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NONLINEAR ERGODIC THEOREMS FOR NONEXPANSIVE MAPPINGS IN GENERAL BANACH SPACES

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Abstract. We prove nonlinear ergodic theorems for nonexpansive mappings and strongly continuous one-parameter semigroups of nonexpansive mappings in general Banach spaces.

1. Introduction

Edelstein [6] studied a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset in a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space, let T be a nonexpansive mapping of C into itself and let $\xi \in C$. Then, for each point x of the closure of convex hull of the ω-limit set $\omega(\xi)$ of ξ, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge to a fixed point of T, where the ω-limit set $\omega(\xi)$ of ξ is the set of cluster points of the sequence $\{T^n\xi : n = 1, 2, \dots\}$.

In 1975, Baillon [3] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction *"an ergodic retraction"*.

In 1981, Takahashi [12, 14] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [10] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space with a Fréchet differentiable norm in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [8]. Lau, Shioji and Takahashi [9] generalized Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach spaces.

Recently, using results of Bruck [4, 5], Atsushiba and Takahashi [2] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1} T^k x$ converge to a fixed point of T. This result was extended to commutative semigroups of nonexpansive mappings by Atsushiba, Lau and Takahashi [1]. On the other hand, Suzuki and Takahashi [11] constructed a nonexpansive mapping of a compact and convex subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means $1/n \sum_{k=0}^{n-1} T^k x$ converge to a point of C, but the limit point is not a fixed point of T.

It is natural to ask whether for a nonexpansive mapping with a compact and convex subset C of a general Banach space and for each $x \in C$, the Cesàro means $1/n \sum_{k=0}^{n-1} T^k x$ converge or not. In this paper, we shall give an affirmative answer to this problem and also show a nonlinear ergodic theorem for one-parameter semigroups of nonexpansive mappings in general Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}_+$ the set of positive integers and the set of non-negative real numbers, respectively. We also denote by E a real Banach space with the topological dual E^*. Then, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between E and E^*. For each $x \in E$ and $r > 0$, we denote by $B(x; r)$ the open ball with center x and radius r.

Let C be a closed and convex subset of a Banach space E and let T be a mapping of C into itself. Then, T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let $\mathcal{S} = \{T(s) : s \in \mathbb{R}_+\}$ be a family of nonexpansive mappings of C into itself. Then, $\mathcal{S}$ is said to be a *strongly continuous one-parameter semigroup* of nonexpansive mappings on C if for each $s, t \in \mathbb{R}_+$, $T(s)T(t) = T(st)$ and for each $x \in C$, the mapping $s \mapsto T(s)x$ is continuous in the norm topology. We also denote by $F(\mathcal{S})$ the set of common fixed points of $T(s)$, $s \in \mathbb{R}_+$.

Let f be a function defined on $\mathbb{R}_+$ with values in a Banach space E. Then, f is said to be *(strongly) measurable* if $f^{-1}(G)$ is a Lebesque measurable subset of $\mathbb{R}_+$ for each open subset G of E. A measurable function f is also said to be *simple* if the range of f is a finite set. Let F be a Lebesque measurable subset of $\mathbb{R}_+$. For a simple function s, we define the Bochner integral $\int_F s(\sigma)\, d\sigma$ of s by

$$\int_F s(\sigma)\, d\sigma = \sum_{k=1}^{n} m(F_k \cap F) s_k,$$

where $s_k = s(\sigma)$ on a Lebesque measurable subset F_k of $\mathbb{R}_+$ $(k = 1, \dots, n)$ and m is the Lebesque measure on $\mathbb{R}_+$. A measurable function f is *Bochner integrable* if there exists a sequence $\{s_n\}$ of simple functions converging almost everywhere to f such that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \|f(\sigma) - s_n(\sigma)\|\, d\sigma = 0.$$

For such a function f, we define the Bochner integral $\int_F f(\sigma)\, d\sigma$ of f by

$$\int_F f(\sigma)\, d\sigma = \lim_{n\to\infty} \int_F s_n(\sigma)\, d\sigma.$$

We know that for each Bochner integrable function f and $x^* \in E^*$,

$$\left\langle \int_F f(\sigma)\, d\sigma, x^* \right\rangle = \int_F \langle f(\sigma), x^* \rangle\, d\sigma$$

and

$$\left\| \int_F f(\sigma)\, d\sigma \right\| \leq \int_F \|f(\sigma)\|\, d\sigma.$$

We also know that a measurable function f is Bochner integrable if and only if $\|f\|$ is Lebesque integrable, that is, $\int_{\mathbb{R}_+} \|f(\sigma)\|\, d\sigma < \infty$. It follows that every strongly continuous function f defined on $\mathbb{R}_+$ with values in E is Bochner integrable. For more details, see Hille and Phillips [7].

3. Main Results

First, we prove a nonlinear ergodic theorem, Theorem 1, for nonexpansive mappings on a compact and convex subset of a general Banach space. The following lemma is crucial in the proof of Theorem 1.

Lemma 1. *Let C be a compact and convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\lim_{n\to\infty} \sup_{h\in\mathbb{N}} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| = 0.$$

Proof. Fix $x \in C$, let $\epsilon > 0$ and let $h \in \mathbb{N}$. Since $\{T^i x : i \in \mathbb{N}\}$ is relatively compact, there exists a finite subset M of $\mathbb{N}$ such that

$$\{T^i x : i \in \mathbb{N}\} \subset \bigcup_{l\in M} B(T^l x; \epsilon/2).$$

Then, there exists a $k \in M$ such that

$$\|T^h x - T^k x\| < \epsilon/2.$$

So, we have

$$\begin{aligned} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x \right\| &\leq \frac{1}{n}\sum_{i=0}^{n-1} \|T^{i+h}x - T^{i+k}x\| \\ &\leq \frac{1}{n}\sum_{i=0}^{n-1} \|T^h x - T^k x\| \\ &= \|T^h x - T^k x\| < \epsilon/2. \end{aligned} \tag{3.1}$$

On the other hand, we have

$$\left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| \leq \frac{1}{n} 2k \sup_{i\in\mathbb{N}} \|T^i x\|$$

and hence

$$\lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| = 0.$$

Then, there exists an $N \in \mathbb{N}$ such that for each $n > N$,

$$\left\| \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x \right\| < \frac{\epsilon}{2}. \tag{3.2}$$

Thus, we have from (3.1) and (3.2) that for each $n > N$,

$$\begin{aligned}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i}x\right\| &\leq \left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x\right\| \\ &\quad + \left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i}x\right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty}\sup_{h\geq 0}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i}x\right\| = 0.$$

This completes the proof. □

Remark. As in the proof of Lemma 1, we obtain the following lemma:

Lemma 2. *Let C be a closed and convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself such that for each $x \in C$, $\{T^n x : n \in \mathbb{N}\}$ is relatively compact. Then, for each $x \in C$,*

$$\lim_{n\to\infty}\sup_{h\in\mathbb{N}}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i}x\right\| = 0.$$

Theorem 1. *Let C be a compact and convex subset of a Banach space E and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x$$

converges uniformly in $h \in \mathbb{N}\cup\{0\}$.

Proof. Fix $x \in C$ and let $\epsilon > 0$. Then, we have from Lemma 1 that there exists an $N_0 \in \mathbb{N}$ such that for each $h \in \mathbb{N}\cup\{0\}$ and $n > N_0$,

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+h}x - \frac{1}{n}\sum_{i=0}^{n-1}T^{i}x\right\| < \frac{\epsilon}{4}. \tag{3.3}$$

Since C is compact, there exists a cluster point y of $1/n\sum_{i=0}^{n-1}T^i x$. We can choice an $N > N_0$ such that

$$\left\|\frac{1}{N}\sum_{i=0}^{N-1}T^{i}x - y\right\| < \frac{\epsilon}{4}. \tag{3.4}$$

So, we have from (3.3) and (3.4) that for each $h \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned}\left\|\frac{1}{N}\sum_{i=0}^{N-1}T^{i+h}x - y\right\| &\leq \left\|\frac{1}{N}\sum_{i=0}^{N-1}T^{i+h}x - \frac{1}{N}\sum_{i=0}^{N-1}T^{i}x\right\| + \left\|\frac{1}{N}\sum_{i=0}^{N-1}T^{i}x - y\right\| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}\end{aligned}$$

and hence

$$\left\|\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{N}\sum_{j=0}^{N-1}T^{i+j}x-y\right\| \le \frac{1}{n}\sum_{i=0}^{n-1}\left\|\frac{1}{N}\sum_{j=0}^{N-1}T^{i+j}-y\right\| \le \sup_{i\ge 0}\left\|\frac{1}{N}\sum_{j=0}^{N-1}T^{i+j}-y\right\| \le \frac{\epsilon}{2}. \tag{3.5}$$

Thus, we have from (3.3) and (3.5) that for each $n > N_0$,

$$\begin{aligned}
&\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i}x-y\right\| \\
&\le \left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i}x-\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{N}\sum_{j=0}^{N-1}T^{i+j}x\right\| + \left\|\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{N}\sum_{j=0}^{N-1}T^{i+j}x-y\right\| \\
&\le \left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i}x-\frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{n}\sum_{i=0}^{n-1}T^{i+j}x\right\| + \frac{\epsilon}{2} \\
&\le \frac{1}{N}\sum_{j=0}^{N-1}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i}x-\frac{1}{n}\sum_{i=0}^{n-1}T^{i+j}x\right\| + \frac{\epsilon}{2} \\
&\le \sup_{j\ge 0}\left\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i}x-\frac{1}{n}\sum_{i=0}^{n-1}T^{i+j}x\right\| + \frac{\epsilon}{2} \\
&\le \frac{\epsilon}{4}+\frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $1/n\sum_{i=0}^{n-1}T^{i}x$ converges to the point y of C. It follows from Lemma 1 that $1/n\sum_{i=0}^{n-1}T^{i+h}x$ converges to y uniformly in $h \in \mathbb{N}\cup\{0\}$. This completes the proof. □

Remark. In [11], Suzuki and Takahashi constructed a nonexpansive mapping T of a compact subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1}T^{k}x$ converge, but the limit point is not a fixed point of T.

Next, we also prove a nonlinear ergodic theorem, Theorem 2, for one-parameter semigroups of nonexpansive mappings on a compact and convex subset of a general Banach space. The following lemmas are crucial in the proof of Theorem 2.

Lemma 3. *Let C be a compact and convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous one-parameter semigroup of nonexpansive mappings of C into itself. Then, for each $x \in C$,*

$$\lim_{t\to\infty}\sup_{h\ge 0}\left\|\frac{1}{t}\int_0^t T(s+h)x\,ds-\frac{1}{t}\int_0^t T(s)x\,ds\right\| = 0.$$

Proof. Fix $x \in C$, let $\epsilon > 0$ and let $h \in \mathbb{R}_+$. Since C is compact, there exists a finite subset M of $\mathbb{R}_+$ such that

$$\{T(s)x : s \in \mathbb{R}_+\} \subset \bigcup_{w \in M} B(T(w)x; \epsilon/2).$$

Then, there exists a $k \in M$ such that

$$\|T(h)x - T(k)x\| < \epsilon/2.$$

So, we have

$$\begin{aligned}
&\left\| \frac{1}{t}\int_0^t T(s+h)x\, ds - \frac{1}{t}\int_0^t T(s+k)x\, ds \right\| \\
&= \sup_{\|x^*\|=1} \left\langle \frac{1}{t}\int_0^t (T(s+h)x - T(s+k)x)\, ds, x^* \right\rangle \\
&= \sup_{\|x^*\|=1} \frac{1}{t}\int_0^t \langle T(s+h)x - T(s+k)x, x^* \rangle\, ds \\
&\le \frac{1}{t}\int_0^t \|T(s+h)x - T(s+k)x\|\, ds \\
&\le \frac{1}{t}\int_0^t \|T(h)x - T(k)x\|\, ds \\
&= \|T(h)x - T(k)x\| < \epsilon/2.
\end{aligned} \tag{3.6}$$

On the other hand, since, for each $t > k$,

$$\begin{aligned}
&\left\| \frac{1}{t}\int_0^t T(s+k)x\, ds - \frac{1}{t}\int_0^t T(s)x\, ds \right\| \\
&= \sup_{\|x^*\|=1} \left| \left\langle \frac{1}{t}\int_0^t T(s+k)x\, ds - \frac{1}{t}\int_0^t T(s)x\, ds, x^* \right\rangle \right| \\
&= \sup_{\|x^*\|=1} \left| \frac{1}{t}\int_0^t \langle T(s+k)x, x^* \rangle\, ds - \frac{1}{t}\int_0^t \langle T(s)x, x^* \rangle\, ds \right| \\
&= \sup_{\|x^*\|=1} \left| \frac{1}{t}\int_0^k \langle T(s+t)x, x^* \rangle\, ds - \frac{1}{t}\int_0^k \langle T(s)x, x^* \rangle\, ds \right| \\
&\le \sup_{\|x^*\|=1} \left| \frac{1}{t}\int_0^k \langle T(s+t)x, x^* \rangle\, ds \right| + \sup_{\|x^*\|=1} \left| \frac{1}{t}\int_0^k \langle T(s)x, x^* \rangle\, ds \right| \\
&\le \frac{1}{t}\int_0^k \|T(s+t)x\|\, ds + \frac{1}{t}\int_0^k \|T(s)x\|\, ds \\
&\le \frac{1}{t} 2k \sup_{w \in \mathbb{R}_+} \|T(w)x\|,
\end{aligned}$$

we have

$$\lim_{t\to\infty} \left\| \frac{1}{t}\int_0^t T(s+k)x\, ds - \frac{1}{t}\int_0^t T(s)x\, ds \right\| = 0.$$

Then, there exists a $T \in \mathbb{R}_+$ such that for each $t > T$,

$$\left\| \frac{1}{t}\int_0^t T(s+k)x\ ds - \frac{1}{t}\int_0^t T(s)x\ ds \right\| < \frac{\epsilon}{2}. \tag{3.7}$$

Thus, we have from (3.6) and (3.7) that for each $t > T$,

$$\begin{aligned} &\left\| \frac{1}{t}\int_0^t T(s+h)x\ ds - \frac{1}{t}\int_0^t T(s)x\ ds \right\| \\ &\leq \left\| \frac{1}{t}\int_0^t T(s+h)x\ ds - \frac{1}{t}\int_0^t T(s+k)x\ ds \right\| \\ &\quad + \left\| \frac{1}{t}\int_0^t T(s+k)x\ ds - \frac{1}{t}\int_0^t T(s)x\ ds \right\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{t\to\infty} \sup_{h \geq 0} \left\| \frac{1}{t}\int_0^t T(s+h)x\ ds - \frac{1}{t}\int_0^t T(s)x\ ds \right\| = 0.$$

This completes the proof. □

Remark. As in the proof of Lemma 3, we obtain the following lemma:

Lemma 4. *Let C be a closed and convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous one-parameter semigroup of nonexpansive mappings of C into itself such that for each $x \in C$, $\{T(s)x : s \in \mathbb{R}_+\}$ is relatively compact. Then, for each $x \in C$,*

$$\lim_{t\to\infty} \sup_{h \geq 0} \left\| \frac{1}{t}\int_0^t T(s+h)x\ ds - \frac{1}{t}\int_0^t T(s)x\ ds \right\| = 0.$$

Lemma 5. *Let C be a compact and convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous one-parameter semigroup of nonexpansive mappings of C into itself. Fix $k \in \mathbb{R}_+$. Then, for each $t > 0$ and $x \in C$,*

$$\frac{1}{t}\int_0^t T(s+k)x\ ds = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T(it/n + k)x.$$

Proof. Since, for each $x \in C$, the function $s \mapsto T(s+k)x$ is strongly continuous, we have that for each $x^* \in E^*$, the real-valued function $s \mapsto \langle T(s+k)x, x^*\rangle$ is continuous. So, we have that for each $x^* \in E^*$,

$$\begin{aligned} \left\langle \frac{1}{t}\int_0^t T(s+k)x\ ds, x^* \right\rangle &= \frac{1}{t}\int_0^t \langle T(s+k)x, x^*\rangle\ ds \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \langle T(it/n + k)x, x^*\rangle \\ &= \lim_{n\to\infty} \langle S_n(x), x^*\rangle, \end{aligned}$$

where $S_n(x) = 1/n \sum_{i=0}^{n-1} T(it/n + k)x$. Since C is compact, there exists a subsequence $\{S_{n_j}(x)\}$ of $\{S_n(x)\}$ converging to a point y of C. Then, we have that for each $x^* \in E^*$,

$$\begin{aligned}\langle y, x^* \rangle &= \lim_{j\to\infty} \langle S_{n_j}(x), x^* \rangle = \lim_{n\to\infty} \langle S_n(x), x^* \rangle \\ &= \left\langle \frac{1}{t} \int_0^t T(s+k)x \; ds, x^* \right\rangle\end{aligned}$$

and hence $y = 1/t \int_0^t T(s+k)x \; ds$. So, $1/n \sum_{i=0}^{n-1} T(it/n + k)x$ converges to $1/t \int_0^t T(s+k)x \; ds$. This completes the proof. □

Theorem 2. *Let C be a compact and convex subset of a Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous one-parameter semigroup of nonexpansive mappings of C into itself. Then, for each $x \in C$,*

$$\frac{1}{t} \int_0^t T(s+h)x \; ds$$

converges uniformly in $h \in \mathbb{R}_+$.

Proof. Let $\epsilon > 0$ and let $x \in C$. Then, we have from Lemma 3 that there exists a $T_0 \in \mathbb{R}_+$ such that for each $h \in \mathbb{R}_+$ and $t > T_0$,

$$\left\| \frac{1}{t} \int_0^t T(s+h)x \; ds - \frac{1}{t} \int_0^t T(s)x \; ds \right\| < \frac{\epsilon}{8}. \tag{3.8}$$

Since C is compact, there exists a cluster point y of $1/t \int_0^t T(s)x \; ds$. We can choice a $T > T_0$ such that

$$\left\| \frac{1}{T} \int_0^T T(s)x \; ds - y \right\| < \frac{\epsilon}{8}. \tag{3.9}$$

On the other hand, let $h \in \mathbb{R}_+$. Since $\{T(s)x : s \in \mathbb{R}_+\}$ is relatively compact, there exists a finite subset M of $\mathbb{R}_+$ such that

$$\{T(s)x : s \in \mathbb{R}_+\} \subset \bigcup_{w\in M} B(T(w)x; \epsilon/16).$$

Then, there exists a $k \in M$ such that

$$\|T(h)x - T(k)x\| < \epsilon/16. \tag{3.10}$$

Since, from Lemma 5, there exists an $N \in \mathbb{N}$ such that

$$\left\| \frac{1}{T} \int_0^T T(s+k)x \; ds - \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + k)x \right\| < \frac{\epsilon}{8}, \tag{3.11}$$

where $t_i = it/N$ for each $i = 0, \dots, N-1$, we have from (3.10) and (3.11) that

$$\begin{aligned}
&\left\| \frac{1}{T} \int_0^T T(s+h)x\, ds - \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + h)x \right\| \\
&\leq \left\| \frac{1}{T} \int_0^T T(s+h)x\, ds - \frac{1}{T} \int_0^T T(s+k)x\, ds \right\| \\
&\quad + \left\| \frac{1}{T} \int_0^T T(s+k)x\, ds - \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + k)x \right\| \\
&\quad + \left\| \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + k)x - \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + h)x \right\| \\
&\leq \frac{1}{T} \int_0^T \| T(s+h)x - T(s+k)x \|\, ds \\
&\quad + \frac{1}{N} \sum_{i=0}^{N-1} \| T(t_i + k)x - T(t_i + h)x \| + \frac{\epsilon}{8} \\
&\leq \frac{1}{T} \int_0^T \| T(h)x - T(k)x \|\, ds + \frac{1}{N} \sum_{i=0}^{N-1} \| T(k)x - T(h)x \| + \frac{\epsilon}{8} \\
&= 2\| T(k)x - T(h)x \| + \frac{\epsilon}{8} \\
&< \frac{2\epsilon}{16} + \frac{\epsilon}{8} = \frac{\epsilon}{4}.
\end{aligned}$$

So, we have that for each $h \in \mathbb{R}_+$,

$$\left\| \frac{1}{T} \int_0^T T(s+h)x\, ds - \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + h)x \right\| < \frac{\epsilon}{4}. \tag{3.12}$$

Then, we have from (3.8), (3.9) and (3.12) that for each $h \in \mathbb{R}_+$,

$$\begin{aligned}
\left\| \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + h)x - y \right\| &\leq \left\| \frac{1}{N} \sum_{i=0}^{N-1} T(t_i + h)x - \frac{1}{T} \int_0^T T(s+h)x\, ds \right\| \\
&\quad + \left\| \frac{1}{T} \int_0^T T(s+h)x\, ds - \frac{1}{T} \int_0^T T(s)x\, ds \right\| \\
&\quad + \left\| \frac{1}{T} \int_0^T T(s)x\, ds - y \right\| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{2},
\end{aligned}$$

and hence

$$\begin{aligned}(3.13)\qquad &\left\|\frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=0}^{N-1} T(t_i+s)x\ ds - y\right\| \\ &\le \frac{1}{t}\int_0^t \left\|\frac{1}{N}\sum_{i=0}^{N-1} T(t_i+s)x - y\right\| ds \\ &\le \sup_{s\in\mathbb{R}_+}\left\|\frac{1}{N}\sum_{i=0}^{N-1} T(t_i+s)x - y\right\| \\ &\le \epsilon/2.\end{aligned}$$

Thus, we have from (3.8) and (3.13) that for each $t > T_0$,

$$\begin{aligned}&\left\|\frac{1}{t}\int_0^t T(s)x\ ds - y\right\| \\ &\le \left\|\frac{1}{t}\int_0^t T(s)x\ ds - \frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=0}^{N-1} T(t_i+s)x\ ds\right\| \\ &\quad + \left\|\frac{1}{t}\int_0^t \frac{1}{N}\sum_{i=0}^{N-1} T(t_i+s)x\ ds - y\right\| \\ &\le \left\|\frac{1}{t}\int_0^t T(s)x\ ds - \frac{1}{N}\sum_{i=0}^{N-1}\frac{1}{t}\int_0^t T(t_i+s)x\ ds\right\| + \frac{\epsilon}{2} \\ &\le \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{t}\int_0^t T(s)x\ ds - \frac{1}{t}\int_0^t T(t_i+s)x\ ds\right\| + \frac{\epsilon}{2} \\ &\le \sup_{h\in\mathbb{R}_+}\left\|\frac{1}{t}\int_0^t T(s)x\ ds - \frac{1}{t}\int_0^t T(h+s)x\ ds\right\| + \frac{\epsilon}{2} \\ &\le \frac{\epsilon}{8} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $1/t\int_0^t T(s)x\ ds$ converges to the point y of C. It follows from Lemma 3 that $1/t\int_0^t T(s+h)x\ ds$ converges to y uniformly in $h \in \mathbb{R}_+$. This completes the proof. □

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