x = \{i \in I, \Psi_i(x) \neq 0\}$. We have, $x \in S_{J_x}$ and then $f(x) = g(x, \Psi(x)) \in g(x, \Delta_{J_x}) \subset Tx$ □

[1]we replace (5) by (8), (3) by (6), (4) by (7) and lemma 2 by lemma 4.

Since the condition (ii) of Theorem 2 is satisfied for multi-valued maps with convex values (remark 1), we deduce the following corollary. A similar result can be found in [2] (proposition 2, page 81).

Corollary 1. *Let $T : X \to 2^Y$ be a multi-valued map with convex*[2] *values having locally continuous selections, i.e. $\forall x \in X, \exists U_x$ a neighborhood of x on which T has a continuous selection.*

Then, T has a continuous selection (on X).

Proof. Since T has convex values, T is locally continuously contractible (from the remark 1). T has, in addition, locally continuous selections, therefore, it satisfies hypotheses of Theorem 2, which guarantees the existence of a continuous selection. □

3. Some deductions

We give in this section some results analogous to those known in the literature under other hypotheses. See [24, 7, 14, 23, 20] for the third first corollaries. For the validity of application of previous selection results, the condition $A1$ is considered in the regularity conditions (required for spaces) in all following statements.

The first corollary is a fixed point result in product spaces.

Corollary 2. *Let I be a countable*[3] *set of indices. For every $i \in I$, let X_i be a metrizable convex subset of a locally convex topological vector space E_i and D_i a compact subset of X_i. Put $X = \prod_{i\in I} X_i$ and let, for every $i \in I$, $S_i : X \to 2^{D_i}$ a multi-valued map satisfying either the condition (c) of theorem 1 or the two conditions (i) and (ii) of theorem 2.*

Then, $\exists \overline{x} \in X$, such that $\overline{x}_i \in S_i\overline{x}, \forall i \in I$. (for $x \in X$, we denote by x_i the projection of x to E_i).

Proof. For all $i \in I$, from Theorem 1 if S_i satisfies (c) or Theorem 2 if S_i satisfies (i) and (ii), S_i possesses a continuous selection, call it f_i. Define $f : X \to \prod_{i\in I} D_i, x \mapsto \prod_{i\in I} f_i(x)$. Using Himmelberg's fixed point theorem (Theorem 2 in [11]), f has a fixed point, let $\overline{x}$. We have : $\overline{x}_i \in f_i(\overline{x}) \in S_i(\overline{x})$ □

The following result deals with the quasi-variational inequality.

Corollary 3. *Let X be a convex metrizable subset of a locally convex topological vector space E, D a compact subset of X, Y a separated locally convex topological vector space, $S : X \to 2^D$ a continuous multi-valued map with convex closed values, $T : X \to 2^Y$ a multi-valued map satisfying either the condition (c) of Theorem 1, either the two conditions (i) and (ii) of Theorem 2 and $f : X \times Y \times X \to \mathbb{R}$ a continuous function.*

Suppose,

1) *The function $z \mapsto f(x, y, z)$ is quasi-convex, $\forall (x, z) \in X \times Y$,*
2) $\forall x \in X, \forall y \in T(x), f(x, y, x) \geq 0$.

---

[2]Here, Y is assumed, in addition to $A1$, to be a topological vector space.

[3]The condition "I is countable" is required for the metrizability of the product space $\prod_{i\in I} X_i$

Then, $\exists \overline{x} \in S(\overline{x}), \exists \overline{y} \in T(\overline{x})$*, such that*

$$f(\overline{x}, \overline{y}, x) \geq 0 \text{ for all } x \in S(\overline{x}).$$

Proof. Let σ be a continuous selection of T (which exists from Theorem 1 if T satisfies (c), or Theorem 2 if T satisfies the conditions (i) and (ii) of this theorem).

Define the function $H : X \to 2^D$, by

$$H(x) = \{z \in S(x), f(x, \sigma(x), z) = \min_{t \in S(x)} f(x, \sigma(x), t)\}$$

From the continuity of the functions f, σ and S and the compactness of the values of S, we conclude that H is *u.s.c* with compact values. Furthermore, we deduce the convexity of the values of H from the convexity of the values of S and the condition 1).

Then the multi-valued map $H : X \to 2^D$ satisfies all the hypotheses of the Himmelberg's fixed point theorem [11], consequently it has a fixed point $\overline{x}$. Let $\overline{y} = \sigma(\overline{x})$. We have,

$$0 \leq f(\overline{x}, \sigma(\overline{x}), \overline{x}) = \min_{t \in S(\overline{x})} f(\overline{x}, \sigma(\overline{x}), t) \leq f(\overline{x}, \overline{y}, x), \forall x \in S(\overline{x}). \qquad \square$$

Now we prove the existence of an equilibrium for abstract economies. By an abstract economy, we mean a quadruple $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$. I is an arbitrary set of indexes (agents), $P_i : X = \prod_{j \in I} X_j \to 2^{X_i}$ is the preference correspondence for $i \in I$. $A_i, B_i : X \to 2^{X_i}$ are the constraint correspondences. An equilibrium for this economy is a point $\overline{x} \in X$ such that $\overline{x}_i \in \overline{B_i(\overline{x})}$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset, \forall i \in I$.

Corollary 4. *Let* $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ *be an abstract economy in which* I *is countable. Let* $X = \prod_{i \in I} X_i$*. Suppose,* $\forall i \in I$,

1) X_i *is a convex metrizable subset of a locally convex topological vector space* E_i,
2) $\forall x \in X, B_i(x)$ *is convex and* $A_i(x) \subset B_i(x) \subset D_i$*, where* D_i *is a compact subset of* X_i.
3) *The map* $x \longmapsto \overline{B_i(x)}$ *is u.s.c. on* X.
4) *The set* $W_i = \{x \in X, A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open.*
5) *The map* $T_i : W_i \to 2^{X_i}, T_i(x) = A_i(x) \cap P_i(x)$ *satisfies the condition* (c) *of theorem 1 or the two conditions* (i) *and* (ii) *of theorem 2.*
6) $\forall x \in X, x_i \notin A_i(x) \cap P_i(x)$.

Then, Γ *possesses an equilibrium, i.e.* $\exists \overline{x} \in X$*, such that* $\overline{x}_i \in \overline{B_i(\overline{x})}$ *and* $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset, \forall i \in I$.

Proof. For every $i \in I$, T_i admits a continuous selection f_i (from Theorem 1 or Theorem 2).

Define, for all $i \in I$, the set-valued map $G_i : X \to X_i$

$$G_i(x) = \begin{cases} f_i(x) & \text{if } x \in W_i, \\ \overline{B_i(x)} & \text{otherwise.} \end{cases}$$

Let us prove that G_i is *u.s.c.* The upper semicontinuity of G_i on W_i is a consequence of the continuity of f_i. Then, we have to show that G_i is *u.s.c* on $X \backslash W_i$ ($X \backslash W_i$ is the complement of W_i in X).

Let $x \in X \backslash W_i$ and O an open set in X_i which contains $G_i(x)$. Then $\overline{B_i(x)} = G_i(x) \subset O$. From 3), $\exists Q$ a neighborhood of x in X such that $\overline{B_i(x')} \subset O, \forall x' \in Q$. From 2), $f_i(x') \subset A_i(x') \subset B_i(x'), \forall x' \in W_i$. Therefore $G_i(x') \subset \overline{B_i(x')}, \forall x' \in X$. We deduce that $\forall x' \in Q, G_i(x') \subset O$. Then G_i is *u.s.c.*

The set-valued map $G : \prod_{i \in I} X_i \to \prod_{i \in I} D_i$ defined by $G(x) = \prod_{i \in I} G_i(x), \forall x \in \prod_{i \in I} X_i$ is *u.s.c.* [10], with closed convex values. G satisfies all the hypotheses of Himmelberg's [11] fixed point theorem. Consequently, it admits a fixed point, call it $\overline{x}$. We have, $\forall i \in I, \overline{x}_i \in G_i(\overline{x})$. Taking 6) into account, we conclude that $\overline{x}_i \in X \backslash W_i, \forall i \in I$ and then, $\overline{x}_i \in \overline{B_i(\overline{x})}$ and $A_i(\overline{x}) \cap P_i(\overline{x}) = \emptyset, \forall i \in I$. □

We can apply Theorem 1 to prove, as in [12], the following coincidence result:

Theorem 3. *Let X be a compact convex metrizable subset of a locally convex topological vector space E, Y a separated locally convex topological vector space and $C : X \to 2^Y$ a multi-valued map satisfying either the condition (c) of Theorem 1 or the two conditions (i) and (ii) of Theorem 2. Then,*

1) *For every continuous function $h : Y \to X$, there exists $y_0 \in Y$ such that $y_0 \in C(h(y_0))$.*
2) *For every multi-valued map $R : X \to 2^Y$ such that R^{-1} has a continuous selection,*

$$\exists x_0 \in X, C(x_0) \cap R(x_0) \neq \emptyset.$$

Proof. Denote by ρ a continuous selection of C (which exists from Theorem 1 if C satisfies (c), or Theorem 2 if C satisfies (i) and (ii)).

From the proof of Theorem 1 or 2 (following the case), ρ is factorized as :

$$\begin{array}{ccccc} X & & \xrightarrow{\rho} & & Y \\ \Phi_1 \searrow & & & & \nearrow g \\ & & X \times \mathcal{N} & & \end{array}$$

where $\mathcal{N}$ is a geometric realization of a complex with a finite number of vertices in this case. In fact, since X is compact, the covering used, in proofs of Theorems 1 and 2, for the construction of a continuous selection for the considered multi-valued map can be taken finite. Φ_1 is continuous. The function g given by Lemma 3 or 5 (following the used theorem) is also continuous, because it is continuous on all subsets of type $X \times \Delta_S$, where Δ_S is a simplex of $\mathcal{N}$, and $\mathcal{N}$ contains a finite number of simplexes. According to $A1$, g has a continuous extension $\widetilde{g} : X \times co(\mathcal{N}) \to Y$. Put

$$\Phi_2 : X \times co(\mathcal{N}) \xrightarrow{\widetilde{g}} Y \xrightarrow{h} X \xrightarrow{\Phi_1} X \times \mathcal{N}. \quad \Phi_2 = \Phi_1 \circ h \circ \widetilde{g}.$$

Φ_2 is continuous, defined from the convex compact locally convex $X \times co(\mathcal{N})$ into itself. So it has a fixed point (apply for example Tychonoff's fixed point theorem) which we denote by $(p_0, q_0) \in X \times \mathcal{N}$. Put $y_0 = \widetilde{g}(p_0, q_0)$. We have : $\Phi_1(h(y_0)) = (p_0, q_0)$ and then, $\widetilde{g}(\Phi_1(h(y_0))) = y_0$, but, $\widetilde{g} \circ \Phi_1 = g \circ \Phi_1 = \rho$, which gives : $\rho(h(y_0)) = y_0$, *i.e.* $y_0 \in C(h(y_0))$.

2) If R is a multi-valued map such that R^{-1} has a continuous selection, denoted by h. From 1), $\exists y_0 \in Y$ such that $y_0 \in C(h(y_0))$. Put $x_0 = h(y_0)$. Then, $y_0 \in R(x_0) \cap C(x_0)$. □

Corollary 5. *Let I, J be two countable sets of indices, for every $i \in I$ (resp. $j \in J$), X_i (resp. Y_j) a convex compact and metrizable subset of a locally convex topological vector space E_i (resp. F_j), $G_i : Y = \prod_{j\in J} Y_j \to 2^{X_i}$ (resp. $H_j : X = \prod_{i\in I} X_i \to 2^{Y_j}$) a multi-valued map satisfying either the condition (c) of Theorem 1 or the conditions (i) and (ii) of Theorem 2.*

Then, $\exists \overline{x} \in X, \exists \overline{y} \in Y$, such that $\overline{y}_j \in H_j(\overline{x})$ and $\overline{x}_i \in G_i(\overline{y}), \forall (i,j) \in I \times J$.

Proof. Consider the two multi-valued maps $C, R : X \to 2^Y$ defined by $C(x) = \prod_{j\in J} H_j(x)$ and $R(x) = \bigcap_{i\in I}\{y \in Y, x_i \in G_i(y)\}$, for every $x \in X$. Then, $R^{-1}(y) = \prod_{i\in I} G_i(y)$, for every $y \in Y$. The multi-valued maps H_j and G_i, $(j,i) \in J \times I$, have (according to Theorem 1 or 2) continuous selections. Consequently, C and R^{-1} have continuous selections. After this, we apply the previous theorem, note that its proof remains true if the multi-valued map C is supposed to have a continuous selection. Then, there exists $\overline{x} \in X$ such that $C(\overline{x}) \cap R(\bar{(x)}) \neq \emptyset$. To end the proof, choose an element $\overline{y} \in C(\overline{x}) \cap R(\overline{x})$. □

An analogous result of the last corollary can be found in [25].

4. Comments and examples

Both of Theorems 1 and 2 are improvements, in a large class of topological spaces, of the theorem of X. Wu and S. K. Shen [23], which is itself a generalization of the theorem of N. C. Yannelis and N. D. Prabhakar [24].

Theorem 4 (Wu et Shen)**.** *Let X be a paracompact subset of a separated topological space E and Y a separated topological vector space. Let S and $T : X \to 2^Y$ be two multi-valued maps satisfying the following conditions :*

(i') $\forall x \in X, S(x)$ is nonempty and $co(S(x)) \subset T(x)$,

(ii') S satisfies the local intersection propriety.

Then, T has a continuous selection.

Deduction of Theorem 4 from Theorem 1: The multi-valued map $co(S(.))$ satisfies the condition (c) of Theorem 1. Indeed, from (ii'), in each point x of X, there exists a neighborhood U_x of x in which the intersection of values of S is nonempty ($\bigcap_{x'\in U_x} S(x') \neq \emptyset$), since the values of $co(S(.))$ are convex, for every open set $V \subset U_x$, $\bigcap_{x'\in V} co(S(x'))$ is nonempty and contractible. Then $co(S(.))$ admits, by virtue of Theorem 1, a continuous selection, which is a selection of T by (i').

Deduction of Theorem 4 from Theorem 2: As it is mentioned in Remark 2, the condition (ii') is stronger than the condition (i) of Theorem 2. Since $S(x) \subset co(S(x)), \forall x \in X$, $co(S(.))$ satisfies the condition (i) of Theorem 2. From Remark 1, the application $co(S(.))$ is locally continuously contractible, *i.e.* satisfies the condition (ii) of Theorem 2. Therefore, $co(S(.))$ admits a continuous selection from Theorem 2, which will be a continuous selection of T from (i').

We are wondering whether the condition (ii) of the theorem 2 can (or not) be implicated by the continuity of the application T and the contractibility of its values.

This question is a subject of our actual interests. But, in the case of Theorem 1, this question has a very easy answer. Precisely, it is very easy to see that the condition (c) can not be necessarily induced by the property of local intersection, the continuity of the application C and the contractibility of its values, as it is shown by the following example:

Example 1. Let $\Gamma : [0,1] \to 2^{\mathbb{R}^2}$, where $\mathbb{R}$ is the real line,

$$\Gamma(x) = [0,1] \times \{0\} \cup [0,1] \times \{1\} \cup \{x\} \times [0,1].$$

Γ is continuous, with contractible values and satisfies the local intersection property. However, Γ dos not satisfies the condition (c).

We can also remark that the condition (c) of Theorem 1 is stronger than the following condition :

(c') $\forall x_0 \in X, \exists U$ a neighborhood of x_0 such that $\underset{x\in U}{\cap} C(x)$ is nonempty and contractible.

It is clear that if the condition (c) is satisfied, (c') is also satisfied. The following example shows that the condition (c) is stronger than the condition (c').

Example 2. $C : [-1,1] \to 2^{\mathbb{R}}$,

$$Cx = \begin{cases} [1,2] \cup [-2,-1], & \text{if } |x| \geq 1/4, \\ [1,2], & \text{else.} \end{cases}.$$

C satisfies (c') but not (c).

Now we give an example which proves that conditions of Theorems 1 and 2 are not a rewriting of the H-convexity of values of the considered multi-valued map, whatever the considered convexity structure defined on the image space.

Example 3. Let C be the unite circle of the euclidien space $\mathbb{R}^2$ and the application defined as:

$$\begin{array}{rccl} S: & C & \longrightarrow & 2^C \\ & x & \longmapsto & \{y \in C, \|x-y\| \geq \sqrt{2}\} \end{array}$$

$\|.\|$ is the euclidien norm.

Consider the function:

$$\begin{array}{rccl} \varphi: & \mathbb{R} & \longrightarrow & C \\ & p & \longmapsto & (\cos(p), \sin(p)), \end{array}$$

Let $x_0 \in C$, $p_0 \in \mathbb{R}$ such that $\varphi(p_0) = x_0$ and σ a continuous selection of S defined on the set $V_0 = \varphi([p_0 - \frac{\pi}{2}, p_0 + \frac{\pi}{2}])$.

$\forall x \in V_0$, there exists a unique $p \in [p_0 - \frac{\pi}{2}, p_0 + \frac{\pi}{2}]$ such that $x = \varphi(p)$. $Sx = \left\{\varphi(r), r \in [p + \frac{\pi}{2}, p + \frac{3\pi}{2}]\right\} = \left\{\varphi(p + \frac{\pi}{2} + \lambda), \lambda \in [0,\pi]\right\}$.

$\forall z \in Sx$, there exists a unique $r \in [p + \frac{\pi}{2}, p + \frac{3\pi}{2}]$ such that $z = \varphi(r)$ with $r = p + \frac{\pi}{2} + \lambda, \lambda \in [0,\pi]$. We denote this r by r_z.

S satisfies the condition (c) of Theorem 1. Indeed, it is not difficult to see that $\forall V$ open contained in V_0, $\cap_{x\in V} S(x)$ is a nonempty arc of C. Prove that S satisfies the hypotheses of theorem 2. Since S satisfies the local intersection property, it suffices to prove that S is locally continuously contractible.

Let $H : [p_0 - \frac{\pi}{2}, p_0 + \frac{\pi}{2}] \times [0, \pi] \longrightarrow Gr_{V_0}S$ defined by

$$H(p, \lambda) = (\varphi(p), \varphi(p + \frac{\pi}{2} + \lambda))$$

H is a continuous bijection, and since $Dom(H)$ is compact, it is an homeomorphism. Then, H^{-1} is continuous from $Gr_{V_0}S$ to $[p_0 - \frac{\pi}{2}, p_0 + \frac{\pi}{2}] \times [0, \pi]$.

If Π_1 is the projection of $\mathbb{R}^2$ to the abscissa axe and Π_2 the projection of $\mathbb{R}^2$ to the ordinate axe, for $x = \varphi(p) \in V_0$ and $z \in Sx$, $z = \varphi(r_z)$, $p = \Pi_1 H^{-1}(x, z)$ and $\lambda = \Pi_2 H^{-1}(x, z)$ and $r_z = p + \lambda + \frac{\pi}{2}$.

Then the map $(x, z) \mapsto r_z$ is continuous.

Let the function $M : Gr_{V_0}S \times [0, 1] \to C$ defined by,

$$M(x, z, t) = \varphi(tr_z + (1 - t)r_{\sigma(x)}$$

It is clear that M is continuous, $M(x, z, 1) = z$ and $M(x, z, 0) = \sigma(x)$ and it is easy to see that $M(x, z, [0, 1]) = \varphi([r_z, r_{\sigma(x)}])$ is the arc $(z, \sigma(z))$ which is contained in $S(x)$.

Now, independently of the chosen convexity structure defined on $\mathbb{R}^2$, prove that the values of S can not be $H-$convex at the same time.

For this reason, we endow $\mathbb{R}^2$ with a given *c*-structure [12] F. Recall that a *c*-structure on a given topological space E is a set-valued map $F : \langle E \rangle \to 2^E$, ($\langle E \rangle$ is the set of all finite subsets of E) satisfying the two axioms:

1)$\forall A \in \langle E \rangle, F(A)$ is contractible,

2)$\forall A, B \in \langle E \rangle, A \subset B \Longrightarrow F(A) \subset F(B)$.

A subset D of E is said to be H-convex [19] (or an F-set [12]) if, $\forall A \in \langle D \rangle, F(A) \subset D$.

Consider the following four points of C: $y_1 = (1, 0), y_2 = (0, 1), y_3 = (-1, 0), y_4 = (0, -1)$. $S(y_i)$ is the half circle containing the points $y_j, j \neq i$.

Suppose that $S(y_i)$ is H-convex, $\forall i \in \{1, ..., 4\}$. Since $S(y_1)$ and $S(y_3)$ are H-convex, we shall have $F(\{y_2, y_4\}) \subset S(y_1)$ and $F(\{y_2, y_4\}) \subset S(y_3)$, and then $F(\{y_2, y_4\}) \subset S(y_1) \cap S(y_3) = \{y_2, y_4\}$. It follows the two possibilities:

$$\text{either } F(\{y_2, y_4\}) = y_2 \text{ either } F(\{y_2, y_4\}) = y_4.$$

The same arguments allow us to obtain

$$\text{either } F(\{y_1, y_3\}) = y_3 \text{ either } F(\{y_1, y_3\}) = y_1.$$

Since F is monotonic (satisfies 2)), we obtain :

$$\text{either } F(\{y_2\}) = F(\{y_4\}) = y_2 \text{ either } F(\{y_2\}) = F(\{y_4\}) = y_4. \tag{9}$$

and

$$\text{either } F(\{y_1\}) = F(\{y_3\}) = y_3 \text{ either } F(\{y_1\}) = F(\{y_3\}) = y_1. \tag{10}$$

Take one possibility of (9) and (10), for example:

$$F(\{y_2\}) = F(\{y_4\}) = y_2 \text{ and } F(\{y_1\}) = F(\{y_3\}) = y_3.$$

We obtain: $F(\{y_1, y_2, y_4\}) \supset \{y_2, y_3\}$, and then $F(\{y_1, y_2, y_4\}) \not\subseteq S(y_3)$. This is an absurdity because $S(y_3) \supset \{y_1, y_2, y_4\}$ and it is supposed to be H-convex. The other possibilities imply analogous absurdities.

We conclude that all the values of S can not be H-convex, at the same time.

References

[1] Askoura Y.,– *Existence d'une sélection continue pour les applications multivoques à images non nécessairement convexes*, C. R. Acad. Sci. Paris, Ser. I, 337(2003),81-83.

[2] Aubin J.P. and Cellina A.,– *Differential Inclusions*, Springer-Verlag, 1984.

[3] Bardaro C. and Ceppitelli R.,– *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem to minimax inequalities*, J. Math. Anal. Appl., 132 (1988), 484-486.

[4] Ben-El-Mechaiekh H., Chebbi S., Florenzano M. and Llinares J. V.,– *Abstract convexity and fixed points*, J. Math. Anal. Appl., 222(1998), 138-150.

[5] Ben-El-Mechaiekh H. and Oudadess M.,– *Some Selection Theorems without Convexity*, J. Math. Anal. App., 195(1995), 614-618.

[6] Bielawski R.,– *Simplicial convexity and its applications*, J. Math. Anal. Appl., 127(1987), 155-171.

[7] Ding X. P., Kim W. K. and Tan K. K.,– *A selection theorem and its applications*, Bull. Austral Math. Soc. 46(1992), 205-212.

[8] Ding X. P. and Park J. Y.,– *Continuous selection theorem, coincidence theorem, and generalized equilibrium in L-convex spaces*, Computers and Mathematics with appl., 44(2002), 95-103.

[9] Dugundji J.,– *An extension of Tietze's theorem*, Pacific J. Math., 1(1951), 353-367.

[10] Fan Ky,– *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U. S. A. 38(1952), 121-126

[11] Himmelberg C. J.,– *Fixed points of compact multifunctions*, J. Math. Anal. Appl., 38(1972), 205-207.

[12] Horvath C. D.,– *Contractibility and Generalized Convexity*, J. Math. Anal. App., 156(1991), pp. 341-357.

[13] Horvath C. D.,– *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse 6^e série, 2 (1993), 253-269.

[14] Lin L. J., Yu Z. T., Ansari Q. H. and Lai L. P.,– *Fixed point and maximal element theorems with applications to abstract economies and minimax inequalities*, J. Math. Anal. Appl., 284(2003), 656-671.

[15] Mao J-F., – *A New Continuous Selection Theorem*, J. Math. Anal. App., 222 (1998), pp. 585-593.

[16] Park S. and Kim H.,– *Foundation of* KKM *Theory on generalized convex spaces*, J. Math. Anal. Appl., 209(1997), 551-571.

[17] Park S.,– *Continuous selection theorems in generalized convex spaces*, Numer. Func. anal. and Optimiz. 20 (1999), 567-583

[18] Park S.,– *Fixed point theorems in locally G-convex spaces*, Nonlinear Anal., 48 (2002), 869-879.

[19] Tarafdar E.,– *A Fixed Point Theorem in H-Space and Related Results*, Bull. Austral. Math. Soc., 42(1990), 133-140.

[20] Tarafdar E.,– *Fixed point theorems in $H-$space and equilibrium points of abstract economies*, J. Austral. Math. Soc. Ser A, 53(1992), 252-260.

[21] Van De Vel M.,– *A Selection Theorem for Topological Convex Structures*, Trans. Amer. Math. Soc., 336(1993), 463-496.

[22] Verma R. U.,– G-H-KKM *type theorems and their applications to a new class of minimax inequalities*, Computers Math. Appl., 37(1999), 45-48.

[23] Wu X. and Shen S. K.,– *A further generalization of Yannelis Prabhakar's continuous selection theorem and its applications*, J. Math. Anal. Appl., 197(1996), 61-74.

[24] Yannelis N. C. and Prabhakar N. D.,– *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom., 12(1983), 233-245.

[25] Yu Z. T. and Lin L. J.,– *Continuous selection and fixed point theorems*, Nonlinear Analysis, 52(2003),445-455.

Manuscript received April 28, 2004
revised July 4, 2006

Youcef Askoura
Université de Bretagne Occidentale-CNRS, UFR Sciences et Techniques, Laboratoire de Mathématiques UMR 6205, 6, avenue Victor Le Gorgeu CS93837-29283 BREST Cedex 3, France
E-mail address: askoura@univ-brest.fr

Christiane Godet-Thobie
Université de Bretagne Occidentale-CNRS, UFR Sciences et Techniques, Laboratoire de Mathématiques UMR 6205, 6, avenue Victor Le Gorgeu CS93837-29283 BREST Cedex 3, France
E-mail address: christiane.godet-thobie@univ-brest.fr