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GENERIC CONVERGENCE OF A CONVEX LYAPOUNOV FUNCTION ALONG TRAJECTORIES OF NONEXPANSIVE SEMIGROUPS IN HILBERT SPACE

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Abstract. We show that while a convex Lyapounov function for a semigroup of contractions on a Hilbert space may not converge to its minimum along the trajectories of the semigroup, it converges generically along the trajectories of the semigroups generated by a class of bounded perturbations of the semigroup generator.

1. Introduction

Let K be a closed convex subset of a real Hilbert space H and let $\{S(t)\}_{t\geq 0}$ be a semigroup of contractions on K generated by a maximal monotone operator A on H. The study of convergence of a trajectory $S(t)x$ as $t \to \infty$ had the attention of several mathematicians in the past. See, for example, [3], [4], [5], [6], [7], [8], [12], [13] and [14]. In general $S(t)x$ does not converge strongly or even weakly as $t \to \infty$, and convergence requires additional conditions. Dafermos and Slemrod [8] obtained strong convergence by assuming that the ω-limit sets are nonempty. Brezis in [5] and [6] studied strong convergence when F, the set of fixed points of $\{S(t)\}_{t\geq 0}$ has a non empty interior. Bruck [7] and Brezis [6] obtained strong convergence for the special case $A = \partial\phi$, where ϕ is a proper l.s.c. convex function from H to $\mathbb{R} \cup \{\infty\}$, under some restrictions on ϕ. Brezis [6] proved strong convergence under the assumption that bounded subsets of level sets of ϕ are relatively compact, i.e. for every M, $\{x : \phi(x) \leq M, ||x|| \leq M\}$ is compact, which ensures the convergence for finite dimensional spaces. Also, Bruck [7] used the condition that ϕ is even to get strong convergence. Bruck [7] showed the weak convergence of $S(t)x$ under an additional condition on the generator A, which he called demipositivity. Also, for the special case $A = \partial\phi$, he showed that $S(t)x$ converges weakly as $t \to \infty$ to a minimizer of ϕ. Baillon [3] constructed a proper l.s.c. convex function ϕ such that the semigroup generated by $\partial\phi$ converges weakly to the minimizer of ϕ but not strongly, demonstrating that weak convergence is the best possibility.

The fact that $S(t)x$ does not converge even weakly, in general, as $t \to \infty$, together with the use of Lyapounov functions to determine the asymptotic behavior of semigroups, inspired us to consider a l.s.c. Lyapounov function $f : H \to \mathbb{R} \cup \{\infty\}$ and then study convergence of $f(S(t)x)$ as $t \to \infty$. Since f, being Lyapounov, is decreasing along the trajectories $S(t)x$, a natural question is whether

$$f(S(t)x) \to \min(f) \quad \text{as} \quad t \to \infty. \tag{1}$$

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In finite dimensional space even without the nonexpansivity of the semigroup and convexity of the Lyapounov function, if a strictly Lyapounov function f exists in the neighbourhood N of an equilibrium point x_0 of the semigroup, meaning f is continuously differentiable on $\overline{N}$, $f(x) > f(x_0)$ for all $x \in \overline{N} \setminus \{x_0\}$ and f strictly decreases on the set $N \setminus \{x_0\}$, then the trajectories converge to its equilibrium point and the Lyapounov function converges along the trajectories to its minimum [9]. We will have a different definition of Lyapounov function and show that this is not the case, in general, in infinite dimensional spaces. We will show in Proposition 1 that, in general, $f(S(t)x) \not\to \min(f)$ as $t \to \infty$, when there is a unique equilibrium point, even if f strictly decreases along the trajectories of $\{S(t)\}_{t\geq 0}$, by assuming $\{S(t)\}_{t\geq 0}$ to be the semigroup given by Baillon[3] and assuming $f : H \to \mathbb{R}$ to be given by $f(x) = \frac{||x||^2}{2}$ $\forall\, x \in H$. So the next question is under what condition equation (1) holds. We note that if f decreases along the trajectory at a particular rate then convergence occurs. More precisely, if for a semigroup $\{S(t)\}_{t\geq 0}$, f decreases along the trajectories at a particular rate we call f regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$. In Theorem 1 we will show equation (1) holds if f is regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$.

Even if f is not converging along $S(t)x$ in the sense of equation (1), can we find a semigroup close to $\{S(t)\}_{t\geq 0}$ and having the convergence property (1)? The affirmative answer to this question further poses the question of how large is the collection of semigroups such that f is decreasing along the trajectories, and having the convergence property (1). The answer to this question is the main concern of this paper. So instead of considering the convergence of $f(S(t))x$ for a single semigroup, we investigate the convergence of $f(S^1(t)x)$, where $\{S^1(t)\}_{t\geq 0}$ are the semigroups generated by a class of bounded perturbations of the generator A of the given semigroup $\{S(t)\}_{t\geq 0}$. In fact, under some mild conditions on A and f, we construct a complete metric space $(\mathcal{A}^1, d)$ of the bounded perturbations of the generator A such that f is Lyapounov for all the semigroups generated by these perturbations. We show that there is a very large subset $\mathcal{F}^1$ of $\mathcal{A}^1$ such that f is regularly Lyapounov for all the semigroups generated by the maximal monotone operators in $\mathcal{F}^1$. When we say a set is very large we mean the complement of it in the complete metric space is very small, more precisely, a σ-porous subset of the metric space. We say a property is a generic property of a complete metric space or a Baire space if the subset for which the property is not true is of first category. In particular, the convex Lyapounov function f for $\{S(t)\}_{t\geq 0}$, generated by A, converges generically along the trajectories of a class of semigroups generated by bounded perturbations of A.

The generic approach, when a property is investigated for the whole space instead of an element of the space, already has many successful applications see, for example, [10]. Reich and Zaslavski [17] investigated the convergence of a continuous convex bounded below function f in the more general setting of a Banach space, along the trajectories given by vector fields on the Banach space, and obtained positive results by restricting f under some conditions. For more recent results on continuous descent methods see [1] and [2]. We study the convergence of a convex l.s.c. Lyapounov function f in Hilbert space, along the trajectories of the semigroups generated by a class of bounded perturbations of a multivalued maximal monotone operator.

2. Preliminaries and Notation

Throughout this paper H stands for a real Hilbert space. Let A be a maximal monotone operator on H such that $A^{-1}\{0\} \neq \emptyset$ and let $\{S(t)\}_{t\geq 0}$ be the semigroup of contractions generated by A on $K = \overline{D(A)}$. Usually $-A$ is called the generator of $\{S(t)\}_{t\geq 0}$ but we find it more convenient to say A is the generator of $\{S(t)\}_{t\geq 0}$ in the same sense as Pazy [11].

Since in a Hilbert space there is one to one correspondence between the maximal monotone operators and the semigroups of contractions [11] we will switch frequently between semigroups and the maximal monotone operators generating them.

Definition 1. Let f be a proper l.s.c. function from H to $\mathbb{R} \cup \{\infty\}$ such that $K \subseteq Dom(f)$ and suppose there exists $x_0 \in A^{-1}\{0\}$ such that $f(x_0) = \min(f) := \min\{f(x) : x \in H\}$. We say f is Lyapounov for $\{S(t)\}_{t\geq 0}$ if

$$f(S(t)x) \leq f(x) \quad \forall\, x \in K, \quad t \geq 0,$$

and strictly Lyapounov if

$$f(S(t)x) < f(x) \quad \forall\, x \in D(A) \setminus A^{-1}\{0\}, \quad t > 0.$$

Note that the function $f(x) = \|x - x_0\|^2$ is a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by a maximal monotone operator A if $x_0 \in A^{-1}\{0\}$.

Let us recall the notion of porosity before proceeding further. Let (X, d) be a complete metric space and $B(x, r)$ the closed ball centered at $x \in X$ and of radius $r > 0$. We say a subset $E \subseteq X$ is porous in (X, d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and for each $x \in X$, there exists $y \in X$ satisfying

$$B(y, \alpha r) \subseteq B(x, r) \setminus E.$$

A subset of the space X is called σ-porous in (X, d) if it is a countable union of porous subsets in (X, d). Several other notions of porosity are available in the literature but we use the strong notion used by Reich and Zaslavski [17]. Also, in the definition of porosity the point x can be assumed to be in E. Since porous sets are nowhere dense sets, all σ-porous sets are of the first category.

3. Convergence theorem

We begin this section with a counterexample showing that even though a continuous convex function f is strictly decreasing along the trajectories of $\{S(t)\}_{t\geq 0}$, for some $a_1 \in H$, $f(S(t)a_1)$ does not converge to $\min(f)$.

Example 1. Let $H = \mathbb{R}$, define $A : \mathbb{R} \to \mathbb{R}$ as

$$Ax = \begin{cases} x - 1 & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad |x| \leq 1 \\ x + 1 & \text{if} \quad x \leq -1. \end{cases}$$

Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^2$. Clearly A is a maximal monotone operator and $A^{-1}\{0\} = \{x : |x| \le 1\}$. The function f is continuous convex function and $f(0) = 0 = \min(f), 0 \in A^{-1}\{0\}$ and f is strictly Lyapounov for the semigroup $\{S(t)\}_{t\ge 0}$ generated by A. For all $a_1 > 1$, $S(t)a_1 \to 1$ so $f(S(t)a_1) \to 1$, not $\min(f)$, as $t \to \infty$.

For this A, and a different continuous convex f, we would hope to get $f(S(t)a_1) \to \min(f)$ if f is minimised at all points of $A^{-1}\{0\}$, i.e. $A^{-1}\{0\} \subseteq \partial f^{-1}\{0\}$. Because if $S(t)a_1 \to x_1 \in A^{-1}\{0\}$ then $f(x_1) = \min(f)$ and by continuity $f(S(t)a_1) \to f(x_1)$. The next proposition shows that even though $A^{-1}\{0\} \subseteq \partial f^{-1}\{0\}$ and f is strictly decreasing along the trajectories of $\{S(t)\}_{t\ge 0}$ and $A^{-1}\{0\} = \{x_0\}$, for some $a_1 \in H$, $f(S(t)a_1)$ does not converge to $\min(f)$.

Proposition 1. *There exists a real valued convex continuous function f on H such that $\min(f) = f(x_0)$, $x_0 \in H$, $f(x) > f(x_0) \,\forall\, x \ne x_0$, and a semigroup $\{S(t)\}_{t\ge 0}$ of contractions on a closed convex subset C of H containing x_0 such that $\{x \in C : S(t)x = x \;\forall\, t \ge 0\} = \{x_0\}$ and f is strictly Lyapounov for $\{S(t)\}_{t\ge 0}$, and $a_1 \in C$ for which $f(S(t)a_1) \not\to \min(f)$ as $t \to \infty$.*

Proof. Let $H = l^2$ and let $f : l^2 \to \mathbb{R}$ be given by $f(x) = \frac{||x||^2}{2} \,\forall\, x \in l^2$. Let $\{S_\alpha(t)\}_{t\ge 0}$ be the semigroup on the positive cone $C = \{x \in l^2 : x = (x_i)_{i\ge 1},\ \text{all } x_i \ge 0\}$ generated by $\partial\phi_\alpha$ as given by Baillon [3]. To recall the definition of $\{S_\alpha(t)\}_{t\ge 0}$ and ϕ_α let us recall a few definitions and results from [3].

The function $f_\lambda : \mathbb{R}^2 \to \mathbb{R}$, for $\lambda > 0$, is defined as

$$f_\lambda(x, y) = \begin{cases} [\arctan(\frac{x}{y})]^\lambda (x^2 + y^2)^{\frac{1}{2}} & \text{if } x \ge 0, y \ge 0, \\ \infty & \text{otherwise.} \end{cases}$$

In polar coordinates

$$f_\lambda(x, y) = \begin{cases} \left(\frac{\pi}{2} - \theta\right)^\lambda \rho & \text{if } \begin{cases} x = \rho\cos\theta \ge 0,\ y = \rho\sin\theta \ge 0 \\ \text{i.e.: } \rho \ge 0 \text{ and } 0 \le \theta \le \pi/2 \end{cases} \\ \infty & \text{otherwise.} \end{cases}$$

For $\lambda \ge 1$, f_λ is a convex l.s.c. function with subdifferential ∂f_λ. For $\alpha = (\alpha_i)_{i\ge 1}$, all $\alpha_i > 0$, and $\lambda = (\lambda_i)_{i\ge 1}$, all $\lambda_i \ge 1$, $\phi_\alpha : l^2 \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\phi_\alpha(x) = \alpha_1 f_{\lambda_1}(x_1, x_2) + \cdots + \alpha_n f_{\lambda_n}(x_n, x_{n+1}) + \cdots .$$

For $a_1 = (1, 0, 0, \ldots)$, and $\lambda_i = \frac{\pi^2}{8}\frac{b}{b-1}b^i$, with $b > 1$, Baillon [3] has chosen α such that

$$\lim_{t\to\infty} ||S_\alpha(t)a_1|| > \frac{1}{6}. \tag{2}$$

Also, 0 is the only fixed point of $\{S_\alpha(t)\}_{t\ge 0}$ as $\phi_\alpha(x) \ge 0 \;\forall\, x \in l^2$, and $\phi_\alpha(x) = 0 \Rightarrow f_{\lambda_i}(x_i, x_{i+1}) = 0 \;\forall\, i \Rightarrow x_i = 0 \;\forall\, i$. To see that f is strictly Lyapounov for $\{S_\alpha(t)\}_{t\ge 0}$, we note that for all $x \in D(\partial\phi_\alpha) \setminus \{0\}$

$$\frac{d^+}{dt} f(S_\alpha(t)x)\Big]_{t=0} = \langle \nabla f(x), -\partial\phi^o_\alpha(x)\rangle = \langle x, -\partial\phi^o_\alpha(x)\rangle \le -\phi_\alpha(x) < 0,$$

where $\partial\phi^o_\alpha(x)$ denotes the element of minimal norm in $\partial\phi_\alpha(x)$. Hence $f(S_\alpha(t)x)$ is a strictly decreasing function of t. Also by (2), for $a_1 \in C$

$$f(S_\alpha(t)a_1) = \frac{||S_\alpha(t)a_1||^2}{2} \not\to 0 = \min(f) \quad \text{as} \quad t \to \infty. \qquad \square$$

The idea of regularity that we use was essentially already given in [17], page 4, and it had previously been given in [16], page 1005, in the study of discrete descent methods.

Definition 2. A Lyapounov function f for a semigroup $\{S(t)\}_{t\geq 0}$ is called regularly Lyapounov for $\{S(t)\}_{t\geq 0}$ if for each positive integer n there exists a positive number $\delta(n)$ (depending on n) and for every x in D_n, where $D_n = \{x \in D(A) : ||x|| \leq n,\ f(x) \geq \min(f) + \frac{1}{n}\}$, there exists $\alpha(x) > 0$ such that

$$f(x) - f(S(t)x) \geq t\delta(n) \quad \forall\, t \in [0, \alpha(x)).$$

Note that a regularly Lyapounov function f for a semigroup $\{S(t)\}_{t\geq 0}$ is also strictly Lyapounov for $\{S(t)\}_{t\geq 0}$ if $f(x) > \min(f)\, \forall\, x \in D(A) \setminus A^{-1}\{0\}$.

Theorem 1. *Let f be a regularly Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$. Then for every $x \in D(A)$*

$$\lim_{t\to\infty} f(S(t)x) = \min(f).$$

Proof. Let if possible, for some $x \in D(A)$,

$$\lim_{t\to\infty} f(S(t)x) \neq \min(f).$$

Then there exists some positive integer N such that

$$f(S(t)x) \geq \min(f) + \frac{1}{N}, \quad \forall\, t \geq 0.$$

Since all the trajectories of the semigroup $\{S(t)\}_{t\geq 0}$ are bounded, $\|S(t)x\|$ is bounded, say by M. Let $m = \max(N, M)$ then

$$S(t)x \in D_m \quad \forall\, t \geq 0.$$

Since f is regularly Lyapounov for $\{S(t)\}_{t\geq 0}$ there exists a positive number $\delta(m)$ such that for each $x \in D_m$ there exists $\alpha > 0$ such that

$$f(x) - f(S(t)x) \geq t\delta(m), \quad \forall\, t \in [0, \alpha].$$

Let $V = \{T : f(x) - f(S(\tau)x) \geq \tau\delta(m) \quad \forall\, \tau \in [0, T]\}$. Then V is a nonempty subinterval of $[0, \infty)$. We claim V is open and closed in $[0, \infty)$. To see V is open in $[0, \infty)$ let $T \in V$ be given. Since $S(T)x \in D_m$ there exists $\alpha' > 0$ such that

$$f(S(T)x) - f(S(t)S(T)x) \geq t\delta(m) \quad \forall\, t \in [0, \alpha']. \tag{3}$$

Also $T \in V$ implies

$$f(x) - f(S(T)x) \geq T\delta(m). \tag{4}$$

Adding the inequalities (3) and (4) we get

$$f(x) - f(S(t+T)x) \geq (t+T)\delta(m) \quad \forall\, t \in [0, \alpha']. \tag{5}$$

Thus $[0, T+\alpha'] \subseteq V$, and V is open.
To see V is closed in $[0,\infty)$, let $\langle t_n \rangle_{n=1}^{\infty} \in V$ and $t_n \nearrow t$ as $n \to \infty$, $t > 0$. Then for every n we have

$$\begin{aligned} f(x) - f(S(t)x) &\geq f(x) - f(S(t_n)x) \ \ (\text{as } t \to f(S(t)x) \text{ is decreasing}) \\ &\geq t_n \delta(m) \ \ (\text{as } t_n \in V), \text{ giving} \\ f(x) - f(S(t)x) &\geq t\delta(m). \end{aligned}$$

Hence $t \in V$. Now V is a nonempty open and closed subinterval of $[0,\infty)$, and therefore $V = [0,\infty)$. Hence for every $t \in [0,\infty)$, $f(x) - f(S(t)x) \geq t\delta(m)$. Therefore by taking the limit as $t \to \infty$, we get $\lim_{t\to\infty} f(S(t)x) = -\infty$, contradicting the fact that f is bounded below. □

Remark 1. If f is uniformly continuous on bounded subsets of K then in Theorem 1, $f(S(t)x) \to \min(f)$ as $t \to \infty\ \forall\, x \in K$.

Remark 2. It is interesting to note that if we assume f to be bounded below and $A^{-1}\{0\} \neq \emptyset$ instead of assuming f to be minimized at $x_0 \in A^{-1}\{0\}$ in the definition of Lyapounov function (Definition 1), and by replacing $\min(f)$ by $\inf(f)$ in Definition 2, then in Theorem 1 for every $x \in D(A)$

$$\lim_{t\to\infty} f(S(t)x) = \inf(f),$$

and this further gives us $f(x_0) = \inf(f)\,\forall\, x_0 \in A^{-1}\{0\}$.

Remark 3. We note that the conclusion of Theorem 1 holds true for a general ω-semigroup (in the sense of Pazy[11]) if all the trajectories are bounded. The trajectories will be bounded if level sets of f $\{x : f(x) \leq c\}$ are bounded.

Proposition 2. *Let f be a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A on $K = \overline{D(A)}$ and suppose $\frac{d^+}{dt} f(S(t)x)\Big]_{t=0}$ exists for every $x \in D(A)$. Then f is regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$ iff for each positive integer n there exists $h_n > 0$ such that*

$$\frac{d^+}{dt} f(S(t)x)\Big]_{t=0} \leq -h_n, \quad \forall\, x \in D_n.$$

Proof. $\Longrightarrow$ Let f be regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$. Let n be a positive integer and $x \in D_n$. Then by Definition 2 there exist positive numbers $\delta(n)$ and $\alpha(x)$ such that

$$f(x) - f(S(t)x) \geq t\delta(n) \quad \forall\, t \in [0, \alpha(x)).$$

Therefore

$$\lim_{t\to 0+} \frac{f(x) - f(S(t)x)}{t} = -\frac{d^+}{dt} f(S(t)x)\Big]_{t=0} \geq \delta(n).$$

$\Longleftarrow$ Let n be a positive number and $x \in D_n$ be given. Then by the given assumption there exists $h_n > 0$ such that $\frac{d^+}{dt} f(S(t)x)\Big]_{t=0} \leq -h_n$. Let $\delta' = \frac{h_n}{2}$. Then there exist $\alpha'(x) > 0$ such that

$$\left| \frac{d^+}{dt} f(S(t)x)\Big]_{t=0} - \frac{f(S(t)x) - f(x)}{t} \right| < \delta' \quad \forall\, t \in [0, \alpha'(x)).$$

Therefore for all $t \in [0, \alpha'(x))$,

$$\frac{f(S(t)x) - f(x)}{t} < \frac{d^+}{dt} f(S(t)x)\Big]_{t=0} + \delta' \leq -h_n + \delta' = -\frac{h_n}{2}.$$

Thus we have positive numbers $\delta(n) = \frac{h_n}{2}$ and $\alpha'(x)$ such that for all $x \in D_n$

$$\frac{f(S(t)x) - f(x)}{t} < -\delta(n) \quad \forall\, t \in [0, \alpha'(x)).$$ □

Corollary 1. *Let f be a proper convex l.s.c. function on H which attains its minimum. We take the maximal monotone operator A to be ∂f, and take $\{S(t)\}_{t\geq 0}$ to be the semigroup generated by ∂f on $\overline{D(\partial f)} = K$. Then f is regularly Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$, and $\forall\, x \in K$*

$$f(S(t)x) \to \min(f) \quad \textit{as} \quad t \to \infty.$$

Proof. It is already shown in [5], Theorem 3.2 that

$$\frac{d^+}{dt} f(S(t)x) = -\|\frac{d^+}{dt} S(t)x\|^2 \leq 0 \quad \forall\, t > 0, \quad \forall\, x \in K.$$

Also for $x \in D(\partial f)$,

$$\frac{d^+}{dt} f(S(t)x)\Big]_{t=0} = -\left\| \left[\frac{d^+}{dt} S(t)x\right]_{t=0} \right\|^2 = -\|\partial f^o(x)\|^2. \tag{6}$$

Let $x_0 \in \{x \in K : f(x) = \min(f)\}$. Then $\forall\, x \in D_n$ we have

$$\begin{aligned} \frac{1}{n} < f(x) - f(x_0) &\leq \langle x - x_0, \partial f^o(x)\rangle \\ &\leq \|x - x_0\| \|\partial f^o(x)\| \\ &\leq (n + \|x_0\|)\|\partial f^o(x)\|. \end{aligned}$$

Thus

$$\|\partial f^o(x)\| \geq \frac{1}{n(n + \|x_0\|)}. \tag{7}$$

Hence by (6) and (7),

$$\left[\frac{d^+}{dt} f(S(t)x)\right]_{t=0} \leq -\left(\frac{1}{n(n + \|x_0\|)}\right)^2 = -h_n \text{ (say)} \quad \forall\, x \in D_n.$$

Therefore by Proposition 2 and Theorem 1,

$$f(S(t)x) \to \min(f) \quad \text{as} \quad t \to \infty \quad \forall\, x \in D(\partial f).$$

Also by [5] Theorem 3.2, $S(t)x \in D(\partial f)\ \forall\, x \in K$ and $t > 0$. Hence for all $x \in K$,

$$f(S(t)x) \to \min(f) \quad \text{as} \quad t \to \infty.$$ □

Remark 4. For $S_\alpha(t)$ and ϕ_α as given by Baillon [3] we have

(1) $S_\alpha(t)x \rightharpoonup 0 =$ point of minimum of $\phi_\alpha \quad \forall\, x \in C$.
(2) $S_\alpha(t)x \not\to 0$ for $x = a_1 \in C$.
(3) $\phi_\alpha(S_\alpha(t)x) \to \phi_\alpha(0) = 0 \quad \forall\, x \in C$.

4. Generic convergence

In this section we investigate the generic convergence of the convex Lyapounov function f along the trajectories of a set of semigroups. This set consists of the semigroups generated by a class of perturbations of the maximal monotone operator A. Instead of working with these semigroups, we will be dealing more with the maximal monotone operators generating them.

Under the mild assumption that $f(P_K x) \leq f(x)$ for all $x \in H$, the following proposition gives many conditions equivalent to f being Lyapounov for $\{S(t)\}_{t\geq 0}$. Brezis has considered these conditions in [5], Theorem 4.4 and [5], Proposition 4.6.

Proposition 3. *Let A be a maximal monotone operator, and $\{S(t)\}_{t\geq 0}$ the semigroup generated by A on $K = \overline{D(A)}$. Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a convex l.s.c. function and $K \subseteq Int\, D(f)$. Consider:*

(1) *$f(S(t)x) \leq f(x)$ for all $x \in K, t \geq 0$.*
(2) *For all $x \in D(A)$ and all $z \in \partial f(x)$, $\langle z, A^\circ x\rangle \geq 0$.*
(3) *For all $x \in D(A)$ there exists $z \in \partial f(x)$ such that $\langle z, A^\circ x\rangle \geq 0$.*
(4) *For all $x \in D(A)$ and all $y \in Ax$, there exists $z \in \partial f(x)$ such that $\langle z, y\rangle \geq 0$.*

Then $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ and $4 \Rightarrow 3$. Suppose $f(P_K x) \leq f(x)$ for all $x \in H$, where P_K is the nearest point projection on K, then $1 \Rightarrow 4$.

Proof. We only show $1 \Rightarrow 2$ and refer to [5] for the rest of the proof. Let $x \in D(A)$ and $z \in \partial f(x)$, then for all $y \in H$

$$f(y) - f(x) \geq \langle z, y - x\rangle.$$

In particular for all $t \geq 0$

$$f(S(t)x) - f(x) \geq \langle z, S(t)x - x\rangle,$$

which with condition 1 implies

$$\langle z, S(t)x - x\rangle \leq 0.$$

Dividing by $t > 0$ and letting t go to 0 we obtain $\langle z, -A^\circ x\rangle \leq 0$. □

Proposition 4. *Let A be a maximal monotone operator, and $\{S(t)\}_{t\geq 0}$ the semigroup generated by A on $K = \overline{D(A)}$. Let $f : H \to \mathbb{R} \cup \{\infty\}$ be a convex l.s.c. function, minimized at a point x_0, and $K \subseteq Int\, D(f)$. Then the following are equivalent:*

(1) *For each positive integer n, there exists $\delta(n) > 0$ such that for every x in D_n, there exists $\alpha(x) > 0$ such that*
$$f(S(t)x) - f(x) \leq -t\delta(n) \quad \forall\, t \in [0, \alpha(x)).$$
(2) *For each positive integer n, there exists $h_n > 0$ such that for all $x \in D_n$ and all $z \in \partial f(x)$, $\langle z, A^\circ x\rangle \geq h_n$.*
(3) *For each positive integer n, there exists $h_n > 0$ such that for each $x \in D_n$ there exists $z \in \partial f(x)$ such that $\langle z, A^\circ x\rangle \geq h_n$.*
(4) *For each positive integer n, there exists $h_n > 0$ such that for all $x \in D_n$*
$$f'(x, A^\circ x) = \max_{z\in\partial f(x)} \langle z, A^\circ x\rangle \geq h_n,$$

where $f'(x, A^\circ x)$ *denotes the right hand directional derivative of* f *at* x *in the direction of* $A^\circ x$.

Proof. To see $1 \Rightarrow 2$, let 1 hold. Let n be a positive integer, and take $\delta(n)$ to satisfy 1. Let $x \in D_n$. By 1, we take $\alpha(x) > 0$ such that

$$f(S(t)x) - f(x) \leq -t\delta(n) \quad \forall\, t \in [0, \alpha(x)). \tag{8}$$

Let $z \in \partial f(x)$ i.e.

$$f(y) - f(x) \geq \langle z, y - x\rangle \quad \forall\, y \in H.$$

In particular for all $t \geq 0$,

$$f(S(t)x) - f(x) \geq \langle z, S(t)x - x\rangle.$$

Dividing by $t > 0$ and using (8) gives for all $t \in (0, \alpha(x))$,

$$-\delta(n) \geq \langle z, \frac{S(t)x - x}{t}\rangle.$$

Taking the limit $t \to 0^+$ gives

$$-\delta(n) \geq \langle z, -A^\circ x\rangle.$$

Obviously, $2 \Rightarrow 3 \Rightarrow 4$. To see $4 \Rightarrow 1$, we follow the proof that (iii) $\Rightarrow$ (i) in Proposition 4.6 of [5]. Let 4 hold. Let n be a given positive integer. By 4, take $h_n > 0$ such that

$$f'(x, A^\circ x) \geq h_n \quad \forall x \in D_{2n}. \tag{9}$$

Let $x \in D_n$. Since f is convex and continuous at x, it is Lipschitz in a neighbourhood of x. Thus, $t \mapsto f(S(t)x)$ is Lipschitz on $[0, \alpha_1(x)]$, $\alpha_1(x) > 0$, and there exists $L > 1$ such that

$$|f(S(t_1)x) - f(S(t_2)x)| \leq L\|S(t_1)x - S(t_2)x\| \quad \forall\, t_1, t_2 \in [0, \alpha_1(x)]. \tag{10}$$

Choose

$$\alpha(x) = \min(\alpha_1(x), \frac{1}{2nL(\|A^\circ x\| + 1)}). \tag{11}$$

Then we claim $S(t)x \in D_{2n}$, $\forall t \in [0, \alpha(x)]$. Let $t \in [0, \alpha(x)]$. Note that

$$\begin{aligned} \|S(t)x\| &\leq \|S(t)x - x\| + \|x\| \\ &\leq t\|A^\circ x\| + \|x\| \\ &\leq \frac{\|A^\circ x\|}{2nL(\|A^\circ x\| + 1)} + n \quad \text{(using (11) and } x \in D_n) \\ &\leq 1 + n \\ &\leq 2n. \end{aligned} \tag{12}$$

Also

$$\begin{aligned} |f(S(t)x) - f(x)| &\leq L\|S(t)x - x\| \quad \text{(using (10))} \\ &\leq Lt\|A^\circ x\| \\ &\leq L\frac{1}{2nL(\|A^\circ x\| + 1)}\|A^\circ x\| \quad \text{(using (11))} \end{aligned} \tag{13}$$

$$\leq \frac{1}{2n}.$$

Therefore,

$$f(S(t)x) - \min(f) = f(S(t)x) - f(x) + f(x) - \min(f) \tag{14}$$
$$\geq \frac{-1}{2n} + \frac{1}{n} \quad \text{(using (13) and } x \in D_n)$$
$$= \frac{1}{2n}.$$

Since $S(t)x \in D(A)$, (12) and (14) establish our claim. Suppose $t \mapsto S(t)x$ and $t \mapsto f(S(t)x)$ differentiable at $t_0 \in (0, \alpha(x))$. For all $z \in \partial f(S(t_0)x)$, and all $v \in H$,

$$f(v) - f(S(t_0)x) \geq \langle z, v - S(t_0)x \rangle.$$

So taking $v = S(t_0 - \epsilon)x$, $\epsilon > 0$,

$$\frac{f(S(t_0 - \epsilon)x) - f(S(t_0)x)}{\epsilon} \geq \langle z, \frac{S(t_0 - \epsilon)x - S(t_0)x}{\epsilon} \rangle.$$

Let $\epsilon \searrow 0$,

$$-\frac{d}{dt} f(S(t)x)\Big]_{t_0} \geq \langle z, A^\circ S(t_0)x \rangle.$$

Thus

$$-\frac{d}{dt} f(S(t)x)\Big]_{t_0} \geq f'(x, A^\circ S(t_0)x)$$
$$\geq h_n \quad \text{(using (9) as } S(t_0)x \in D_{2n}).$$

Then integrating gives

$$f(S(t)x) - f(x) \leq -th_n \quad \forall\, t \in [0, \alpha(x)].$$

□

Note that these equivalent conditions give us the flexibility to use the definition of Lyapounov and regularly Lyapounov function for the semigroup $\{S(t)\}_{t \geq 0}$ in terms of its generator A.

5. Assumptions

The following assumptions A(4.1)–A(4.5) will be assumed when specified.

A(4.1) A is a maximal monotone operator on H.

A(4.2) f is a convex Lyapounov function for the semigroup generated by A, X_0 a nonempty closed bounded convex subset of $A^{-1}\{0\}$, $f(x_0) = \min(f)$ for all $x_0 \in X_0$, and $\overline{D(A)} = K \subseteq \text{Int } D(f)$.

A(4.3) $D(A)$ is a convex subset of H.

A(4.4) For each $x \in D(A)$, Ax does not contain a line.

A(4.5) For all $x \in H$, $f(P_K x) \leq f(x)$.

We give examples of a maximal monotone operator A, set X_0, and function f, for which either all the assumptions A(4.1)–A(4.5) hold or one of the assumptions does not hold.

Example 2. We take $H = L^2[0,1]$, where $L^2[0,1]$ denote the set of equivalence classes of all Lebesgue measurable functions $u : [0,1] \to \mathbb{R}$ satisfying

$$\int_0^1 |u(x)|^2 dx < \infty; \tag{15}$$

we identify the functions that are equal almost everywhere on $[0,1]$. Let $D(A) = \{u \in L^2[0,1] : u', u'' \in L^2[0,1], u'(0) = u'(1) = 0\}$, and $A(u) = -u''$ for $u \in D(A)$. Thus A is defined on the set of equivalence classes each containing a function $u : [0,1] \to \mathbb{R}$ which is differentiable, with derivative u' which is absolutely continuous on $[0,1]$ and satisfies $u'(0) = u'(1) = 0$, and whose derivative, which exists a.e., is in $L^2[0,1]$, not just $L^1[0,1]$. Let $f : L^2[0,1] \to \mathbb{R}$ be given by $f(u) = \|u\|^2$. Let $X_0 = \{0\}$. Then A, X_0, and f satisfy assumptions A(4.1)–A(4.5).

Proof. One checks that A is densely defined, linear (giving $A0 = 0$), and monotone (since for $u \in D(A)$, $\langle u, Au\rangle = -\langle u, u''\rangle = \langle u', u'\rangle \geq 0$). One checks $R(I + A) = H$. Therefore, A satisfies A(4.1). The function f is convex and $f(0) = 0 = \min(f)$. Also $K = \overline{D(A)} = H \subseteq \operatorname{Int} D(f) = H$ and f is a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A as for all $u \in D(A)$,

$$\frac{d^+}{dt} f(S(t)u)\bigg]_{t=0} = -\langle \nabla f(u), Au\rangle = \langle 2u, u''\rangle = -2\langle u', u'\rangle \leq 0.$$

Therefore, f and X_0 satisfy A(4.2). Since A is linear and single valued, A(4.3) and A(4.4) hold. For each $u \in H$, as $K = H$, $P_K(u) = u$, and therefore A(4.5) holds. ☐

Example 3. Let $H = \mathbb{R}$, define $A : [0,1] \to \mathbb{R}$ as

$$Ax = \begin{cases} (-\infty, 0] & \text{if } x = 0 \\ x & \text{if } 0 < x < 1 \\ [1, \infty) & \text{if } x = 1. \end{cases}$$

Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^2$. Let $X_0 = \{0\}$. Then A, X_0 and f satisfy A(4.1)–A(4.5).

Proof. Clearly A is a maximal monotone operator and satisfies A(4.1), A(4.3) and A(4.4). f is a convex function and $f(0) = 0 = \min(f)$, $0 \in A^{-1}\{0\}$. The function f is Lyapounov for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A as for each $x \in [0,1]$, noting $\{S(t)\}_{t\geq 0}$ is a semigroup of contractions, $f(S(t)x) = \|S(t)x\|^2 \leq \|x\|^2 = f(x)$. Therefore f and X_0 satisfy A(4.2). To check A(4.5), we note that

$$P_K(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Thus $f(P_K(x)) \leq f(x)$ and A(4.1)–A(4.5) all hold. ☐

In our next example we use the maximal monotone operator A, whose domain is not a convex set, as given by Simons[18].

Example 4. Let $H = \mathbb{R}^2$, define $\phi : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ as

$$\phi(x_1, x_2) = \begin{cases} |x_2| \vee (1 - \sqrt{1 - x_1^2}), & \text{if } |x_1| \vee |x_2| \leq 1 \\ \infty & \text{otherwise.} \end{cases}$$

Let $A = \partial\phi$ and define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x) = \frac{\|x\|^2}{2}$. Let $X_0 = \{0\}$. Then A, X_0 and f satisfy assumptions A(4.1)–A(4.5) except A(4.3).

Proof. Note that ϕ is a proper convex l.s.c. function on $\mathbb{R}^2$. Therefore $\partial\phi$ is a maximal monotone operator. Also $(0,0) \in A^{-1}((0,0))$. We note that $D(\phi) = \{(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1\}$, $D(\partial\phi) = D(\phi) \setminus \{(x_1, x_2) : x_1 = \pm 1, |x_2| < 1\}$, and $K = \overline{D(\partial\phi)} = D(\phi)$. Clearly A satisfies A(4.1), A(4.4) but not A(4.3). The function f is convex and $f(0) = 0 = \min(f)$. Also $K \subseteq \text{Int } D(f) = H$ and f is a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A as for each $x \in K$,

$$\left.\frac{d^+}{dt} f(S(t)x)\right]_{t=0} = \langle \nabla f(x), -A^\circ x\rangle = -\langle x, A^\circ x\rangle \leq 0.$$

Therefore, f and X_0 satisfy A(4.2). To see A(4.5), we note that, for each $x \in \mathbb{R}^2$, $||P_K(x)\| \leq \|x\|$ as $(0,0) \in K$ and P_K is a contraction. Therefore, for each $x \in \mathbb{R}^2$,

$$f(P_K(x)) = \frac{\|P_K(x)\|^2}{2} \leq \frac{\|x\|^2}{2} = f(x).$$

Therefore, A(4.5) holds. □

Example 5. Let $H = \mathbb{R}^2$, $K = \{(x, 0) : 0 \leq x \leq 1\}$. Let $A = \partial I_K$ where, we recall,

$$I_K(x, y) = \begin{cases} 0 & \text{if } (x, y) \in K \\ \infty & \text{otherwise.} \end{cases}$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x, y) = \|(x, y)\|^2.$$

Let $X_0 = \{0\}$. Then A(4.4) does not hold but all of A(4.1), A(4.2), A(4.3) and A(4.5) hold.

Proof. Note that $X_0 = \{0\}$ and A is a maximal monotone operator and $(0,0) \in A(x, y)\, \forall\, (x, y) \in K$. Clearly A satisfies A(4.1), A(4.3) but not A(4.4). f is a convex function and $f(0,0) = 0 = \min(f)$. Also $K = \overline{D(A)} \subseteq \text{Int } D(f) = H$ and f is a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A as for each $(x, y) \in K$, $f(S(t)(x, y)) = f(x, y)$. Therefore, f and X_0 satisfy A(4.2). To see A(4.5), we note that, for each $(x, y) \in \mathbb{R}^2$, $||P_K(x, y)\| \leq \|(x, y)\|$ as $(0,0) \in K$ and P_K is a contraction. Therefore, for each $(x, y) \in \mathbb{R}^2$,

$$f(P_K(x, y)) = \|P_K(x, y)\|^2 \leq \|(x, y)\|^2 = f(x, y).$$

Therefore, A(4.5) holds. □

Example 6. Let $H = \mathbb{R}^2$, $K = \{(x, y) : x - 1 \leq y \leq x + 1, x \in \mathbb{R}\}$ and $A = \partial I_K$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x, y) = y^2.$$

Let X_0 be any nonempty closed bounded convex set of $\{(x, 0) : |x| \leq 1\}$. Then A(4.5) does not hold but all others, A(4.1)–A(4.4), hold.

Proof. A is a maximal monotone operator and $A(x,y) = \{(0,0)\}$ for all $(x,y) \in \operatorname{Int} K$. Clearly A satisfies A(4.1), A(4.3) and A(4.4). f is a convex function and $f(x,0) = 0 = \min(f)\,\forall x : |x| \leq 1$. Also $K = \overline{D(A)} \subseteq \operatorname{Int} D(f) = H$ and f is a Lyapounov function for the semigroup $\{S(t)\}_{t\geq 0}$ generated by A as for each $(x,y) \in K$, $f(S(t)(x,y)) = f(x,y)$. Therefore, f and X_0 satisfy A(4.2). To see A(4.5) does not hold, we note that $P_K(3,0) = (2,1)$ and $f(P_K(3,0)) = f(2,1) = 1 > f(3,0) = 0$. □

The following proposition is a simple perturbation result which will help us to define the collection of bounded perturbations of A. To prove it we use [5], Theorem 2.4, Lemma 2.5, and the argument used in [5], Proposition 2.10. Let us recall a single valued operator $A : C \to H$, C a convex subset of H is said to be hemicontinuous if for each $x, y \in C$, $A((1-t)x + ty) \rightharpoonup Ax$ (weak convergence) as $t \to 0^+$.

Proposition 5. *Let A be a maximal monotone operator satisfying A(4.3). Let $A' : D(A') \subseteq H \to H$ be such that :*

(1) *$D(A) \subseteq D(A')$,*
(2) *A' is single valued, hemicontinuous and monotone on $D(A)$, and*
(3) *A' maps bounded subsets of $D(A)$ to bounded sets.*

Then $A + A'$ is a maximal monotone operator.

Proof. Since A' is monotone on $D(A)$ there exist a maximal monotone extension $\tilde{A}'$ of $A' |_{D(A)}$ such that $D(\tilde{A}') \subseteq \overline{D(A)} = K$. To show $A + A'$ is maximal monotone we show that

(a) $A + \tilde{A}'$ is maximal monotone, and
(b) $A + \tilde{A}' = A + A'$.

To see (a), let $y \in H$ be arbitrary. We write $\tilde{A}'_\lambda = \frac{I-(I+\lambda\tilde{A}')^{-1}}{\lambda}$ for the Yosida approximation of $\tilde{A}'$. Since A and $\tilde{A}'$ are maximal monotone, for $\lambda > 0$ there is $x_\lambda \in D(A)$ such that $y \in x_\lambda + Ax_\lambda + \tilde{A}'_\lambda x_\lambda$. Since $D(A) \bigcap D(\tilde{A}') \neq \emptyset$, $\|x_\lambda\|$ is bounded by Lemma 2.5 of [5]. Therefore, by condition 3 of this Proposition, $\|A' x_\lambda\|$ is bounded as $\lambda \to 0$. Denoting the element of minimal norm in the closed convex set $\tilde{A}' x_\lambda$ by $(\tilde{A}')^o x_\lambda$ we get

$$\|\tilde{A}'_\lambda x_\lambda\| \leq \|(\tilde{A}')^o x_\lambda\| \leq \|A' x_\lambda\|.$$

Therefore $\|\tilde{A}'_\lambda x_\lambda\|$ is bounded as $\lambda \to 0$ and by [5], Theorem 2.4, $A + \tilde{A}'$ is maximal monotone. To establish (b), we use the argument used in [5], Proposition 2.10. Clearly, $A + A' \subseteq A + \tilde{A}'$. For the reverse inclusion we first show that for each $x \in D(A)$, $\tilde{A}' x \subseteq A' x + \partial I_K x$. Let $x \in D(A)$ and $z \in \tilde{A}' x$. Since $\tilde{A}'$ is monotone we have

$$\langle A' y - z, y - x \rangle \geq 0 \quad \forall\, y \in D(A).$$

In particular putting $y = y_t = (1-t)x + tu$, $t \in (0,1)$, $u \in D(A)$ in the above inequality we get

$$\langle A' y_t - z, u - x \rangle \geq 0 \quad \forall\, u \in D(A),\, t \in (0,1).$$

Taking the limit as $t \to 0$ and using the hemicontinuity of A' on $D(A)$ we get

$$\langle A' x - z, u - x \rangle \geq 0 \quad \forall\, u \in D(A),$$

which holds true $\forall\, u \in K$, implying $z \in A'x + \partial I_K x$. Thus $A + \tilde{A}' \subseteq A + (A' + \partial I_K) = A' + (A + \partial I_K)$. Since $A + \partial I_K$ is a monotone extension of A, $A + \partial I_K = A$, establishing (b). □

Definition 3. Let A, X_0 and f satisfy A(4.1), A(4.2) and A(4.3). Let $\mathcal{A}^1$ be the set of monotone operators A^1 which may be written as $A^1 = A + A'$ where $A' : D(A) \subseteq H \to H$ satisfy the following:

(1) A' is single valued, hemicontinuous and monotone on $D(A)$,
(2) A' maps bounded subsets of $D(A)$ to bounded sets,
(3) $(A + A')x_0 \ni 0$ for some x_0 (depending on A') in X_0,
(4) for each $x \in D(A)$ and $z \in \partial f(x)$, $\langle z, (A + A')^o x\rangle \geq 0$.

Definition 4. Let $\mathcal{F}^1$ be the set of monotone operators $A^1 = A + A'$ of $\mathcal{A}^1$ such that for each positive integer n there exist a positive number $h_n > 0$ such that for each x in D_n, where we recall $D_n = \{x \in D(A) : \|x\| \leq n,\ f(x) \geq \min(f) + \frac{1}{n}\}$, and all $z \in \partial f(x)$,

$$\langle z, (A + A')^o x\rangle \geq h_n.$$

Proposition 6. *If A satisfies A(4.1), A(4.3) and f satisfies A(4.2) then f is Lyapounov for each semigroup $\{S^1(t)\}_{t\geq 0}$ generated by A^1 in $\mathcal{A}^1$ and regularly Lyapounov for each semigroup $\{S^1(t)\}_{t\geq 0}$ generated by A^1 in $\mathcal{F}^1$.*

Proof. Let $A^1 = A + A' \in \mathcal{A}^1$ be given. Since A satisfies A(4.1), A(4.3) then by Proposition 5, A^1 is a maximal monotone operator. Also $A^1 x_0 = Ax_0 + A'x_0 \ni 0$ for some x_0 (depending on A') in X_0. Let $\{S^1(t)\}_{t\geq 0}$ be the semigroup generated by A^1. By Proposition 3, 4 of Definition 3 implies

$$f(S^1(t)x) \leq f(x) \quad \forall\, x \in K = \overline{D(A)}, \forall\, t \geq 0.$$

Hence f is Lyapounov for the semigroup $\{S^1(t)\}_{t\geq 0}$. Now let $A^1 = A + A' \in \mathcal{F}^1$ and n be a positive integer. Then there exists a positive number $h_n > 0$ such that for each x in D_n and $z \in \partial f(x)$,

$$\langle z, (A + A')^o x\rangle \geq h_n.$$

Hence by Proposition 4, f is regularly Lyapounov for the semigroup $\{S^1(t)\}_{t\geq 0}$. □

The next proposition is a simple but interesting geometrical result which will help us to equip $\mathcal{A}^1$ with a metric, so that we can see that the subset $\mathcal{F}^1$ of $\mathcal{A}^1$ is large.

Proposition 7. *Let C be a nonempty convex subset of a linear space X. Then there exists $0 \neq x \in X$ such that $C + x = C \Leftrightarrow C = \bigcup_{c\in C} L + c$ for a nonzero linear subspace L of X.*

Proof. $\Longrightarrow$ Let there exist a nonzero x in X such that $C + x = C$. Let $L = \{\lambda x : \lambda \in \mathbb{R}\}$. Then for each $c \in C$, $L + c$ is an affine set containing c and

$$C \subseteq \bigcup_{c\in C} L + c.$$

For the reverse inclusion we show that for each $c \in C$, $L + c \subseteq C$. For each $c \in C$ and for each integer n, using $C + x = C$, we get

$$c + nx \in C. \tag{16}$$

Let $\lambda x+c$, for a real number λ, be any arbitrary element of $L+c$. Choose a positive integer n such that $|\lambda|\leq n$ and let $\mu=\frac{|\lambda|}{n}$. Then $\mu\in[0,1]$ and $\lambda=\pm n\mu$. Then by (16) and the convexity of C we get

$$\lambda x+c=\pm n\mu x+c-\mu c+\mu c=(1-\mu)c+\mu(c\pm nx)\in C.$$

Hence,

$$C=\bigcup_{c\in C}L+c.$$

$\Longleftarrow$ Let $C=\bigcup_{c\in C}L+c$, for a non zero linear subspace L of X. Let x be a non zero element of L. We have

$$C+x=(\bigcup_{c\in C}L+c)+x=\bigcup_{c\in C}L+c=C.$$ □

Proposition 8. *Assume A(4.1) to A(4.5). Let* $d:\mathcal{A}^1\times\mathcal{A}^1\to\mathbb{R}$ *be defined as*

$$d(A^1,A^2)=\sum_{n=1}^{\infty}\frac{1}{2^n}.\frac{d_n(A^1,A^2)}{1+d_n(A^1,A^2)}$$

for $A^1=A+A'$ *and* $A^2=A+A''$ *in* $\mathcal{A}^1$, *where for each integer* n, $d_n(A^1,A^2)$ *is defined as*

$$d_n(A^1,A^2)=\sup\{\|A'x-A''x\|:\|x\|\leq n\,,\,x\in D(A)\}.$$

Then $(\mathcal{A}^1,d)$ *is a complete metric space.*

Proof. In view of Proposition 7 and A(4.4), note that if $A^1\in\mathcal{A}^1$ can be represented as $A^1=A+A'$ and $A^1=A+A''$ then $A'=A''$. It is easy to see that $(\mathcal{A}^1,d)$ is a metric space. For completeness, let $\langle A^n=A+A^{(n)}\rangle_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{A}^1,d)$, where $A^{(n)}$ satisfy 1-4 of Definition 3. Then for each $x\in D(A)$, $\langle A^{(n)}x\rangle_{n=1}^{\infty}$ is a Cauchy sequence in H and hence converges. We define $B:D(A)\to H$ as

$$Bx=\lim_{n\to\infty}A^{(n)}x.$$

In fact $A^{(n)}\to B$ uniformly on bounded subsets of $D(A)$ and $d(A+A^{(n)},A+B)\to 0$ as $n\to\infty$. We check B satisfies 1-4 of Definition 3. Note B is a single valued monotone operator on $D(A)$. To see B is hemicontinuous on $D(A)$, let $x,y\in D(A)$. Let z be an arbitrary nonzero element of H. Then $A^{(n)}\to B$ uniformly on the bounded subset $\{x+t(y-x):0\leq t<1\}$ of $D(A)$, so for $\epsilon>0$ there exists a positive integer p such that

$$\|A^{(n)}(x+t(y-x))-B(x+t(y-x))\|<\frac{\epsilon}{3\|z\|}\quad\forall\,n\geq p,\,0\leq t<1.$$

In particular,

$$\|A^{(p)}(x+t(y-x))-B(x+t(y-x))\|<\frac{\epsilon}{3\|z\|}\quad\text{for }\ 0\leq t<1. \tag{17}$$

Also $A^{(p)}$ is hemicontinuous so there exists $\delta'>0$ such that

$$|\langle A^{(p)}(x+t(y-x))-A^{(p)}x,z\rangle|<\frac{\epsilon}{3}\quad\forall\,t:0\leq t<\delta'. \tag{18}$$

Choose $\delta = \min(1, \delta')$, then by (17), (18) we have for all $t \in [0, \delta)$

$$\begin{aligned}
&|\langle B(x+t(y-x)) - Bx, z\rangle| \\
&= |\langle B(x+t(y-x)) - A^{(p)}(x+t(y-x)) \\
&\quad + A^{(p)}(x+t(y-x)) - A^{(p)}x + A^{(p)}x - Bx, z\rangle| \\
&\le \|B(x+t(y-x)) - A^{(p)}(x+t(y-x))\|\|z\| \\
&\quad + |\langle A^{(p)}(x+t(y-x)) - A^{(p)}x, z\rangle| + \|A^{(p)}x - Bx\|\|z\| \\
&< \frac{\epsilon}{3\|z\|}\|z\| + \frac{\epsilon}{3} + \frac{\epsilon}{3\|z\|}\|z\| = \epsilon.
\end{aligned}$$

Hence B is hemicontinuous. Also B is monotone, so 1 holds. We check 2. Let S be a bounded subset of $D(A)$. Then $A^{(n)} \to B$ uniformly on S and there exists a positive integer m such that

$$\|A^{(n)}x - Bx\| < 1 \quad \forall\, x \in S, \forall\, n \ge m. \tag{19}$$

Since $A^{(m)}x$ is bounded on bounded subsets of $D(A)$, there exists $M > 0$ such that

$$\|A^{(m)}x\| \le M \quad \forall\, x \in S. \tag{20}$$

By (19) and (20), B is bounded on bounded subset of $D(A)$.

We check 3. By 3 of Definition 3, for each n there exists $x_n \in X_0$ such that $A^n x_n = (A + A^{(n)})x_n \ni 0$. Since X_0 is a bounded set, $\langle x_n \rangle$ has a weakly convergent subsequence, say $\langle x_{n_k} \rangle$, converging to $x_0 \in X_0$. Since $A^n = A + A^{(n)}$ is monotone, for all $x \in D(A), y \in Ax$,

$$\langle 0 - (y + Bx), x_0 - x\rangle = \lim_{n\to\infty} \langle 0 - (y + A^{(n)}x), x_{n_k} - x\rangle \ge 0.$$

Thus $(A + B)x_0 \ni 0$ since $A + B$ is maximal monotone, by Proposition 5. For 4, let $x \in D(A)$ and let $(A + B)^o x = y^o + Bx$, for some $y^o \in Ax$. In view of Proposition 3 it suffices to show that there exists $z \in \partial f(x)$ such that $\langle z, (A + B)^o x\rangle \ge 0$. Also for each n, $A + A^{(n)} = A^n \in \mathcal{A}^1 \Rightarrow$ for all $z \in \partial f(x)$, $\langle z, (A + A^{(n)})^o x\rangle \ge 0$. This implies, by 4 of Proposition 3, for each $y^\circ + A^{(n)}x \in A^n x$ there exists $z_n \in \partial f(x)$ such that

$$\langle z_n, y^o + A^{(n)}x\rangle \ge 0.$$

Since $z_n \in \partial f(x)$ and $\partial f(x)$ is bounded, $\langle z_n \rangle$ has a weakly convergent subsequence, say $\langle z_{n_k} \rangle$, converging to $z \in \partial f(x)$. Thus

$$\langle z, (A + B)^o x\rangle = \lim_{k\to\infty} \langle z_{n_k}, y^o + A^{(n_k)}x\rangle \ge 0.$$

Hence, $A + B \in \mathcal{A}^1$ and $(\mathcal{A}^1, d)$ is a complete metric space. □

We show the significance of the condition that X_0 is bounded. Here we give an example of A, f and X_0 satisfying A(4.1) to A(4.5) except that X_0 is unbounded, and show that $(\mathcal{A}^1, d)$ is not a complete metric space. Let $H = \mathbb{R}$, define $A : \mathbb{R} \to \mathbb{R}$ as $Ax = 0$ for every $x \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be any convex continuous monotone decreasing function satisfying $f(x) = 0 \quad \forall\, x \in [0, \infty)$. We assume $X_0 = [0, \infty)$. Let $B : \mathbb{R} \to \mathbb{R}$ be a continuous monotone increasing function such that $Bx \to 0$ as $x \to \infty$. Let $A^{(n)} : \mathbb{R} \to \mathbb{R}$ be given as $A^{(n)} = B + \frac{1}{n}I$. Then $A + A^{(n)} \in \mathcal{A}^1$,

$A + A^{(n)} \to A + B$ as $n \to \infty$, but $A + B \notin \mathcal{A}^1$ as there does not exist any $x \in \mathbb{R}$ such that $(A + B)x = 0$.

Theorem 2. *Assume A(4.1) to A(4.5) and let ∂f be bounded on bounded subsets of $D(A)$. Then $\mathcal{A}^1 \setminus \mathcal{F}^1$ is a σ-porous subset of the complete metric space $(\mathcal{A}^1, d)$.*

Proof. For each positive integer n, we recall $D_n = \{x \in D(A) : \|x\| \leq n,\ f(x) \geq \min(f) + \frac{1}{n}\}$, and define

$$\Omega_n = \{A^1 = A + A' \in \mathcal{A}^1 : \inf_{\substack{x \in D_n \\ z \in \partial f(x)}} \langle z, (A^1)^o x \rangle = 0\}.$$

Then

$$\bigcup_{n=1}^{\infty} \Omega_n = \mathcal{A}^1 \setminus \mathcal{F}^1.$$

Therefore, it suffices to show that, for each positive integer n, Ω_n is a porous subset of $\mathcal{A}^1$. Let n be given. So we need to show there exists $r_o > 0$ and $\alpha \in (0, 1)$ such that for each $r \in (0, r_o]$ and for each $A^1 \in \Omega_n$, there exists $A^1_\gamma \in \mathcal{A}^1$ satisfying

$$B(A^1_\gamma, \alpha r) \subseteq B(A^1, r) \setminus \Omega_n \tag{21}$$

where $B(A^1_\gamma, \alpha r)$ and $B(A^1, r)$ are closed balls of the metric space $(\mathcal{A}^1, d)$. Since ∂f is bounded on bounded subsets of $D(A)$ choose $M_n > 1$ such that

$$\|z\| \leq M_n \quad \forall\, z \in \partial f(x),\ x \in D(A),\ \|x\| \leq n. \tag{22}$$

Let

$$\|X_0\| = \sup\{\|x\| : x \in X_0\}. \tag{23}$$

Let $S = \sum_{i=1}^{\infty} \frac{i}{2^i}$ and choose $\alpha \in (0, 1)$ such that

$$2^{n+3} \alpha M_n < \frac{(1 - \alpha)(S + \|X_0\| + 1)^{-1}}{n}. \tag{24}$$

Let $r_0 = 1$ and $r \in (0, r_0]$ be given. Let

$$\gamma = \frac{(1 - \alpha) r}{2(S + \|X_0\| + 1)}. \tag{25}$$

Clearly, $\gamma \in (0, 1)$. Let $A^1 = A + A' \in \Omega_n$ be given. By 3 of Definition 3, there exists $x_0 \in X_0$ such that

$$(A + A')x_0 \ni 0. \tag{26}$$

Define $A'_\gamma : D(A) \to H$ as

$$A'_\gamma x = A'x + \gamma(x - x_0), \text{ or } A^1_\gamma = A' + \gamma(I - x_0).$$

We define $A^1_\gamma = A + A'_\gamma$, and show that $A^1_\gamma \in \mathcal{A}^1$. Clearly, A'_γ is a single valued hemicontinuous monotone operator on $D(A)$, and is bounded on bounded subsets of $D(A)$. Also, (26) gives $A^1_\gamma x_0 = (A + A'_\gamma)x_0 = (A + A' + \gamma(I - x_0))x_0 \ni 0$. Thus A^1_γ satisfies 1-3 of Definition 3. Now we check 4 of Definition 3. Let $x \in D(A)$ and $(A^1_\gamma)^o x = y_\gamma + A'x + \gamma(x - x_0)$ for $y_\gamma \in Ax$. Since $A^1 \in \mathcal{A}^1$, $\langle z, (A^1)^o x\rangle \geq$

$0\ \forall\, x \in D(A)$ and $\forall\, z \in \partial f(x)$ which in turn implies, by Proposition 3, there exists $z \in \partial f(x)$ such that

$$\langle z, y_\gamma + A'x\rangle \geq 0.$$

Hence

$$\langle z, (A^1_\gamma)^o x\rangle = \langle z, y_\gamma + A'x + \gamma(x - x_0)\rangle \geq \langle z, \gamma(x - x_0)\rangle \geq 0.$$

Hence, by Proposition 3, 4 of Definition 3 holds and $A^1_\gamma \in \mathcal{A}^1$. To show (21), we need to show that

(A) $B(A^1_\gamma, \alpha r) \subseteq B(A^1, r)$, and
(B) $B(A^1_\gamma, \alpha r) \cap \Omega_n = \emptyset$.

To see (A), let $B^1 = A + B'$ be an arbitrary element of $B(A^1_\gamma, \alpha r)$. Let us first estimate the distance between A^1 and A^1_γ. For each i, $i \geq 1$ we have

$$\begin{aligned} d_i(A^1_\gamma, A^1) &= \sup\{\|A'_\gamma x - A'x\| : \|x\| \leq i\,,\, x \in D(A)\} \\ &= \sup\{\|\gamma(x - x_0)\| : \|x\| \leq i\,,\, x \in D(A)\} \leq \gamma(i + \|x_0\|). \end{aligned}$$

Therefore,

$$\begin{aligned} d(A^1_\gamma, A^1) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(A^1_\gamma, A^1)}{1 + d_i(A^1_\gamma, A^1)} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(A^1_\gamma, A^1) \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \gamma(i + \|x_0\|) = \gamma(S + \|x_0\|) \\ &= \frac{(1-\alpha) r (S + \|x_0\|)}{2(S + \|X_0\| + 1)} \text{ (by using (25))} \\ &< \frac{(1-\alpha) r}{2} \text{ (by using (23)).} \end{aligned}$$

So using the above estimate we get

$$d(B^1, A^1) \leq d(B^1, A^1_\gamma) + d(A^1_\gamma, A^1) < \alpha r + \frac{(1-\alpha) r}{2} \leq r,$$

which establishes (A).

To establish (B), let $B^1 = A + B'$ be an arbitrary element of $B(A^1_\gamma, \alpha r)$. Let $x \in D_n$ be given. We show there exists $z \in \partial f(x)$ such that $\langle z, (B^1)^o x\rangle \geq \frac{\gamma}{2n}$. Let $(B^1)^o x = y' + B'x$ for $y' \in Ax$. Since $y' + A'x \in A^1 x$ and $A^1 \in \mathcal{A}^1$, by 4 of Proposition 3, there exists $z \in \partial f(x)$ such that

$$\langle z, y' + A'x\rangle \geq 0. \tag{27}$$

We can write

$$\langle z, (B^1)^o x\rangle = \langle z, (B^1)^o x - (y' + A'_\gamma x)\rangle + \langle z, y' + A'_\gamma x\rangle. \tag{28}$$

Now

$$\begin{aligned} \langle z, y' + A'_\gamma x\rangle &= \langle z, y' + A'x + \gamma(x - x_0)\rangle \\ &\geq \langle z, \gamma(x - x_0)\rangle \quad \text{by (27)} \\ &\geq \gamma(f(x) - f(x_0)) \end{aligned} \tag{29}$$

$$\geq \frac{\gamma}{n}.$$

Also

$$\|(B^1)^o x - (y' + A'_\gamma x)\| = \|A'_\gamma x - B'x\| \tag{30}$$
$$\leq \sup\{\|A'_\gamma x - B'x\| : \|x\| \leq n\,,\, x \in D(A)\} = d_n(A^1_\gamma, B^1).$$

Since

$$\frac{1}{2^n}\frac{d_n(A^1_\gamma, B^1)}{1 + d_n(A^1_\gamma, B^1)} \leq d(A^1_\gamma, B^1) \leq \alpha r,$$

noting $2^n\alpha r < 1$ by (24) and (25), since $M_n > 1$, we get

$$d_n(A^1_\gamma, B^1) \leq \frac{2^n\alpha r}{1 - 2^n\alpha r}. \tag{31}$$

Combining (30) and (31) gives

$$\|(B^1)^o x - (y' + A'_\gamma x)\| \leq \frac{2^n\alpha r}{1 - 2^n\alpha r}. \tag{32}$$

Using (22) and (32) we get

$$|\langle z, (B^1)^o x - (y' + A'_\gamma x)\rangle| \leq M_n \frac{2^n\alpha r}{1 - 2^n\alpha r}. \tag{33}$$

Substituting (29) and (33) in (28) yields

$$\begin{aligned}\langle z, (B^1)^o x\rangle &\geq -M_n\frac{2^n\alpha r}{1 - 2^n\alpha r} + \frac{\gamma}{n}\\ &\geq -2M_n 2^n\alpha r + \frac{\gamma}{n} \quad (\text{as } 2^n\alpha r < \tfrac{1}{2} \text{ by (24), (25)})\\ &> -\frac{\gamma}{2n} + \frac{\gamma}{n} \quad (\text{by (24), (25)})\\ &= \frac{\gamma}{2n} > 0.\end{aligned}$$

Hence, by Proposition 4, there exists $h_n > 0$ such that

$$\langle z, (B^1)^o x\rangle \geq h_n \quad \forall\, x \in D_n, \forall\, z \in \partial f(x).$$

Hence $B^1 \notin \Omega_n$, which establishes (B). □

Remark 5. Note that, in the definition of Lyapounov function, the assumption that there exists $x_0 \in A^{-1}\{0\}$ such that $f(x_0) = \min(f)$ has two different roles to play. In Theorem 1 we use $A^{-1}\{0\} \neq \emptyset$ but f need not be minimized (see Remark 2). Theorem 2 can be proved assuming f is minimized but not necessarily in $A^{-1}\{0\}$. In fact we can assume $A^{-1}\{0\} = \emptyset$ and can redefine the class $\mathcal{A}^1$ by dropping the condition 3 in Definition 3, and then Theorem 2 holds true for this class $\mathcal{A}^1$ too. Since we wish to use Theorem 1 and Theorem 2 together, we need to assume that for each of the semigroups generated by the maximal monotone operators in the class $\mathcal{A}^1$, there is an equilibrium point x_0 such that f is minimized at this x_0.

Another class $\mathcal{A}^2$ of bounded perturbations of A can be defined by replacing 4 of Definition 3 by a condition that $\forall\, x \in D(A)$, $\forall z \in \partial f(x)$, $\langle z, A'x\rangle \geq 0$. That means $\mathcal{A}^2 = \{A + A' : A' \in \mathcal{A}''\}$ where $A' \in \mathcal{A}''$ satisfies 1-3 of Definition 3 and $\forall\, x \in D(A)$, $\forall z \in \partial f(x)$, $\langle z, A'x\rangle \geq 0$. Correspondingly we can define the subset

$\mathcal{F}^2$ of $\mathcal{A}^2$ as $\mathcal{F}^2 = \{A + A' : A' \in \mathcal{F}''\}$ where $\mathcal{F}''$ is the subset of $\mathcal{A}''$ consisting of A' such that for each positive integer n there exist a positive number $h_n > 0$ such that $\langle z, A'x \rangle \geq h_n \quad \forall\, x \in D_n$, and $\forall z \in \partial f(x)$.

Remark 6. Note that if f satisfies A(4.5) then $\mathcal{A}^2 \subseteq \mathcal{A}^1$ and the conclusions of Proposition 6 and Theorem 2 hold for $\mathcal{A}^2$ and $\mathcal{F}^2$.

Remark 7. Proposition 8 for $\mathcal{A}^2$ holds true even without assumption A(4.5). Also without A(4.5), $\mathcal{A}''$ is a convex cone and $\mathcal{A}'' \setminus \mathcal{F}''$ forms a face of the convex cone, which means if $A' \in \mathcal{A}''$, $A'' \in \mathcal{F}''$ then $\lambda A' + (1-\lambda)A'' \in \mathcal{F}''$ for all $\lambda \in [0, 1)$.

Remark 8. Under the conditions of Theorem 2, we have (1) holding for all semigroups generated by operators in a very large subset of $\mathcal{A}^1$, and for all x in K by Remark 1 after Theorem 1.

Thus, a convex continuous Lyapounov function for a semigroup converges generically along the trajectories of the semigroups generated by a class of bounded perturbations of the semigroup generator.

Remark 9. Let us assume $Ax = 0 \ \forall\, x \in H$ and denote $\mathcal{A}^1$ by $\mathcal{A}_{hm}$ and $\mathcal{F}^1$ by $\mathcal{F}_{hm}$. One can see that $\mathcal{A}_{hm}$ is the collection of single valued everywhere defined hemicontinuous monotone bounded operators such that f is Lyapounov for all the semigroups generated by $V \in \mathcal{A}_{hm}$. By Theorem 2, $\mathcal{F}_{hm}$ is a very large subset of $\mathcal{A}_{hm}$ and (1) holds for all the semigroups generated by operators in this very large subset. Thus a convex continuous Lyapounov function f for a semigroup generated by single valued everywhere defined hemicontinuous monotone bounded operator on a Hilbert space H converges generically along the trajectories of the semigroups in $\mathcal{A}_{hm}$.

Remark 10. The results of [17] are not directly comparable with the results of this paper, although there are common approaches in these two papers. We do not give the details of a comparison here. If f satisfies A(4.2) then using our notation and assumptions, the set $\mathcal{A}$ given by Reich and Zaslavski [17] is the set of all the vector fields $V : H \to H$ such that V is bounded on bounded subsets of H and $\langle z, Vx \rangle \geq 0 \quad \forall\, x \in H$ and $\forall\, z \in \partial f(x)$. One can see that the metric ρ in [17] is the same as our metric d and $\mathcal{A}_{hm}$ is a closed subset of the complete metric space $(\mathcal{A}, d)$. By Remark 9, [17] Theorem 2.2 holds true for $\mathcal{A}_{hm}$. Also $V \in \mathcal{A}_{hm}$ ensures the existence of $u : [0, T] \longrightarrow H$ such that $u'(t) = -V(u(t))$ a.e. $t \in [0, T]$. Moreover [17] Theorem 3.2 holds for $\mathcal{A}_{hm}$.

The paper [17] has a point of view in which one wishes to minimize the convex function f, and finds that one can do so by using a very large subset of the vector fields for which f is a Lyapounov function. In this paper, however, we have started with the traditional viewpoint of wishing to study stability of an equilibrium point of a semigroup $\{S(t)\}_{t \geq 0}$, using a Lyapounov function, then proceeding to consider convergence of $f(S(t)x)$ as a question of interest in its own right. However, the viewpoint of [17] is appropriate for this paper too.

Remark 11. We would like to draw attention to the monotone operator A^1_γ in the proof of Theorem 2. Note that $A^1_\gamma = A + A' + \gamma(I - x_0)$ is a strongly maximal monotone operator. We recall that B is a strongly monotone operator if there exists $\gamma > 0$ such that for all $(x_i, y_i) \in B,\ i = 1, 2,\ \langle x_1 - x_2, y_1 - y_2 \rangle \geq \gamma \|x_1 - x_2\|^2$. Let $\{S^1_\gamma(t)\}_{t \geq 0}$ be the semigroup generated by A^1_γ. By [5] Theorem 3.9, for each $x \in K$, $S^1_\gamma(t)x \to x_0$ and hence $f(S^1_\gamma(t)x) \to f(x_0) = \min(f)$ as $t \to \infty$. One can easily show that the subset $\mathcal{A}^1_s$ of $\mathcal{A}^1$ containing all strongly monotone operators is a dense subset of the complete metric space $(\mathcal{A}^1, d)$ and for each semigroup $\{S^1(t)\}_{t \geq 0}$ generated by the operators in A^1_s we rather have a stronger result than (1), that is, for each $x \in K$, $S^1(t)x \to x_0$ and hence $f(S^1(t)x) \to f(x_0) = \min(f)$ as $t \to \infty$. We found difficulties that did not allow us to show that $\mathcal{A}^1_s$ is a very large subset of $\mathcal{A}^1$.

5.1. **Not So Generic Convergence.** We wish to understand the significance of A(4.3) and in this subsection we drop A(4.3), the assumption that $D(A)$ is convex. We define $\mathcal{A}_1$ and $\mathcal{F}_1$ as a replacement for $\mathcal{A}^1$ and $\mathcal{F}^1$. We obtain some results which are weaker than those of Theorem 2. We show the density of $\mathcal{F}_1$ in $\mathcal{A}_1$.

Definition 5. Let A, X_0 and f satisfy A(4.1) and A(4.2). Let $\mathcal{A}_1$ be the set of perturbations of A which can be written as $A_1 = A + V_1$ where $V_1 : H \to H$ satisfy the following:

(1) V_1 is a single valued, everywhere defined, hemicontinuous, monotone operator,
(2) V_1 is bounded on bounded subsets of H,
(3) $(A + V_1)x_0 \ni 0$ for some x_0 (depending upon V_1) in X_0, and
(4) for each $x \in D(A)$, and $\forall\, z \in \partial f(x)$, $\langle z, (A + V_1)^o x \rangle \geq 0$.

Definition 6. Let $\mathcal{F}_1$ be the set of perturbations $A_1 = A + V_1$ in $\mathcal{A}_1$ such that for each positive integer n there exist a positive number $h_n > 0$ such that for each $x \in D_n$ and $\forall\, z \in \partial f(x)$,

$$\langle z, (A + V_1)^o x \rangle \geq h_n.$$

Recall that the definition of $\mathcal{A}^1$ needs assumption A(4.3). Therefore, $\mathcal{A}_1 \subseteq \mathcal{A}^1$ and $\mathcal{F}_1 \subseteq \mathcal{F}^1$ if A satisfies A(4.3).

Proposition 9. *f is Lyapounov for each semigroup $\{S^1(t)\}_{t \geq 0}$ generated by the operators in $\mathcal{A}_1$ and regularly Lyapounov for each semigroup $\{S^1(t)\}_{t \geq 0}$ generated by the operators in $\mathcal{F}_1$.*

Proof. Omitted. □

Proposition 10. *If A and f satisfy A(4.4) and A(4.5) then $(\mathcal{A}_1, d)$ is a metric space and $\mathcal{F}_1$ is a dense subset of $(\mathcal{A}_1, d)$ where*

$$d(A_1, A_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} . \frac{d_n(A_1, A_2)}{1 + d_n(A_1, A_2)}$$

for $A_1 = A + V_1$ and $A_2 = A + V_2$ in $\mathcal{A}_1$, and for each integer n, $d_n(A_1, A_2)$ is defined as

$$d_n(A_1, A_2) = \sup\{\|V_1 x - V_2 x\| : \|x\| \leq n\,,\ x \in D(A)\}.$$

Proof. In view of Proposition 7 and A(4.4), note that if $A_1 \in \mathcal{A}_1$ can be represented as $A_1 = A + V_1$ and $A_1 = A + V_2$ then $V_1 x = V_2 x$ for all $x \in D(A)$. It is easy to see that $(\mathcal{A}_1, d)$ is a metric space. To see $\mathcal{F}_1$ is a dense subset of $(\mathcal{A}_1, d)$ let $A_1 = A + V_1$, where V_1 satisfies 1-4 of Definition 5, be any arbitrary element of $(\mathcal{A}_1, d)$. Let $\epsilon > 0$ be given and let $S = \sum_{i=1}^{\infty} \frac{i}{2^i}$. Let $x_0 \in X_0$ be such that $(A + V_1)x_0 \ni 0$. For $\gamma = \frac{\epsilon}{(S+\|x_0\|+1)} > 0$ define $V_\gamma : H \to H$ as

$$V_\gamma x = V_1 x + \gamma(x - x_0),$$

and

$$A_\gamma = A + V_\gamma.$$

Then clearly 1-3 of Definition 5 hold for V_γ. Let $x \in D(A)$ be given and let $A^o_\gamma x = y_\gamma + V_1 x + \gamma(x - x_0)$ for some $y_\gamma \in Ax$. Since $A_1 \in \mathcal{A}_1$ and A(4.5) holds, by 4 of Proposition 3, there exists $z \in \partial f(x)$ such that

$$\langle z, y_\gamma + V_1 x\rangle \geq 0,$$

implying

$$\langle z, A^o_\gamma x\rangle \geq \langle z, \gamma(x - x_0)\rangle \geq 0.$$

Therefore, by Proposition 3, for each $z \in \partial f(x)$

$$\langle z, A^o_\gamma x\rangle \geq 0.$$

Hence $A_\gamma \in \mathcal{A}_1$. Also for each positive integer n and for all $x \in D_n$ we have

$$\begin{aligned}\langle z, A^o_\gamma x\rangle &\geq \langle z, \gamma(x - x_0)\rangle \\ &\geq \gamma(f(x) - f(x_0)) \\ &\geq \frac{\gamma}{n} > 0.\end{aligned}$$

Therefore, by Proposition 4, there exists $h_n > 0$ such that $\forall z \in \partial f(x)$,

$$\langle z, A^o_\gamma x\rangle \geq h_n.$$

Therefore $A_\gamma \in \mathcal{F}_1$. Now let us estimate the distance between A_1 and A_γ. For each $i \geq 1$ we have

$$d_i(A_1, A_\gamma) = \sup\{\|\gamma(x - x_0)\| : \|x\| \leq i,\ x \in D(A)\} \leq \gamma(i + \|x_0\|).$$

Therefore,

$$\begin{aligned}d(A_1, A_\gamma) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(A_1, A_\gamma)}{1 + d_i(A_1, A_\gamma)} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(A_1, A_\gamma) \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \gamma(i + \|x_0\|) = \gamma(S + \|x_0\|) \\ &= \frac{\epsilon}{(S + \|x_0\| + 1)} (S + \|x_0\|) \text{ (using the value of } \gamma \text{)} \\ &< \epsilon.\end{aligned}$$

Hence $\mathcal{F}_1$ is a dense subset of $(\mathcal{A}_1, d)$. □

Note that each element of $\mathcal{A}_1$ is a maximal monotone operator by [5], Corollary 2.5 and Corollary 2.7.

Remark 12. Under the conditions of Proposition 10, we have (1) holding for all the semigroups generated by operators in a dense subset of $\mathcal{A}_1$, and for all x in $D(A)$.

Remark 13. There are two main difficulties in obtaining a complete metric space of bounded perturbations of A, namely,

(a) the sum of two maximal monotone operators may not be a maximal monotone operator, and
(b) the limit of maximal monotone operators may not be a maximal monotone operator.

The assumption A(4.3) helps us to overcome these difficulties as in Proposition 5 we obtain rather a strong result, that the sum $A + A'$ is a maximal monotone operator, although A' is not a maximal monotone operator. The properties of A' are further helpful in defining the class $\mathcal{A}^1$ and showing in Proposition 8 that $(\mathcal{A}^1, d)$ is a complete metric space. By dropping A(4.3) and defining a new class $\mathcal{A}_1$ instead of $\mathcal{A}^1$ we overcame (a). Since an extension of a single valued monotone operator may not be a single valued maximal monotone operator, we could not show that $(\mathcal{A}_1, d)$ is a complete metric space.

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