Journal of Nonlinear and Convex Analysis
Volume 7, Number 2, 2006, 269–275
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

WOLFE-TYPE DUALITY INVOLVING UNIVEX FUNCTIONS FOR A MINIMAX PROGRAMMING PROBLEM

SHASHI K. MISHRA* AND NORMA G. RUEDA

ABSTRACT. A sufficient optimality theorem is proved for a certain minimax programming problem under the assumptions of proper (b,η,ϕ)-univex conditions on the functions involved for such a problem. A dual is defined for this problem and duality theorems relating the primal and the dual are proved. The results presented in this paper extend earlier works in the literature to a wider class of functions.

1. INTRODUCTION

The concept of invexity was introduced by Hanson [4] as a generalization of convexity. Later Bector and Singh [2] defined B-vexity and discussed properties of functions satisfying this condition. In 1992 Bector *et al.* [3] introduced the class of univex functions and obtained sufficient optimality conditions and duality results. Recently, Bector [1] defined properly (b,η)-invex functions and established a sufficient optimality theorem and duality results for a class of minimax programming problems. In this paper we generalize the results given in [1] to a new class of functions, called properly (b,η,ϕ)-univex functions.

2. PRELIMINARIES

In this section we give some definitions, the main problem considered in the paper, and some results needed in the sequel.

We will distinguish between $\leq$ and $\leqq$, and between $\geq$ and $\geqq$. For $x, y \in R^n$ we define $x \leqq y$ iff $x_i \leqq y_i$ for $i = 1, 2, \ldots, n$; $x \leq y$ iff $x \leqq y$ and $x \neq y$; $x \geqq y$ iff $x_i \geqq y_i$ for $i = 1, 2, \ldots, n$; and $x \geq y$ iff $x \geqq y$ and $x \neq y$.

Let $X \subseteq R^n$ be an open set. The next two definitions below are due to Hanson [4].

Definition 2.1. A differentiable function $f : X \to R$ is said to be η-invex if there exists a function $\eta : X \times X \to R^n$ such that for each $x, u \in X$,

$$f(x) - f(u) \geqq \eta(x, u)^T \nabla f(u).$$

Definition 2.2. A differentiable function $f : X \to R$ is said to be *strictly η-invex* if there exists a function $\eta : X \times X \to R^n$ such that for each $x, u \in X$,

$$f(x) - f(u) > \eta(x, u)^T \nabla f(u).$$

2000 *Mathematics Subject Classification.* 90C47, 90C46.

Key words and phrases. Minimax programming, duality, generalized convex functions.

*The research of this author is financially supported by the University Grants Commission of India through grant No. F. 8 – 33/2001(SR – I)

The following definition is due to Bector [1].

Definition 2.3. A differentiable function $f : X \to R$ is said to be *properly* (b, η)-*invex* if there exist functions $\eta : X \times X \to R^n$and $b : X \times X \to R^+ \backslash \{0\}$ such that for each $x, u \in X$,

$$b(x, u)[f(x) - f(u)] \geqq \eta(x, u)^T \nabla f(u).$$

The concept of univexity was introduced by Bector et al.[3] in 1992.

Definition 2.4. A differentiable function $f : X \to R$ is said to be *properly* (b, η, ϕ)-*univex* if there exist functions $\eta : X \times X \to R^n$, $b : X \times X \to R^+ \backslash \{0\}$and $\phi : R \to R$, such that for each $x, u \in X$,

$$b(x, u)\phi[f(x) - f(u)] \geqq \eta(x, u)^T \nabla f(u).$$

If ϕ is the identity function, then the above definition reduces to the definition of (b, η)-invexity [1]. There exist functions that are properly (b, η, ϕ)-univex, but they are not properly (b, η)-invex.

Example 2.1. Let $f : [1, \infty) \to R$ defined by $f(x) = -x + 1$. This function is properly (b, η, ϕ)-univex with respect to $\eta(x, u) = -1/x + u,\ b = 1,\ \phi(x) = -x$ at u =1, but it is not properly (b, η)-invex.

Definition 2.5. A differentiable function $f : X \to R$ is said to be *properly strictly* (b, η, ϕ)-*univex* if there exist functions $\eta : X \times X \to R^n$, $b : X \times X \to R^+ \backslash \{0\}$and $\phi : R \to R$, such that for each $x, u \in X$,

$$b(x, u)\phi[f(x) - f(u)] > \eta(x, u)^T \nabla f(u).$$

The following theorem, which we shall use in the sequel, is easy to prove; therefore we state it without proof. This theorem is an extension of Theorem 2.1 [1].

Theorem 2.1. *Let f_i and each of $h_{ij}, j = 1,\ 2, \cdots,\ m$,be properly (b_i, η, ϕ)-univex on X. If $\lambda_i \geqq 0$ and $y_{ij} \geqq 0, j = 1,\ 2, \ldots,\ m$,then $\lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij}$ is properly (b_i, η, ϕ)-univex on X. If f_i is properly strictly (b_i, η, ϕ)-univex and $\lambda_i > 0$, or at least one of h_{ij} for which the corresponding $y_{ij} > 0$, is properly strictly (b_i, η, ϕ)-univex, then $\lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij}$is properly strictly (b_i, η, ϕ)-univex on X.*

Primal problem. In this paper we consider the following generalized minimax programming problem as the primal problem.

$$q^* = \min_{x \in X} \max_{1 \leq i \leq p} [f_i(x)],$$

subject to $h_{ij}(x) \leqq 0,\ i = 1,\ 2, \ldots,\ p,,\ j = 1,\ 2, \ldots,\ m.$

We assume that $f_i, i = 1,\ 2, \ldots,\ p$, and $h_{ij}(x),\ i = 1,\ 2, \ldots, p;\ j = 1,\ 2, \ldots,\ m$ are differentiable on X.

We now consider the following programming problem (E) which is equivalent to (P) in the sense of the Lemmas 2.1 and 2.2 given below.

$$\min_{x,q} q$$

subject to

$$f_i(x) \leqq q,\ i = 1,\ 2, \ldots,\ p \tag{1}$$

$$h_{ij}(x) \leqq 0,\ i = 1,\ 2, \ldots,\ p;\ j = 1,\ 2, \ldots,\ m, \tag{2}$$

$$x \in X.$$

Lemma 2.1. *Let x be (P)-feasible. Then there exists $q \in R$ such that $(x, q)$is (E)-feasible, and if $(x, q)$is (E)-feasible, then x is (P)-feasible.*

Lemma 2.2. *Let x^* be (P)-optimal. Then there exists $q^* \in R$ such that $(x^*, q^*)$is (E)-optimal, and if (x^*, q^*) is (E)-optimal then $x^*$is (P)-optimal with q^* as the optimal value of the (P)-objective.*

3. Optimality conditions

The Kuhn-Tucker conditions are necessary in the solution of a nonlinear programming problem if a constraint qualification is satisfied. Most of the constraint qualifications that appear in the literature do not involve the objective function. See Mangasarian [5] for several examples of constraint qualifications. One of them, known as Slater's, requires the set of feasible points to contain an interior point.

Lemma 3.1 (Necessary optimality conditions [1])**.** *Let x^* be (P)-optimal. Let a constraint qualification hold for (P). Then there exist $q^* \in R$, $\lambda^* \in R^p$ and a matrix $Y^* \in R^{p\times m}$, such that $(x^*, q^*, \lambda^*, Y^*)$satisfies*

$$\sum_{i=1}^{p} \nabla[\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*)] = 0 \tag{3}$$

$$\lambda_i^*[f_i(x^*) - q^*] = 0,\ i = 1,\ 2, \ldots,\ p, \tag{4}$$

$$y_{ij}^* h_{ij}(x^*) = 0,\ \ i = 1,\ 2, \ldots, p;\ j = 1,\ 2, \ldots,\ m, \tag{5}$$

$$f_i(x^*) \leqq q^*,\ i = 1,\ 2, \ldots,\ p, \tag{6}$$

$$h_{ij}(x^*) \leqq 0,\ i = 1, 2, \ldots, p; j = 1,\ 2, \ldots,\ m, \tag{7}$$

$$\sum_{i=1}^{p} \lambda_i^* = 1, \tag{8}$$

$$q^* \in R,\ \lambda^* \in R^p, Y^* \in R^{p\times m}, \lambda^* \geq 0,\ Y^* \geqq 0. \tag{9}$$

Theorem 3.1 (Sufficient optimality conditions)**.** *If* $(x^*, q^*, \lambda^*, Y^*)$ *satisfies (3)-(9),* ϕ *is a linear mapping, and the following conditions are satisfied: Either* (i) *Each* $f_i,\ i = 1,\ 2, \ldots,\ p,$ *and* $h_{ij}(x),\ i = 1,\ 2, \ldots,\ p,\ j = 1,\ 2, \ldots,\ m,$*is a properly* (b_i, η, ϕ)*-univex function on* X*; or* (ii) $\lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij}$ *is properly* (b_i, η, ϕ)*-univex on* X *for* $i = 1,\ 2, \ldots,\ p,$ *and* $\lambda_i \geqq 0,\ y_{ij} \geqq 0,\ \ i = 1,\ 2, \ldots,\ p; j = 1,\ 2, \ldots,\ m,$ *and* (iii) $\phi(a) \geqq 0 \Rightarrow a \geqq 0,$ *then* x^* *is (P)-optimal.*

Proof. First we prove that (x^*, q^*) is (E)-optimal. Since $(x^*, q^*, \lambda^*, Y^*)$ satisfies (3), therefore we have

$$\eta(x, x^*)^T \sum_{i=1}^{p} \nabla[\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*)] = 0 \tag{10}$$

for all (P)-feasible solutions x.

From Theorem 2.1 and either (i) or (ii), $\lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij}$ is properly (b_i, η, ϕ)-univex on X. Therefore,

$$\begin{aligned} b_i(x, x^*)\phi[(\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - (\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*))] \\ \geqq \eta(x, x^*)^T \nabla[\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*)]. \end{aligned} \tag{11}$$

By taking summation over i in (11), we obtain

$$\begin{aligned} \sum_{i=1}^{p} b_i(x, x^*)\phi[(\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - (\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*))] \\ \geqq \eta(x, x^*)^T \sum_{i=1}^{p} \nabla[\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*)]. \end{aligned} \tag{12}$$

Now (10) and (12) yield

$$\sum_{i=1}^{p} b_i(x, x^*)\phi[(\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - (\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*))] \geqq 0.$$

From (iii) and the linearity of ϕ we get

$$\sum_{i=1}^{p} b_i(x, x^*)[(\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - (\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*))] \geqq 0. \tag{13}$$

Now multiplying both sides of (1) by $\lambda_i^* \geq 0$, we obtain

$$\lambda_i^* f_i(x) \leqq \lambda_i^* q, \quad i = 1,\ 2, \ldots,\ p. \tag{14}$$

Multiplying both sides of (2) by y_{ij}^* and summing over $j = 1, 2, \dots, m$, we obtain

$$\sum_{j=1}^{m} y_{ij}^* h_{ij}(x) \leqq 0, \quad i = 1, 2, \dots, p. \tag{15}$$

Adding (14) and (15) we get

$$\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x) \leqq \lambda_i^* q. \tag{16}$$

Since $(x^*, q^*, \lambda^*, Y^*)$satisfies (4) and (5) we have

$$\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) = \lambda_i^* q^*. \tag{17}$$

Using (16) and (17) in (13) we obtain

$$(q - q^*)[\sum_{i=1}^{p} b_i(x, x^*)\lambda_i^*] \geqq 0. \tag{18}$$

Since $\lambda_i^* \geq 0, \sum_{i=1}^{p} \lambda_i^* = 1$, and $b_i(x, x^*) > 0,\ i = 1, 2, \dots, p,\quad \sum_{i=1}^{p} b_i(x, x^*)\lambda_i^* > 0$. Hence, from (18), we have $q \geq q^*$ for (x^*, q^*) and for all (E)-feasible points (x, q). Thus (x^*, q^*) is (E)-optimal. Hence, by Lemma 2.2, x^* is (P)-optimal with q^* as the optimal value of the (P)-objective. □

The previous theorem is an extension of Theorem 3.2 [1].

4. Duality theorems

We now consider the following dual (D) to (E).

$$\operatorname{Max} v$$

subject to

$$\sum_{i=1}^{p} \nabla[\lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u)] = 0 \tag{19}$$

$$\lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \geqq \lambda_i v,\ i = 1, 2, \dots, p \tag{20}$$

$$\sum_{i=1}^{p} \lambda_i = 1, \tag{21}$$

$$u \in X, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, Y \geqq 0. \tag{22}$$

We shall denote the set of (E)-feasible solutions by W and the set of (D)-feasible solutions by T.

Theorem 4.1 (Weak duality)**.** *If* $(x, q) \in W$,$(u, v, \lambda, Y) \in T, \phi$ *is a linear mapping, and* $\phi(a) \geqq 0 \Rightarrow a \geqq 0$, *then* $q \geqq v$.

Proof. If $(x, q) \in W$ and $(u, v, \lambda, Y) \in T$, we have, from (19),

$$\eta(x,u)^T \sum_{i=1}^{p} \nabla[\lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u)] = 0. \tag{23}$$

Using Theorem 2.1 (as in Theorem 3.1) and (23), we obtain for $(x, q) \in W$ and $(u, v, \lambda, Y) \in T$,

$$\sum_{i=1}^{p} b_i(x,u)\phi[(\lambda_i f_i(x) + \sum_{j=1}^{m} y_{ij} h_{ij}(x)) - (\lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u))] \geqq 0.$$

From the hypothesis on ϕ and the linearity of this function we get

$$\sum_{i=1}^{p} b_i(x,u)[(\lambda_i f_i(x) + \sum_{j=1}^{m} y_{ij} h_{ij}(x)) - (\lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u))] \geqq 0. \tag{24}$$

Now, using the constraints (1) and (2) of (E), and the constraints (20) and (22) in (24), we obtain for $(x, q) \in W$ and $(u, v, \lambda, Y) \in T$,

$$(q - v) \sum_{i=1}^{p} b_i(x,u)\lambda_i \geqq 0.$$

Since $\sum_{i=1}^{p} b_i(x,u)\lambda_i > 0$ it follows that $q \geq v$ for$(x, q) \in W$ and $(u, v, \lambda, Y) \in T$. □

The previous theorem is an extension of Theorem 4.1 [1].

Corollary 4.1. *For $(x^*, q^*) \in W$ and $(u^*, v^*, \lambda^*, Y^*) \in T$ let $q^* = v^*$. Then (x^*, q^*) is (E)-optimal and $(u^*, v^*, \lambda^*, Y^*)$ is (D)-optimal.*

Theorem 4.2 (Direct duality)**.** *Let $(x^*, q^*) \in W$, at which a constraint qualification holds, be (E)-optimal. Then there exist $\lambda^* \in R^p$ and $Y^* \in R^{p\times m}$ such that $(x^*, q^*, \lambda^*, Y^*) \in T$, the (D)-objective value is equal to the (E)-objective value at $(x^*, q^*, \lambda^*, Y^*)$, and $(x^*, q^*, \lambda^*, Y^*)$ is (D)-optimal.*

Proof. Since (x^*, q^*) is (E)-optimal, therefore there exist $\lambda^* \in R^p$ and $Y^* \in R^{p\times m}$ such that conditions (3)-(9) are satisfied at $(x^*, q^*, \lambda^*, Y^*)$. From (3)-(5), (8), and (9) it follows that $(x^*, q^*, \lambda^*, Y^*)$ is (D)-feasible. The (D)-objective value is equal to q^*, which is the same as the (E)-objective. Using Corollary 4.1, we get that $(x^*, q^*, \lambda^*, Y^*)$ is (D)-optimal. □

The following theorem is an extension of Theorem 4.3 [1].

Theorem 4.3 (Strict converse duality)**.** *Let $(x^*, q^*) \in W$, at which a constraint qualification holds. Let (x^*, q^*) be (E)-optimal, and let $(u^*, v^*, \lambda^*, Y^*) \in T$ be (D)-optimal. Assume that ϕ is a linear mapping, and for $i = 1, 2, \ldots, p$, and for every (E)-feasible solution, at least one of the f_i, for which the corresponding $\lambda_i > 0$, is properly strictly (b_i, η, ϕ)-univex, or at least one of the h_{ij}, for which the corresponding $y_{ij} > 0$, is properly strictly (b_i, η, ϕ)-univex, then $(x^*, q^*) = (u^*, v^*)$.*

Proof. We assume that $(x^*, q^*) \neq (u^*, v^*)$ and exhibit a contradiction. Since (x^*, q^*) is (E)-optimal, there exist $\lambda^0 \in R^p$ and $Y^0 \in R^{p\times m}$ such that $(x^*, q^*, \lambda^0, Y^0) \in T$ and is (D)-optimal. Also $(u^*, v^*, \lambda^*, Y^*) \in T$ is (D)-optimal, therefore,

$$q^* = v^*. \tag{25}$$

$$\eta(x^*, u^*)^T \sum_{i=1}^{p} \nabla[\lambda_i^* f_i(u^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(u^*)] = 0. \tag{26}$$

Similar to steps followed in the proof of Theorem 3.1, from (26) and Theorem 2.1, we get

$$\begin{aligned}\sum_{i=1}^{p} b_i(x^*, u^*)[(\lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*)) \\ -(\lambda_i^* f_i(u^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(u^*))] > 0.\end{aligned} \tag{27}$$

As in Theorem 3.1, (27) yields $(q^* - v^*) \sum_{i=1}^{p} b_i(x^*, u^*)\lambda_i^* > 0$. Since $\sum_{i=1}^{p} b_i(x^*, u^*)\lambda_i^* > 0$ then $q^* > v^*$, which contradicts (25). Therefore $(x^*, q^*) = (u^*, v^*)$. □

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Manuscript received June 26, 2002
revised March 9, 2006

Shashi K. Mishra
Department of Mathematics, Statistics and Computer Science, College of Basic Sciences and Humanities, G. B. Pant University of Agriculture and Technology, Pantnagar-26145, India

Norma G. Rueda
Department of Mathematics, Merrimack College, North Andover MA 01845, U. S. A.