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ϵ-OPTIMALITY FOR MINIMAX PROGRAMMING PROBLEMS

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Dedicated to late Professor M. C. Puri

Abstract. In this paper, we establish Karush-Kuhn-Tucker (for short, we call KKT) type necessary and sufficient optimality conditions for an ε-optimal solution of a nondifferentiable general minimax programming problem. These optimality conditions are then utilized to derive ε-optimality conditions for a generalized fractional programming problem with nondifferentiable convex inequality constraints, linear equality constraints and abstract constraints, by employing ε-parametric technique.

1. Introduction and prerequisites

Study for approximate solutions of nonlinear programming problems has gained momentum in recent years. This is because of the fact that most of the mathematical models formulated for practical problems are not precise copies of the original problems, but only approximate ones. Also, from a computational viewpoint, algorithms developed in the literature to solve nonlinear programming problems compute only approximate solutions for such problems. Above all, this notion seems to be useful for the problems that otherwise have no optimal solutions. Hence, study for approximate solutions is of great interest.

Several authors have turned their attention to properties for approximate solutions of nonlinear optimization problems. Notably, Strodiot, Nguyen, and Heukemes [13] derived ε-optimality conditions of KKT type for points which are within ε of being optimal, i.e., ε-approximate optimal to the problem of minimizing a nondifferentiable convex objective function subject to nondifferentiable convex inequality constraints, linear constraints and abstract constraints with Slater's constraint qualification. Subsequently, they employed these optimality conditions to illustrate the mechanism of a bundle algorithm. In [10], Loridan discussed some properties of ε-efficient solutions for vector minimization problems. Later, Liu [8] adapted similar approach to obtain ε-duality results for nondifferentiable nonconvex multiobjective programming problems. Using an ε-parametric approach, Liu and Yokoyama [9] established necessary and sufficient optimality conditions and ε-duality theorems for an ε-Pareto optimal solution of a nonlinear multiobjective fractional programming problem.

In this paper, we wish to extend some of the results of [9] to generalized minimax programming problems.

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The paper is organized as follows. In Section 2, we obtain necessary and sufficient optimality conditions of KKT type for an ε-optimal solution of a minimax programming problem, while in Section 3, we concentrate on the generalized fractional programming and employ ε-parametric approach to transform the problem into a minimax nonlinear parametric programming problem. Results of Section 2 are then utilized to derive ε-optimality conditions for generalized fractional programming problem. In Section 4, we show ε-KKT conditions of the generalized fractional programming and give an example. In Section 5, associated with an ε-KKT condition, we show a duality result under some restrictive assumption.

Throughout the paper, $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and $\varepsilon > 0$ is the permissible error. We first recall the following definitions and results that are needed in the sequel.

Definition 1.1 ([7])**.** Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, finite at $\overline{x}$. The ε-subdifferential of ψ at $\overline{x}$ is the set defined by

$$\partial_\varepsilon \psi(\overline{x}) = \{\xi \in \mathbb{R}^n \mid \psi(y) \geqq \psi(\bar{x}) - \varepsilon + \langle \xi, y - \overline{x}\rangle \text{ for any } y \in \mathbb{R}^n\}.$$

Definition 1.2 ([7])**.** Let C be a nonempty closed convex set in $\mathbb{R}^n$. The ε-normal cone of C at $\overline{x}$ is the set defined by

$$N_\varepsilon(C; \overline{x}) = \{\xi \in \mathbb{R}^n \mid \langle \xi, y - \overline{x}\rangle \leqq \varepsilon \text{ for any } y \in C\}.$$

Lemma 1.1 ([13, Theorem 4.2])**.** *Let Y be a compact topological vector space, and let $\phi(x, y)$ be a real-valued function defined on $\mathbb{R}^n \times Y$, convex in x for every $y \in Y$, and upper semi-continuous in y for every $x \in \mathbb{R}^n$. Set $\psi(x) = \max_{y\in Y} \phi(x, y)$, and let $\overline{x} \in \mathbb{R}^n$. Then, $x^* \in \partial_\varepsilon \psi(\overline{x})$ if and only if there exist $n+1$ points $y^1, \ldots, y^{n+1} \in Y$, scalars $\alpha_i \geqq 0$, $\varepsilon_i \geqq 0$, $1 \leqq i \leqq n+1$ such that*

(a) $x^* \in \displaystyle\sum_{i=1}^{n+1} \partial_{\varepsilon_i}(\alpha_i \phi(\overline{x}, y^i))$

(b) $0 \leqq \psi(\overline{x}) - \displaystyle\sum_{i=1}^{n+1} \alpha_i \phi(\overline{x}, y^i) \leqq \varepsilon - \sum_{i=1}^{n+1} \varepsilon_i$,

(c) $\displaystyle\sum_{i=1}^{n+1} \alpha_i = 1.$

2. Minimax programming problems

Minimax programming is a growing area of research. Over the years, considerable attention has been paid to develop the theory and solution methods for minimax programming problems that arise frequently in practice. These problem area spread over a wide range of fields from approximate problems to more difficult facility allocation problems. For more applications and comprehensive theory of minimax problems, see Danskin [2], Dem'yanov and Malzemov [3], Du and Pardalos [4] and references cited therein.

In this section, we study the following nondifferentiable general minimax programming problem

$$\text{(P)} \qquad \min_{x\in S} \ f(x)$$

$$\text{where } f(x) = \max_{y \in Y} \phi(x, y),$$

$S = \{x \in \mathbb{R}^n \,|\, h_j(x) \leqq 0, 1 \leqq j \leqq m \,; Ax = b; x \in C\}$, is a nonempty feasible set; $A \in \mathbb{R}^{k \times n}$ and $b \in \mathbb{R}^k$; $C \subseteq \mathbb{R}^n$ is a nonempty convex set, and $Y \subseteq \mathbb{R}^m$ is a nonempty compact set ; $\phi : \mathbb{R}^n \times Y \to \mathbb{R}$; $h_j : \mathbb{R}^n \to \mathbb{R}$, $1 \leqq j \leqq m$; $h_j(\cdot)$, $1 \leqq j \leqq m$, are convex on $\mathbb{R}^n$ and $\phi(\cdot, y)$ is convex on S for any $y \in Y$; $\phi(x, \cdot)$ is upper semicontinuous on Y for each $x \in S$. Also, minimization means obtaining an ε-optimal solution in the sense defined below.

Definition 2.1. $\overline{x} \in S$ is said to be an ε-optimal solution of (P) if

$$f(x) + \varepsilon \geqq f(\bar{x}) \text{ for any } x \in S.$$

In order to establish necessary and sufficient optimality conditions to obtain an ε-optimal solution of (P), we have to assume constraint qualification of Slater's type for (P).

(CQ) $\quad \exists\, x_0 \in \mathbb{R}^n$ such that $h_j(x_0) < 0,\ 1 \leqq j \leqq m;\ Ax_0 = b;\ x_0 \in \operatorname{int} C$,

where, $\operatorname{int} C$ denotes the interior of C in the space $\mathbb{R}^n$.

We now derive the ε-optimality conditions for the problem (P).

Theorem 2.1. *Suppose (CQ) holds for (P). $\overline{x} \in S$ is an ε-optimal solution of (P), if and only if there exist $n+1$ points $\overline{y}^1, \ldots, \overline{y}^{n+1} \in Y$, scalars $\alpha_i \geqq 0$, $\varepsilon_i \geqq 0$, $1 \leqq i \leqq n+1$, $\mu_j \geqq 0$, $\hat{\varepsilon}_j \geqq 0$, $1 \leqq j \leqq m$, $\varepsilon_c \geqq 0$ and a vector $\eta \in \mathbb{R}^k$ such that*

$$0 \in \sum_{i=1}^{n+1} \partial_{\varepsilon_i}(\alpha_i \phi(\overline{x}, \overline{y}^i))(\overline{x}) + \sum_{j=1}^{m} \partial_{\hat{\varepsilon}_j}(\mu_j h_j)(\overline{x}) + A^T \eta + N_{\varepsilon_c}(C; \overline{x}),$$

$$\sum_{i=1}^{n+1} \varepsilon_i + \sum_{j=1}^{m} \hat{\varepsilon}_j + \varepsilon_c - \sum_{i=1}^{n+1} \alpha_i \phi(\overline{x}, \overline{y}^i) + \max_{y \in Y} \phi(\overline{x}, y) - \varepsilon \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0,$$

$$\sum_{i=1}^{n+1} \alpha_i = 1\,.$$

Proof. From [13, Theorem 2.4], it follows that, $\overline{x} \in S$ is an ε-optimal solution of (P) if and only if there exist scalars $\varepsilon_0 \geqq 0$, $\mu_j \geqq 0, \hat{\varepsilon}_j \geqq 0$ $(1 \leqq j \leqq m)$, $\varepsilon_c \geqq 0$, and a vector $\eta \in \mathbb{R}^k$

$$0 \in \partial_{\varepsilon_0} f(\overline{x}) + \sum_{j=1}^{m} \partial_{\hat{\varepsilon}_j}(\mu_j h_j)(\overline{x}) + A^T \eta + N_{\varepsilon_c}(C; \overline{x})$$

$$\varepsilon_0 + \sum_{j=1}^{m} \hat{\varepsilon}_j + \varepsilon_c - \varepsilon \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0\,.$$

Now, using the characterization of ε_0-subgradient of max function, stated in Lemma 1.1, we get that, $\overline{x} \in S$ in an ε-optimal solution of (P) only if there exist

$n+1$ points $\overline{y}^1, \ldots, \overline{y}^{n+1} \in Y$, scalars $\alpha_i \geqq 0$, $\varepsilon_i \geqq 0$, $1 \leqq i \leqq n+1$, such that

$$0 \in \sum_{i=1}^{n+1} \partial_{\varepsilon_i}(\alpha_i \phi(\overline{x}, \overline{y}^i))(\overline{x}) + \sum_{j=1}^{m} \partial_{\hat{\varepsilon}_j}(\mu_j h_j)(\overline{x}) + A^T \eta + N_{\varepsilon_c}(C; \overline{x}),$$

$$\sum_{i=1}^{n+1} \varepsilon_i + \sum_{j=1}^{m} \hat{\varepsilon}_j + \varepsilon_c - \sum_{i=1}^{n+1} \alpha_i \phi(\overline{x}, \overline{y}^i) + \max_{y \in Y} \phi(\overline{x}, y) - \varepsilon \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0,$$

$$\sum_{i=1}^{n+1} \alpha_i = 1\,.$$

3. Generalized fractional programming problems

A particular case of minimax programming problem is a generalized fractional programming problem in which the objective is to minimize the largest of the several ratios. Such problems arise in management applications of goal programming, resource allocation problems, financial and economic planning problems and problems in numerical mathematics. Research surveys by Schaible and Ibaraki [11], Schaible and Shi [12] contains a large bibliography on generalized fractional programming problems, its applications and solutions methods. Various algorithms for solving generalized fractional programs are also reported in [1].

We concentrate on the following generalized fractional programming problem

$$\text{(FP)} \qquad \min_{x} \max_{1 \leqq i \leqq p} \frac{f_i(x)}{g_i(x)} \quad \text{subject to} \quad x \in S$$

where $f_i, g_i : \mathbb{R}^n \to \mathbb{R}$ and the feasible set S is the same as defined in Section 2. We assume the following:

$$f_i(\cdot), -g_i(\cdot),\ 1 \leqq i \leqq p \text{ are convex on } S,\ \inf_{x \in S} g_i(x) > 0,\ 1 \leqq i \leqq p.$$

Moreover, minimization means obtaining an ε-optimal solution, that is, to find $\overline{x} \in S$ such that

$$\max_{1 \leqq i \leqq p} f_i(x)/g_i(x) + \varepsilon \geqq \max_{1 \leqq i \leqq p} f_i(\overline{x})/g_i(\overline{x}) \text{ for any } x \in S.$$

Using the ε-parametric approach, we associate the following minimax nonlinear parametric programming problem, involving a parameter $v \in \mathbb{R}$, with the primal problem (FP):

$$(\mathrm{P}_v) \qquad \min_{x} \max_{1 \leqq i \leqq p} (f_i(x) - v g_i(x)) \quad \text{subject to} \quad x \in S.$$

The next theorem establishes the relationship between the ε-optimal solutions of (FP) and $(\mathrm{P}_{\overline{v}})$, for some parameter $\overline{v} \in \mathbb{R}$.

Theorem 3.1. *Let* $\overline{x} \in S$ *be an* ε*-optimal solution of* (FP)*. Then, there exists* $\overline{v} = \max_{1 \leqq i \leqq p} \ f_i(\overline{x})/g_i(\overline{x}) - \varepsilon$ *such that* $\overline{x}$ *is an* $\overline{\varepsilon}$*-optimal solution of* $(P_{\overline{v}})$ *where* $\overline{\varepsilon} = \varepsilon \ \max_{1 \leqq i \leqq p} \ g_i(\overline{x})$.

Proof. Since $\overline{x} \in S$ is an ε-optimal solution of (FP), hence

$$\text{(3.1)} \qquad \max_{1 \leqq i \leqq p} f_i(x)/g_i(x) + \varepsilon \geqq \max_{1 \leqq i \leqq p} f_i(\overline{x})/g_i(\overline{x}) \qquad \text{for any } x \in S$$

$$\Rightarrow \max_{1 \leqq i \leqq p} f_i(x)/g_i(x) \geqq \overline{v} \qquad \text{for any } x \in S$$

$$\Rightarrow \max_{1 \leqq i \leqq p} (f_i(x) - \overline{v} g_i(x)) \geqq 0 \qquad \text{for any } x \in S.$$

Also, since $\overline{v} = \max_{1 \leqq i \leqq p} \ f_i(\overline{x})/g_i(\overline{x}) - \varepsilon$, hence

$$f_i(\overline{x}) - \overline{v} g_i(\overline{x}) \leqq \varepsilon \, g_i(\overline{x}) \leqq \varepsilon \ \max_{1 \leqq i \leqq p} (g_i(\overline{x})) \qquad \text{for any } 1 \leqq i \leqq p\,.$$

Thus,

$$f_i(\overline{x}) - \overline{v} g_i(\overline{x}) \leqq \overline{\varepsilon} \qquad \text{for any } 1 \leqq i \leqq p\,.$$

The above inequality along with (3.1) yield

$$\max_{1 \leqq i \leqq p} (f_i(x) - \overline{v} g_i(x)) \geqq \max_{1 \leqq i \leqq p} (f_i(\overline{x}) - \overline{v} g_i(\overline{x})) - \overline{\varepsilon} \qquad \text{for any } x \in S.$$

Therefore, $\overline{x}$ is an $\overline{\varepsilon}$-optimal solution of $(\mathrm{P}_{\overline{v}})$. □

Remark 3.1. Converse conclusion of the above theorem needs not follow, i.e., if $\overline{x} \in S$ is an $\overline{\varepsilon}$-optimal solution of $(\mathrm{P}_{\overline{v}})$ where $\overline{v} = \max_{1 \leqq i \leqq p} \ f_i(\overline{x})/g_i(\overline{x}) - \varepsilon \geqq 0$ and $\overline{\varepsilon} = \max_{1 \leqq i \leqq p} g_i(\overline{x})$, then $\overline{x}$ needs not be an ε-optimal soltion of (FP), as illustrated by the following example.

Example 3.1. Let $f_1(x) = 2x - 1,\ f_2(x) = 3x^2 - 2,\ g_1(x) = x + 1,\ g_2(x) = 6,$ $h_1(x) = x(x-2),\ x \in \mathbb{R}$. Then, (FP) is given by

$$\min_x \ \max \big((2x-1)/(x+1), (3x^2-2)/6\big)$$
$$\text{subject to } \ x \in S = [0, 2]\,.$$

Let $\overline{x} = 1$ and $\overline{\varepsilon} = 2$. Then, $\overline{\varepsilon} = \varepsilon \max(g_1(\overline{x}), g_2(\overline{x}))$ give $\varepsilon = 1/3$ and $\overline{v} = \max(f_1(\overline{x})/g_1(\overline{x}), f_2(\overline{x})/g_2(\overline{x})) - \varepsilon = 1/6$. The parametric minimax problem $(\mathrm{P}_{\overline{v}})$ is given by

$$\min_{x \in S} \ \max(f_1(x) - \overline{v} g_1(x), f_2(x) - \overline{v} g_2(x)) = \min_{x \in S} \ \max \big((11x - 7)/6, 3x^2 - 3\big)$$

and we have for any $x \in S$,

$$\max(f_1(x) - \overline{v} g_1(x), f_2(x) - \overline{v} g_2(x)) \geqq -7/6.$$

so, we have

$$\max(f_1(x) - \overline{v} g_1(x), f_2(x) - \overline{v} g_2(x)) + \overline{\varepsilon}$$
$$\geqq \max(f_1(\overline{x}) - \overline{v} g_1(\overline{x}), f_2(\overline{x}) - \overline{v} g_2(\overline{x})) \quad \text{for any } x \in S.$$

Hence, $\overline{x} = 1$ is an $\overline{\varepsilon}$-optimal solution of $(\mathrm{P}_{\overline{v}})$. But, since

$$\max(f_1(0)/g_1(0), f_2(0)/g_2(0)) = -1/3$$

we have

$$\max(f_1(0)/g_1(0), f_2(0)/g_2(0)) + \varepsilon = -1/3 + 1/3 \ngeqq 1/2 = \max(f_1(\overline{x})/g_1(\overline{x}), f_2(\overline{x})/g_2(\overline{x})).$$

Therefore, $\overline{x} = 1$ is not an ε-optimal solution of (FP).

4. ε-optimality conditions for fractional programs

Utilizing ε-optimality conditions for minimax programming problem, as developed in Section 2 of the present paper, we now derive the necessary and sufficient optimality conditions of KKT type for an ε-optimal solution of a generalized fractional programming problem (FP).

Theorem 4.1 (Necessary Optimality Condition)**.** *Suppose that (CQ) holds and* $\bar{v} = \max_{1\leqq i\leqq p} f_i(\bar{x})/g_i(\bar{x}) - \epsilon$. *Let* $\overline{x} \in S$ *be an* ε*-optimal solution of (FP), with* $0 < \varepsilon < \max_{1\leqq i\leqq p} f_i(\overline{x})/g_i(\overline{x})$. *Then, there exist scalars* $\overline{v} \geqq 0, \alpha_i \geqq 0, \varepsilon_{1i} \geqq 0, \varepsilon_{2i} \geqq 0$ $(1 \leqq i \leqq p), \varepsilon_{3j} \geqq 0, \mu_j \geqq 0$ $(1 \leqq j \leqq m), \varepsilon_c \geqq 0$ *and a vector* $\eta \in \mathbb{R}^k$ *such that*

$$0 \in \sum_{i=1}^{p}(\partial_{\varepsilon_{1i}}(\alpha_i f_i)(\overline{x}) + \partial_{\varepsilon_{2i}}(\alpha_i\overline{v}(-g_i))(x)) + \sum_{j=1}^{m}\partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_{\varepsilon_c}(C;\overline{x}) \tag{4.1}$$

$$\sum_{i=1}^{p}(\varepsilon_{1i}+\varepsilon_{2i}) + \sum_{j=1}^{m}\varepsilon_{3j} + \varepsilon_c - \varepsilon\max_{1\leqq i\leqq p} g_i(\overline{x}) + \max_{1\leqq i\leqq p}(f_i(\overline{x}) - \overline{v}g_i(\overline{x})) - \sum_{i=1}^{p}\alpha_i(f_i(\overline{x}) - \overline{v}g_i(\overline{x})) \leqq \sum_{j=1}^{m}\mu_j h_j(\overline{x}) \leqq 0\,, \tag{4.2}$$

$$\sum_{i=1}^{p}\alpha_i = 1. \tag{4.3}$$

Proof. Since $\overline{x} \in S$ is an ε-optimal solution of (FP) it follows, from Theorem 3.1, that $\overline{x}$ is an $\overline{\varepsilon}$-optimal solution of $(\mathrm{P}_{\overline{v}})$ where $\overline{\varepsilon} = \varepsilon\ \max_{1\leqq i\leqq p}(g_i(\overline{x}))$. Rewrite $(\mathrm{P}_{\overline{v}})$ as $\min_{x\in S}\max_{1\leqq i\leqq p}\ \phi(x,i)$ where $\phi(x,i) = f_i(x) - \overline{v}g_i(x)$. Now, by Theorem 2.1, there exist scalars $\alpha_i \geqq 0, \varepsilon_{0i} \geqq 0$ $(1 \leqq i \leqq p)$, with $\sum_{i=1}^{p}\alpha_i = 1$, $\varepsilon_{3j} \geqq 0, \mu_j \geqq 0 (1 \leqq j \leqq m), \varepsilon_c \geqq 0$ and a vector $\eta \in \mathbb{R}^k$ such that

$$0 \in \sum_{i=1}^{p}\partial_{\varepsilon_{0i}}(\alpha_i(f_i - \overline{v}g_i))(\overline{x}) + \sum_{j=1}^{m}\partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_{\varepsilon_c}(C;\overline{x}),$$

$$\sum_{i=1}^{p}\varepsilon_{0i} + \sum_{j=1}^{m}\varepsilon_{3j} + \varepsilon_c - \sum_{i=1}^{p}\alpha_i(f_i(\overline{x}) - \overline{v}g_i(\overline{x})) + \max_{1\leqq i\leqq p}(f_i(\overline{x}) - \overline{v}g_i(\overline{x})) - \varepsilon\ \max_{1\leqq i\leqq p}\ g_i(\overline{x}) \leqq \sum_{j=1}^{m}\mu_j h_j(\overline{x}) \leqq 0\,,$$

which, on using [7, Theorem 2.1], yield (4.1), (4.2), and (4.3). □

We give an example of ε-KKT conditions of the generalized minimax programming problem (FP) in which (CQ) does not hold necessarily.

Example 4.1.

$$\text{(FP)} \qquad \min_{x_1,\, x_2} \quad \max\left((0.5^{x_1}+4)/g_1(x_1,x_2), f_2(x_1,x_2)/0.1\right)$$
$$\text{subject to } \ |x_1| - x_2 - 1 \leqq 0$$
$$-x_1 + x_2 \leqq 0$$
$$-x_1 + 1/2 \leqq 0$$
$$\text{where} \quad g_1(x_1,x_2) = \begin{cases} 4 & \text{if } x_2 \geqq 2 \\ -(x_2-2)^2+4 & \text{if } 0 \leqq x_2 \leqq 2 \\ 4x_2 & \text{if } x_2 \leqq 0 \end{cases}$$
$$f_2(x_1,x_2) = \begin{cases} -1.2 & \text{if } x_2 \geqq 2 \\ -x_1+0.8 & \text{if } x_2 \leqq 2 \end{cases}.$$

There does not exist an exact solution of (FP).

Let $\varepsilon = 0.1$. Then, we have $\bar{x} = (2,2)$ is an ε-solution of (FP) and $\bar{v} = \max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) - \varepsilon = \max((0.5^2+4)/4, -1.2) - 0.1 = 0.96$.

From ε-subdifferential calculus, we have that

$$\begin{aligned}
&\partial_{0.0225}(9/10 * f_1)(2,2) = ([-0.47, 0], 0), \\
&\partial_{0.0225}(9/10 * 0.96 * (-g_1))(2,2) = (0, [-0.31, 0]), \\
&\partial_{0.005}(1/10 * f_2)(2,2) = ([-10, 0], 0), \\
&\partial_{0}(1/10 * 0.96 * (-g_2))(2,2) = (0,0), \\
&\partial_{0.05}(1 * h_1)(2,2) = ([1 - 0.05/2, 1], -1), \\
&\partial_{0}(1 * h_2)(2,2) = (-1, 1) \\
&\partial_{0}(0 * h_3)(2,2) = (0,0).
\end{aligned}$$

Then, there exist

$$\begin{aligned}
&\bar{v} = 0.96, \alpha_1 = 9/10, \alpha_2 = 1/10, \\
&\varepsilon_{11} = \varepsilon_{21} = 0.0225, \varepsilon_{12} = 0.005, \varepsilon_{22} = 0, \varepsilon_{31} = 0.05, \varepsilon_{32} = \varepsilon_{33} = 0 \\
&\mu_1 = \mu_2 = 1, \ \mu_3 = 0
\end{aligned}$$

such that

$$\begin{aligned}
(0,0) \in\ &\partial_{0.0225}(9/10 * f_1)(2,2) + \partial_{0.0225}(9/10 * 0.96 * (-g_1))(2,2) \\
&+ \partial_{0.005}(1/10 * f_2)(2,2) + \partial_{0}(1/10 * 0.96 * (-g_2))(2,2) \\
&+ \partial_{0.05}(1 * h_1)(2,2) + \partial_{0}(1 * h_2)(2,2),
\end{aligned}$$

$$\begin{aligned}
&(0.0225 + 0.0225) + (0.005 + 0) + (0.05 + 0) - 0.1 * \max(4, 0.1) \\
&\quad + \max(0.5^2 + 4 - 0.96 * 4, -1.2 - 0.96 * 0.1)
\end{aligned}$$

$$- (9/10) * (0.5^2 + 4 - 0.96 * 4) - (1/10) * (-1.2 - 0.96 * 0.1)$$
$$\leqq 0$$
$$= 1 * h_1(2,2) + 1 * h_2(2,2) + 0 * h_3(2,2),$$
$$9/10 + 1/10 = 1.$$

Theorem 4.2 (Sufficient Optimality Condition)**.** *Let $\overline{x} \in S$, and let there exist scalars $\overline{v} \geqq 0$, $\varepsilon \geqq 0$, $\alpha_i \geqq 0$, $\varepsilon_{1i} \geqq 0$, $\varepsilon_{2i} \geqq 0$, $1 \leqq i \leqq p$, $\varepsilon_{3j} \geqq 0$, $\mu_j \geqq 0$, $1 \leqq j \leqq m$, $\varepsilon_c \geqq 0$ and a vector $\eta \in \mathbb{R}^k$ such that*

$$(4.4) \quad 0 \in \sum_{i=1}^{p} (\partial_{\varepsilon_{1i}}(\alpha_i f_i)(\overline{x}) + \partial_{\varepsilon_{2i}}(\alpha_i \overline{v}(-g_i))(\overline{x})) + \sum_{j=1}^{m} \partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_c(C; \overline{x}),$$

$$(4.5) \quad \sum_{i=1}^{p} (\varepsilon_{1i} + \varepsilon_{2i}) + \sum_{j=1}^{m} \varepsilon_{3j} + \varepsilon_c - \varepsilon \sum_{i=1}^{p} \alpha_i g_i(\overline{x}) \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0\,,$$

$$(4.6) \quad f_i(\overline{x}) - \overline{v} g_i(\bar{x}) = \varepsilon\ g_i(\overline{x}), \quad 1 \leqq i \leqq p,$$

$$(4.7) \quad \sum_{i=1}^{p} \alpha_i = 1\,.$$

Then, $\overline{x}$ is an ε-optimal solution of (FP).

Proof. In view of (4.6), condition (4.5) can be written as

$$(4.8) \quad \sum_{i=1}^{p} (\varepsilon_{1i} + \varepsilon_{2i}) + \sum_{j=1}^{m} \varepsilon_{3j} + \varepsilon_c - \sum_{i=1}^{p} \alpha_i (f_i(\overline{x}) - \overline{v} g_i(\overline{x})) + \max_{1 \leqq i \leqq p} (f_i(\overline{x}) - \overline{v} g_i(\overline{x})) - \varepsilon \max_{1 \leqq i \leqq p} g_i(\overline{x}) \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0\,.$$

Also, from (4.4), we get

$$(4.9) \quad \sum_{i=1}^{p} (\partial_{\varepsilon_{1i}}(\alpha_i f_i)(\overline{x}) + \partial_{\varepsilon_{2i}}(\alpha_i \overline{v}(-g_i))(\overline{x})) + \sum_{j=1}^{m} \partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_{\varepsilon_c}(C; \overline{x})$$
$$\subset \sum_{i=1}^{p} \partial_{\varepsilon_{0i}}(\alpha_i (f_i - \overline{v} g_i))(\overline{x}) + \sum_{j=1}^{m} \partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_{\varepsilon_c}(C; \overline{x})$$

where $\varepsilon_{0i} = \varepsilon_{1i} + \varepsilon_{2i}$, as $\partial_v \psi + \partial_\omega \phi \subset \partial_{v+\omega}(\psi + \phi)$. Letting

$$(4.10) \quad \varepsilon_0 = \sum_{i=1}^{p} \varepsilon_{0i} - \sum_{i=1}^{p} \alpha_i (f_i - \overline{v} g_i)(\overline{x}) + \max_{1 \leqq i \leqq p} (f_i(\overline{x}) - \overline{v} g_i(\overline{x}))$$

in (4.8), we get

$$\varepsilon_0 + \sum_{j=1}^{m} \varepsilon_{3j} + \varepsilon_c - \overline{\varepsilon} \leqq \sum_{j=1}^{m} \mu_j h_j(\overline{x}) \leqq 0 \,. \tag{4.11}$$

From (4.7), (4.9) and (4.10), and on using Lemma 1.1, we get

$$0 \in \partial_{\varepsilon_0}\left(\max_{1\leqq i\leqq p} (f_i - \overline{v} g_i) \right)(\overline{x}) + \sum_{j=1}^{m} \partial_{\varepsilon_{3j}}(\mu_j h_j)(\overline{x}) + A^T\eta + N_{\varepsilon_c}(C;\overline{x}) \,,$$

which, together with (4.11), give that $\overline{x}$ is an $\overline{\varepsilon}$-optimal solution of ($P_{\overline{v}}$). Thus,

$$\max_{1\leqq i\leqq p} (f_i(x) - \overline{v} g_i(x)) \geqq \max_{1\leqq i\leqq p} (f_i(\overline{x}) - \overline{v} g_i(\overline{x})) - \overline{\varepsilon} \qquad \text{for any } x \in S \,,$$

which, in view of (4.6), implies

$$\max_{1\leqq i\leqq p} f_i(x)/g_i(x) \geqq \max_{1\leqq i\leqq p} f_i(\overline{x})/g_i(\overline{x}) - \varepsilon \qquad \text{for any } x \in S \,.$$

Hence, $\overline{x}$ is an ε-optimal solution of (FP). □

5. Duality result

From the necessary condition, we are interestd in a dual problem:

(D) Maximize v

$$\text{(5.1) subject to} \quad 0 \in \sum_{i=1}^{p} (\partial_{\epsilon_{1i}}(\alpha_i f_i(x)) + \partial_{\epsilon_{2i}}(\alpha_i v(-g_i))(x) + \sum_{j=1}^{m} \partial_{\epsilon_{3j}}(\mu_j h_j)(x) + A^\top \eta + N_{\epsilon_c}(C;x),$$

$$f_i(x) - v g_i(x) \geqq \epsilon g_i(x) \ (1 \leqq i \leqq p), \tag{5.2}$$

$$\sum_{i=1}^{p} (\epsilon_{1i} + \epsilon_{2i}) + \sum_{j=1}^{m} \epsilon_{3j} + \epsilon_c - \epsilon \max_{1\leqq i\leqq p} g_i(x) + \max_{1\leqq i\leqq p} (f_i(x) - v g_i(x)) - \sum_{i=1}^{p} \alpha_i (f_i(x) - v g_i(x)) \leqq \sum_{j=1}^{m} \mu_j h_j(x) \leqq 0, \tag{5.3}$$

$$A^\top x = b, \tag{5.4}$$

$$\sum_{i=1}^{p} \alpha_i = 1, \tag{5.5}$$

$$\alpha_i \geqq 0, \epsilon_{1i} \geqq 0, \epsilon_{2i} \geqq 0 \ (1 \leqq i \leqq p), \epsilon_{3j} \geqq 0, \mu_j \geqq 0 \ (1 \leqq j \leqq m). \tag{5.6}$$

Let S_D be the set of all feasible points $(x, v, \alpha_1, \dots, \alpha_p, \epsilon_{11}, \dots, \epsilon_{1p}, \epsilon_{21}, \dots, \epsilon_{2p}, \epsilon_{31}, \dots, \epsilon_{3m}, \epsilon_c, \mu_1, \dots, \mu_m, \eta)$ for the dual problem (D).

Definition 5.1. We call that $(\bar{x}, \bar{v}, \bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{\epsilon}_{11}, \ldots, \bar{\epsilon}_{1p}, \bar{\epsilon}_{21}, \ldots, \bar{\epsilon}_{2p}, \bar{\epsilon}_{31}, \ldots, \bar{\epsilon}_{3m}, \bar{\epsilon}_c, \bar{\mu}_1, \ldots, \bar{\mu}_m, \bar{\eta}) \in S_D$ is an ϵ-optimal solution of (D) if $\bar{v} \geqq v - \epsilon$ for all $(x, v, \alpha_1, \ldots, \alpha_p, \epsilon_{11}, \ldots, \epsilon_{1p}, \epsilon_{21}, \ldots, \epsilon_{2p}, \epsilon_{31}, \ldots, \epsilon_{3m}, \epsilon_c, \mu_1, \ldots, \mu_m, \eta) \in S_D$.

We suppose some restrictive assumption and show a duality result.

Theorem 5.1 (Duality Theorem)**.** *Let $\bar{x} \in S$. Suppose that*

$$f_i(\bar{x})/g_i(\bar{x}) \ (1 \leqq i \leqq p) \ \text{are constant.}$$

If $\bar{x}$ is an ϵ-optimal solution of (FP), then there exist scalars $\bar{v} \geqq 0$, $\bar{\alpha}_i \geqq 0, \bar{\epsilon}_{1i} \geqq 0, \bar{\epsilon}_{2i} \geqq 0$ for $1 \leqq i \leqq p$, $\bar{\epsilon}_{3j} \geqq 0, \bar{\mu}_j \geqq 0$ for $1 \leqq j \leqq m$ and a vector $\bar{\eta} \in \mathbb{R}^k$ such that $(\bar{x}, \bar{v}, \bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{\epsilon}_{11}, \ldots, \bar{\epsilon}_{1p}, \bar{\epsilon}_{21}, \ldots, \bar{\epsilon}_{2p}, \bar{\epsilon}_{31}, \ldots, \bar{\epsilon}_{3m}, \bar{\epsilon}_c, \bar{\mu}_1, \ldots, \bar{\mu}_m, \bar{\eta})$ is an ϵ-optimal solution of (D).

Proof. Let $\bar{v} = \max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) - \epsilon$. From the assumption, we have that for any $1 \leqq i \leqq p$, $f_i(\bar{x}) - \bar{v} g_i(\bar{x}) = f_i(\bar{x}) - (f_i(\bar{x})/g_i(\bar{x}) - \epsilon) g_i(\bar{x}) = \epsilon g_i(\bar{x})$, i.e., (5.2) holds. Then we have that from Theorem 4.1 $(\bar{x}, \bar{v}, \bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{\epsilon}_{11}, \ldots, \bar{\epsilon}_{1p}, \bar{\epsilon}_{21}, \ldots, \bar{\epsilon}_{2p}, \bar{\epsilon}_{31}, \ldots, \bar{\epsilon}_{3m}, \bar{\epsilon}_c, \bar{\mu}_1, \ldots, \bar{\mu}_m, \bar{\eta}) \in S_D$.

Let $K(x) = Ax - b$. From (5.1), for any $(x, v, \alpha_1, \ldots, \alpha_p, \epsilon_{11}, \ldots, \epsilon_{1p}, \epsilon_{21}, \ldots, \epsilon_{2p}, \epsilon_{31}, \ldots, \epsilon_{3m}, \epsilon_c, \mu_1, \ldots, \mu_m, \eta) \in S_D$, there exist $\xi_{1i} \in \partial_{\epsilon_{1i}}(\alpha_i f_i(x)), \xi_{2i} \in \partial_{\epsilon_{2i}}(\alpha_i v(-g_i))(x)$ for $1 \leqq i \leqq p$, $\xi_{3j} \in \partial_{\epsilon_{3j}}(\mu_j h_j(x))$ for $1 \leqq j \leqq m$ and $\xi \in N_{\epsilon_c}(C; x)$ such that

$$\sum_{i=1}^{p}(\xi_{1i} + \xi_{2i}) + \sum_{j=1}^{m} \xi_{3j} + \nabla K^\top(x)\eta + \xi = 0.$$

From the definition of ϵ-subdifferential, we have that

$$\begin{aligned}
\alpha_i f_i(\bar{x}) &\geqq \alpha_i f_i(x) + \langle \xi_{1i}, \bar{x} - x \rangle - \epsilon_{1i} \ (1 \leqq i \leqq p), \\
\alpha_i v(-g_i)(\bar{x}) &\geqq \alpha_i v(-g_i)(x) + \langle \xi_{2i}, \bar{x} - x \rangle - \epsilon_{2i} \ (1 \leqq i \leqq p), \\
\mu_j h_j(\bar{x}) &\geqq \mu_j h_j(x) + \langle \xi_{3j}, \bar{x} - x \rangle - \epsilon_{3j} \ (1 \leqq j \leqq m), \\
0 &\geqq \langle \xi, \bar{x} - x \rangle - \epsilon_c, \\
\nabla^\top K(x)(\bar{x} - x) &= 0.
\end{aligned}$$

So, we have that

$$\begin{aligned}
&\sum_{i=1}^{p}(\alpha_i f_i(\bar{x}) + \alpha_i v(-g_i)(\bar{x})) + \sum_{j=1}^{m} \mu_j h_j(\bar{x}) \\
&\geqq \sum_{i=1}^{p}(\alpha_i f_i(x) + \alpha_i v(-g_i)(x)) + \sum_{j=1}^{m} \mu_j h_j(x) \\
&\quad + \langle \sum_{i=1}^{p}(\xi_{1i} + \xi_{2i}) + \sum_{j=1}^{m} \xi_{3j} + \xi, \bar{x} - x \rangle - \sum_{i=1}^{p}(\epsilon_{1i} + \epsilon_{2i}) - \sum_{j=1}^{m} \epsilon_{3j} - \epsilon_c \\
&\geqq \sum_{i=1}^{p}(\epsilon_{1i} + \epsilon_{2i}) + \sum_{j=1}^{m} \epsilon_{3j} + \epsilon_c - \epsilon \max_{1 \leqq i \leqq p} g_i(x) + \max_{1 \leqq i \leqq p}(f_i(x) - v g_i(x)) \\
&\quad + \langle -\nabla^\top K(x)\eta, \bar{x} - x \rangle - \sum_{i=1}^{p}(\epsilon_{1i} + \epsilon_{2i}) - \sum_{j=1}^{m} \epsilon_{3j} - \epsilon_c \ \text{(from (5.3))}
\end{aligned}$$

$$= -\epsilon \max_{1 \leqq i \leqq p} g_i(x) + \max_{1 \leqq i \leqq p} (f_i(x) - v g_i(x))$$
$$\geqq 0 \text{ (since (5.2) holds).}$$

Since $\bar{x} \in S$, we have $\sum_{i=1}^{p}(\alpha_i f_i(\bar{x}) + \alpha_i v(-g_i)(\bar{x})) \geqq 0$ that is

$$\sum_{i=1}^{p}(\alpha_i f_i)(\bar{x}) / \sum_{i=1}^{p}(\alpha_i g_i)(\bar{x}) \geqq v. \tag{5.7}$$

We next show that

$$\max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) = \sum_{i=1}^{p}(\alpha_i f_i)(\bar{x}) / \sum_{i=1}^{p}(\alpha_i g_i)(\bar{x}). \tag{5.8}$$

Let $\max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) = f_{\bar{i}}(\bar{x})/g_{\bar{i}}(\bar{x})$. So, from the assumption, $f_{\bar{i}}(\bar{x})/g_{\bar{i}}(\bar{x}) = f_i(\bar{x})/g_i(\bar{x})$ that is $f_{\bar{i}}(\bar{x})g_i(\bar{x}) - f_i(\bar{x})g_{\bar{i}}(\bar{x}) = 0$ for any $i = 1, \dots, p$. Then, we have $\sum_{i=1}^{p} \alpha_i (f_{\bar{i}} g_i - f_i g_{\bar{i}})(\bar{x}) = 0$ that is $f_{\bar{i}}(\bar{x}) \sum_{i=1}^{p}(\alpha_i g_i)(\bar{x}) - g_{\bar{i}}(\bar{x}) \sum_{i=1}^{p}(\alpha_i f_i)(\bar{x}) = 0$. Thus, (5.8) holds.

Finally, we have that

$$\begin{aligned}
\bar{v} + \epsilon &= \max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) - \epsilon + \epsilon \quad (\text{since } \bar{v} = \max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) - \epsilon) \\
&= \max_{1 \leqq i \leqq p} f_i(\bar{x})/g_i(\bar{x}) \\
&= \sum_{i=1}^{p}(\alpha_i f_i)(\bar{x}) / \sum_{i=1}^{p}(\alpha_i g_i)(\bar{x}) \\
&\geqq v \quad (\text{from (5.7)}).
\end{aligned}$$

6. Concluding remarks

Based on Ekeland's variational principle [5], another approach of characterizing ε-optimal solutions for nondifferentiable optimization problems has emerged in recent years. Hamel [6] followed this approach to extend the Lagrange multiplier rule for an ε-minimal solution of nondifferentiable mathematical programming problem on a real Banach space. The optimality conditions so obtained yield more information on ε-optimal solutions. It would be interesting to derive ε-optimality conditions for minimax programming problems using Ekeland's variational principle and some of its new variants. This will be the subject of investigation in the subsequent research work by the authors.

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