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MOND-WEIR TYPE MIXED SYMMETRIC FIRST AND SECOND ORDER DUALITY IN NON-DIFFERENTIABLE MATHEMATICAL PROGRAMMING

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Abstract. In this paper, we have formulated first order as well as second order Mond-Weir type mixed symmetric dual problems for a class of non-differentiable nonlinear mathematical programming problems with multiple arguments. We have established weak duality theorems for each pair of problems under generalized pseudo-convexity and second order generalized pseudo-convexity assumptions. Several known results including Hou and Yang [Journal of Mathematical Analysis and Applications 255 (2001) 491], Mishra [European Journal of Operational Research 127 (2000) 507] and Mond and Schechter [Bulletin of the Australian Mathematical Society 53 (1996) 177] as well as others are obtained as special cases.

1. Introduction

Dorn [4] introduced symmetric duality in nonlinear programming by defining a program and its dual to be symmetric if the dual of the dual is the original problem. The symmetric duality for scalar programming has been studied extensively in the literature, one can refer to Dantzig et al. [3], Mishra [12-14], Mond [17], and Nanda and Das [19].

Mond and Schechter [18] studied non-differentiable symmetric duality for a class of optimization problems in which the objective functions consists of support functions. Following Mond and Schechter [18], Chen [2], Hou and Yang [8], and Yang *et al.* [22], studied symmetric duality for such problems.

Very recently, Yang *et al.* [23] presented a mixed symmetric dual formulation for a non-differentiable nonlinear programming problem. However, the models given by Yang *et al.* [23] do not allow the further weakening of generalized convexity assumptions on a part of the objective functions.

In this paper, we introduce two models of mixed symmetric duality for a class of non-differentiable multi-objective programming problems with multiple arguments. The first model is Mond-Weir type mixed symmetric dual model for a class of non-differentiable mathematical programming problems and the second model is second order case of the first model. Mixed symmetric duality for this model has not been given so far by any other author. The advantage of the first model over the model given by Yang *et al.* [23] is that it allows further weakening of convexity on the functions involved. Furthermore, Mangasarian [10, p. 609] and Mond [17, p. 493] have indicated possible computational advantages of the second order duals over the first order duals. We establish weak duality theorems for these two models

Key words and phrases. Second order mixed symmetric dual model; Non-differentiable programming; Duality; Second order generalized pseudo-convexity; Support functions.

under generalized pseudo-convexity and generalized second order pseudo-convexity assumptions and discuss several special cases of these models. The results of Hou and Yang [3], Mishra [12-14], Mond and Schechter [18], Nanda and Das [19], as well as Yang *et al.* [23] are particular cases of the results obtained in the present paper.

2. Preliminaries

Let $f(x, y)$ be real valued twice-differentiable function defined on $R^n \times R^m$. Let $\nabla_x f(\bar{x}, \bar{y})$ and $\nabla_y f(\bar{x}, \bar{y})$ denote the partial derivatives of $f(x, y)$ with respect to x and y at $(\bar{x}, \bar{y})$. The symbols $\nabla_{xy} f(\bar{x}, \bar{y})$, $\nabla_{yx} f(\bar{x}, \bar{y})$ and $\nabla_{yy} f(\bar{x}, \bar{y})$ are defined similarly.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\left\{x^T y : y \in C\right\}.$$

A support function, being convex and everywhere finite, has a sub-differential [20], that is, there exists $z \in R^n$ such that

$$s(y|C) \geq s(x|C) + z^T(y - x), \forall y \in C.$$

The sub-differential of $s(x|C)$ is given by

$$\partial s(x|C) = \left\{z \in C : z^T x = s(x|C)\right\}.$$

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \left\{y \in R^n : y^T(z - x) \leq 0,\ \forall z \in D\right\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

The following definitions will be needed in the sequel:

Definition 1. Let $X \subset R^n$. A functional $F : X \times X \times R^n \to R$ is said to be *sublinear* with respect to its third argument if for any $x, y \in X$

(A) $F(x, y; a_1 + a_2) \leq F(x, y; a_1) + F(x, y; a_2)$ for any $a_1, a_2 \in R^n$;
(B) $F(x, y; \alpha a) = \alpha F(x, y; a)$ for any $\alpha \in R_+$ and $a \in R^n$.

Definition 2. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot, y)$ is said to be *F-convex* at $\bar{x} \in X$, for fixed $y \in Y$, if

$$f(x, y) - f(\bar{x}, y) \geq F(x, \bar{x}; \nabla_x f(\bar{x}, y)), \forall x \in X.$$

Definition 3. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x, \cdot)$ is said to be *F-concave* at $\bar{y} \in Y$, for fixed $x \in X$, if

$$f(x, \bar{y}) - f(x, y) \geq F(y, \bar{y}; -\nabla_y f(x, \bar{y})), \forall y \in Y.$$

Definition 4. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot, y)$ is said to be *F-pseudoconvex* at $\bar{x} \in X$, for fixed $y \in Y$, if

$$F(x, \bar{x}; \nabla_x f(\bar{x}, y)) \geq 0 \Rightarrow f(x, y) \geq f(\bar{x}, y), \forall x \in X.$$

Definition 5. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x, \cdot)$ is said to be *F-pseudoconcave* at $\bar{y} \in Y$,for fixed $x \in X$, if

$$F(y, \bar{y}; -\nabla_y f(x, \bar{y})) \geq 0 \Rightarrow f(x, \bar{y}) \geq f(x, y), \forall y \in Y.$$

Definition 6. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(\cdot, y)$ is said to be *second order-convex* at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if

$$f(x, y) - f(\bar{x}, y) + \frac{1}{2} p^T \nabla_{xx} f(\bar{x}, y) p \geq F(x, \bar{x}; \nabla_x f(\bar{x}, y) + \nabla_{xx} f(\bar{x}, y) p), \forall x \in X.$$

$f$is said to be second order $F-$*concave* at $\bar{x} \in X$,with respect to $p \in R^n$, for fixed $y \in Y$, if $-f$ is second order F-*convex* at $\bar{x} \in X$, with respect to $p \in R^n$.

Definition 7. Let $X \subset R^n, Y \subset R^m$ and $F : X \times Y \times R^n \to R$ be sublinear with respect to its third argument. $f(x, \cdot)$ is said to be *second order$F$-pseudo-convex* at $\bar{x} \in X$, with respect to $p \in R^n$, for fixed $y \in Y$, if

$$F(x, \bar{x}; \nabla_x f(\bar{x}, y) + \nabla_{xx} f(\bar{x}, y) p) \geq 0$$
$$\Rightarrow f(x, y) \geq f(\bar{x}, y) + \frac{1}{2} p^T \nabla_{xx} f(\bar{x}, y) p, \forall x \in X.$$

f is said to be second order F-*pseudo-concave* at $\bar{x} \in X$, with respect to $p \in R^n$,for fixed $y \in Y$, if $-f$ is second order F-*pseudo-convex* at $\bar{x} \in X$, with respect to $p \in R^n$.

Remark 1. (i) The second order $F-$pseudo-convexity reduces to the $F-$pseudo-convexity introduced by Hanson and Mond [7] when $p = 0$.

(ii) For $F(x, \bar{x}; a) = \eta(x, \bar{x})^T a$, where$\eta : X \times X \to R^n$, the second order$F-$convexity reduces to the second order invexity introduced by Hanson [6], and second order $F-$pseudo-convexity reduces to the second order pseudo-invexity introduced by Yang [21].

(iii) For $F(x, \bar{x}; a) = \eta(x, \bar{x})^T a$, and $p = 0$,where$\eta : X \times X \to R^n$, the second order$F-$convexity reduces to the invexity introduced by Hanson [5], and second order $F-$pseudo-convexity reduces to the pseudo-invexity introduced by Kaul and Kaur [9].

3. Mond-Weir type mixed first and second order symmetric duality

In this section, we state two Mond-Weir type mixed symmetric dual pairs and establish duality theorems under generalized convexity assumptions. The advantage of these models are that they allow further weakening of the convexity assumptions and the advantage of the second order dual may be used to give a tighter lower bound than the first order dual for the value of the primal objective function, one can see Mishra [11].

First order model.

Primal problem (NMP)

Min $f\left(x^1,y^1\right)+g\left(x^2,y^2\right)+s\left(x^1\left|C^1\right.\right)+s\left(x^2\left|C^2\right.\right)-\left(y^1\right)^Tz^1-\left(y^2\right)^Tz^2$

subject to $\left(x^1,x^2,y^1,y^2,z^1,z^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|K_1|}\times R^{|K_2|}$

$$\nabla_{y^1}f\left(x^1,y^1\right)-z^1\leq 0, \tag{1}$$

$$\nabla_{y^2}g\left(x^2,y^2\right)-z^2\leq 0, \tag{2}$$

$$\left(y^1\right)^T\left[\nabla_{y^1}f\left(x^1,y^1\right)-z^1\right]\geq 0, \tag{3}$$

$$\left(y^2\right)^T\left[\nabla_{y^2}g\left(x^2,y^2\right)-z^2\right]\geq 0, \tag{4}$$

$$z^1\in D^1, z^2\in D^2. \tag{5}$$

Dual Problem (NMD)

Max $f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^Tw^1+\left(u^2\right)^Tw^2$

subject to $\left(u^1,u^2,v^1,v^2,w^1,w^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|J_1|}\times R^{|J_2|}$

$$\nabla_{x^1}f\left(u^1,v^1\right)+w^1\geq 0, \tag{6}$$

$$\nabla_{x^2}g\left(u^2,v^2\right)+w^2\geq 0, \tag{7}$$

$$\left(u^1\right)^T\left[\nabla_{x^1}f\left(u^1,v^1\right)+w^1\right]\leq 0, \tag{8}$$

$$\left(u^2\right)^T\left[\nabla_{x^2}g\left(u^2,v^2\right)+w^2\right]\leq 0, \tag{9}$$

$$w^1\in C^1,\ \text{and } w^2\in C^2, \tag{10}$$

where C^1 is a compact and convex subsets of $R^{|J_1|}$ and C^2 is a compact and convex subsets of $R^{|J_2|}$, similarly, D^1 is a compact and convex subsets of $R^{|K_1|}$ and D^2 is a compact and convex subsets of $R^{|K_2|}$.

The following model is Mond-Weir type *second order mixed symmetric dual* model for a class of non-differentiable mathematical programming problems:

Second order Model.

Primal Problem (SNP)

Min $f\left(x^1,y^1\right)+g\left(x^2,y^2\right)+s\left(x^1\left|C^1\right.\right)+s\left(x^2\left|C^2\right.\right)-\left(y^1\right)^Tz^1-\left(y^2\right)^Tz^2$
$-\frac{1}{2}\left(p^1\right)^T\nabla_{y^1y^1}f\left(x^1,y^1\right)p^1-\frac{1}{2}\left(p^2\right)^T\nabla_{y^2y^2}g\left(x^2,y^2\right)p^2$

subject to

$\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2\right)\in R^{|J_1|}\times R^{|J_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|K_1|}\times R^{|K_2|}\times R^{|K_1|}\times R^{|K_2|}$

$$\left[\nabla_{y^1}f\left(x^1,y^1\right)-z^1+\nabla_{y^1y^1}f\left(x^1,y^1\right)p^1\right]\leq 0, \tag{11}$$

$$\left[\nabla_{y^2}g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2}g\left(x^2,y^2\right)p^2\right]\leq 0, \tag{12}$$

$$\left(y^1\right)^T\left[\nabla_{y^1}f\left(x^1,y^1\right)-z^1+\nabla_{y^1y^1}f\left(x^1,y^1\right)p^1\right]\geq 0, \tag{13}$$

$$\left(y^2\right)^T\left[\nabla_{y^2}g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2}g\left(x^2,y^2\right)p^2\right]\geq 0, \tag{14}$$

$$z^1\in D^1,\ \text{and } z^2\in D^2, \tag{15}$$

Dual Problem (SND)
Max $f\left(u^1, v^1\right) + g\left(u^2, v^2\right) - s\left(v^1 \left| D^1\right.\right) - s\left(v^2 \left| D^2\right.\right) + \left(u^1\right)^T w^1 + \left(u^2\right)^T w^2$
$-\frac{1}{2}\left(q^1\right)^T \nabla_{y^2 y^2} f\left(u^1, v^1\right) q^1 - \frac{1}{2}\left(q^2\right)^T \nabla_{y^2 y^2} g\left(u^2, v^2\right) q^2$
subject to
$\left(u^1, u^2, v^1, v^2, w^1, w^2, q^1, q^2\right) \in R^{|J_1|} \times R^{|J_2|} \times R^{|K_1|} \times R^{|K_2|} \times R^{|J_1|} \times R^{|J_2|} \times R^{|J_1|} \times R^{|J_2|}$

$$\left[\nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1 x^1} f\left(u^1, v^1\right) q^1\right] \geq 0, \tag{16}$$

$$\left[\nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2 x^2} g\left(u^2, v^2\right) q^2\right] \geq 0, \tag{17}$$

$$\left(u^1\right)^T \left[\nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1 x^1} f\left(u^1, v^1\right) q^1\right] \leq 0, \tag{18}$$

$$\left(u^2\right)^T \left[\nabla_{x^2} g\left(u^2, v^2\right) + w^2 + \nabla_{x^2 x^2} g\left(u^2, v^2\right) q^2\right] \leq 0, \tag{19}$$

$$w^1 \in C^1, \text{ and } w^2 \in C^2, \tag{20}$$

where C^1 is a compact and convex subsets of $R^{|J_1|}$ and C^2 is a compact and convex subsets of $R^{|J_2|}$, similarly, D^1 is a compact and convex subsets of $R^{|K_1|}$ and D^2 is a compact and convex subsets of $R^{|K_2|}$.

4. Mixed duality theorems

In this section, we establish duality theorems for the pair of problems (NMP) and (NMD) as well as (SNP) and (SND) under the F-pseudo-convexity and second order F-pseudo-convexity assumptions.

Theorem 1 (Weak duality). *Let $\left(x^1, x^2, y^1, y^2, z^1, z^2\right)$ be feasible for (NMP) and $\left(u^1, u^2, v^1, v^2, w^1, w^2\right)$ be feasible for (NMD). Suppose that $f\left(\cdot, v^1\right) + \cdot^T w^1$ is F_1-pseudo-convex for fixed v^1, $f\left(x^1, \cdot\right) - \cdot^T z^1$ is F_2-pseudo-concave for fixed x^1, $g\left(\cdot, y^2\right) + \cdot^T w^2$ is G_1-pseudo-convex for fixed v^2 and $g\left(y^2, \cdot\right) - \cdot^T z^2$ is G_2-pseudo-concave for fixed x^2, and the following conditions are satisfied:*

(i) $F_1\left(x^1, u^1; \nabla_{x^1} f\left(u^1, v^1\right) + w^1\right) + \left(u^1\right)^T \left(\nabla_{x^1} f\left(u^1, v^1\right) + w^1\right) \geq 0$;
(ii) $G_1\left(x^2, u^2; \nabla_{x^2} g\left(u^2, v^2\right) + w^2\right) + \left(u^2\right)^T \left(\nabla_{x^2} g\left(u^2, v^2\right) + w^2\right) \geq 0$;
(iii) $F_2\left(y^1, v^1; \nabla_{y^1} f\left(x^1, y^1\right) - z^1\right) + \left(y^1\right)^T \left(\nabla_{y^1} f\left(x^1, y^1\right) - z^1\right) \leq 0$; *and*
(iv) $G_2\left(y^2, v^2; \nabla_{y^2} g\left(x^2, y^2\right) - z^2\right) + \left(y^2\right)^T \left(\nabla_{y^2} g\left(x^2, y^2\right) - z^2\right) \leq 0$.

Then, $\inf(\mathrm{NMP}) \geq \sup(\mathrm{NMD})$.

Proof. Suppose $\left(x^1, x^2, y^1, y^2, z^1, z^2\right)$ be feasible for (NMP) and $\left(u^1, u^2, v^1, v^2, w^1, w^2\right)$ be feasible for (NMD). By the dual constraint (6), we have $\nabla_{x^1} f\left(u^1, v^1\right) + w^1 \geq 0$, and by condition (i), we get

$$F_1\left(x^1, u^1; \nabla_{x^1} f\left(u^1, v^1\right) + w^1\right) \geq -\left(u^1\right)^T \left[\nabla_{x^1} f\left(u^1, v^1\right) + w^1\right] \geq 0$$

(using (8)). Then by F_1-pseudo-convexity of$f\left(\cdot, v^1\right) + \cdot^T w^1$, we get

$$f\left(x^1, v^1\right) + \left(x^1\right)^T w^1 \geq f\left(u^1, v^1\right) + \left(u^1\right)^T w^1. \tag{21}$$

Similarly, by using the constraint (1), condition (iii) and constraint (3), we get

$$F_2\left(y^1, v^1; \nabla_{y^1} f\left(x^1, y^1\right) - z^1\right) \leq -\left(y^1\right)^T \left[\nabla_{y^1} f\left(x^1, y^1\right) - y^1\right] \leq 0.$$

Then by F_2-pseudo-concavity of$f\left(x^1,\cdot\right)-\cdot^T z^1$, we get

$$f\left(x^1,v^1\right)-\left(v^1\right)^T z^1 \leq f\left(x^1,y^1\right)-\left(y^1\right)^T z^1. \tag{22}$$

Now rearranging (21) and (22), we get

$$f\left(x^1,y^1\right)+\left(x^1\right)^T w^1-\left(y^1\right)^T z^1 \geq f\left(u^1,v^1\right)+\left(u^1\right)^T w^1-\left(v^1\right)^T z^1.$$

Using $\left(v^1\right)^T z^1 \leq s\left(v^1\left|D^1\right.\right)$ and $\left(x^1\right)^T w^1 \leq s\left(x^1\left|C^1\right.\right)$, we have

$$f\left(x^1,y^1\right)+s\left(x^1\left|C^1\right.\right)-\left(y^1\right)^T z^1 \geq f\left(u^1,v^1\right)-s\left(v^1\left|D^1\right.\right)+\left(u^1\right)^T w^1. \tag{23}$$

Similarly, using constraints (7), condition (ii), constraint (9) and G_1-pseudo-convexity of the function $g\left(\cdot,y^2\right)+\cdot^T w^2$ and constraint (2), condition (iv), constraint (4) and G_2-pseudo-concavity of $g\left(y^2,\cdot\right)-\cdot^T z^2$ and using $\left(x^2\right)^T w^2 \leq s\left(x^2\left|C^2\right.\right)$ and $\left(v^2\right)^T z^2 \leq s\left(v^2\left|D^2\right.\right)$, finally rearranging the resultants, we get

$$g\left(x^2,y^2\right)+s\left(x^2\left|C^2\right.\right)-\left(y^2\right)^T z^2 \geq g\left(u^2,v^2\right)-s\left(v^2\left|D^2\right.\right)+\left(u^2\right)^T w^2. \tag{24}$$

Finally, from (23) and (24), we have

$$\begin{aligned} &f\left(x^1,y^1\right)+g\left(x^2,y^2\right)+s\left(x^1\left|C^1\right.\right)+s\left(x^2\left|C^2\right.\right)-\left(y^1\right)^T z^1-\left(y^2\right)^T z^2 \\ &\quad \geq f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2. \end{aligned}$$

That is, inf (NMP) $\geq$ sup (NMD). □

The weak duality for the pair (SNP) and (SND) is established in the following theorem.

Theorem 2 (Weak duality). *Let $\left(x^1,x^2,y^1,y^2,z^1,z^2,p^1,p^2\right)$ be feasible for (SNP) and $\left(u^1,u^2,v^1,v^2,w^1,w^2,q^1,q^2\right)$ be feasible for (NMD). Suppose there exist sub-linear functionals F_1, F_2, G_1 and G_2 satisfying:*

(i) $F_1\left(x^1,u^1;\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla x^1 x^1 f\left(u^1,v^1\right)q^1\right)$
$+\left(u^1\right)^T\left(\nabla_{x^1} f\left(u^1,v^1\right)+w^1+\nabla_{x^1x^1} f\left(u^1,v^1\right)q^1\right) \geq 0;$

(ii) $G_1\left(x^2,u^2;\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right)q^2\right)$
$+\left(u^2\right)^T\left(\nabla_{x^2} g\left(u^2,v^2\right)+w^2+\nabla_{x^2x^2} g\left(u^2,v^2\right)q^2\right) \geq 0;$

(iii) $F_2\left(y^1,v^1;\nabla_{y^1} f\left(x^1,y^1\right)-z^1+\nabla_{y^1y^1} f\left(x^1,y^1\right)p^1\right)$
$+\left(y^1\right)^T\left(\nabla_{y^1} f\left(x^1,y^1\right)-z^1+\nabla_{y^1y^1} f\left(x^1,y^1\right)p^1\right) \leq 0;$

and

(iv) $G_2\left(y^2,v^2;\nabla_{y^2} g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2} g\left(x^2,y^2\right)p^2\right)$
$+\left(y^2\right)^T\left(\nabla_{y^2} g\left(x^2,y^2\right)-z^2+\nabla_{y^2y^2} g\left(x^2,y^2\right)p^2\right) \leq 0.$

Furthermore, assume that $f\left(\cdot,v^1\right)+\cdot^T w^1$ is second order F_1-pseudo-convex for fixed v^1, $f\left(x^1,\cdot\right)-\cdot^T z^1$ is second order F_2-pseudo-concave for fixed x^1, $g\left(\cdot,y^2\right)+\cdot^T w^2$ is second order G_1-pseudo-convex for fixed v^2 and $g\left(y^2,\cdot\right)-\cdot^T z^2$ is second order G_2-pseudo-concave for fixed x^2, with respect to $q^1,p^1,q^2$and p^2, respectively. Then, inf (SNP) $\geq$ sup (SND).

Proof. Suppose $\left(x^1, x^2, y^1, y^2, z^1, z^2, p^1, p^2\right)$ be feasible for (SNP) and $\left(u^1, u^2, v^1, v^2, w^1, w^2, q^1, q^2\right)$ be feasible for (NMD). By the dual constraint (16), we have $\left[\nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1\right] \geq 0$, and by condition (i), we get

$$F_1\left(x^1, u^1; \nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1\right)$$
$$\geq -\left(u^1\right)^T\left(\nabla_{x^1} f\left(u^1, v^1\right) + w^1 + \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1\right) \geq 0,$$

(using (18)). Then by second order F_1-pseudo-convexity of$f\left(\cdot, v^1\right) + \cdot^T w^1$, we get

$$f\left(x^1, v^1\right) + \left(x^1\right)^T w^1 \geq f\left(u^1, v^1\right) + \left(u^1\right)^T w^1 - \frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1. \tag{25}$$

Similarly, by using the constraint (11), condition (iii) and constraint (13), we get

$$F_2\left(y^1, v^1; \nabla_{y^1} f\left(x^1, y^1\right) - z^1 + \nabla_{y^1y^1} f\left(x^1, y^1\right) p^1\right)$$
$$\leq -\left(y^1\right)^T\left[\nabla_{y^1} f\left(x^1, y^1\right) - y^1 + \nabla_{y^1y^1} f\left(x^1, y^1\right) p^1\right] \leq 0.$$

Then by second order F_2-pseudo-concavity of$f\left(x^1, \cdot\right) - \cdot^T z^1$, we get

$$f\left(x^1, v^1\right) - \left(v^1\right)^T z^1 \leq f\left(x^1, y^1\right) - \left(y^1\right)^T z^1 - \frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p^1. \tag{26}$$

Now rearranging (25) and (26), we get

$$f\left(x^1, y^1\right) + \left(x^1\right)^T w^1 - \left(y^1\right)^T z^1 - \frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p^1$$
$$\geq f\left(u^1, v^1\right) + \left(u^1\right)^T w^1 - \left(v^1\right)^T z^1 - \frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1.$$

Using $\left(v^1\right)^T z^1 \leq s\left(v^1 \left| D^1\right.\right)$ and $\left(x^1\right)^T w^1 \leq s\left(x^1 \left| C^1\right.\right)$, we have

$$\begin{aligned} f\left(x^1, y^1\right) + s\left(x^1 \left| C^1\right.\right) - \left(y^1\right)^T z^1 - \frac{1}{2}\left(p^1\right)^T \nabla_{y^1y^1} f\left(x^1, y^1\right) p^1 & \\ \geq f\left(u^1, v^1\right) - s\left(v^1 \left| D^1\right.\right) + \left(u^1\right)^T w^1 - \frac{1}{2}\left(q^1\right)^T \nabla_{x^1x^1} f\left(u^1, v^1\right) q^1. & \end{aligned} \tag{27}$$

Similarly, using constraints (17), condition (ii), constraint (19) and second order G_1-pseudo-convexity of the function $g\left(\cdot, y^2\right) + \cdot^T w^2$ and constraint (12), condition (iv), constraint (14) and second order G_2-pseudo-concavity of $g\left(y^2, \cdot\right) - \cdot^T z^2$ and using $\left(x^2\right)^T w^2 \leq s\left(x^2 \left| C^2\right.\right)$ and $\left(v^2\right)^T z^2 \leq s\left(v^2 \left| D^2\right.\right)$, finally rearranging the resultants, we get

$$\begin{aligned} g\left(x^2, y^2\right) + s\left(x^2 \left| C^2\right.\right) - \left(y^2\right)^T z^2 - \frac{1}{2}\left(p^2\right)^T \nabla_{y^2y^2} g\left(x^2, y^2\right) p^2 & \\ \geq g\left(u^2, v^2\right) - s\left(v^2 \left| D^2\right.\right) + \left(u^2\right)^T w^2 - \frac{1}{2}\left(q^2\right)^T \nabla_{x^2x^2} g\left(u^2, v^2\right) q^2. & \end{aligned} \tag{28}$$

Finally, from (27) and (28), we have

$$\begin{aligned}
&f\left(x^1,y^1\right)+g\left(x^2,y^2\right)+s\left(x^1\left|C^1\right.\right)+s\left(x^2\left|C^2\right.\right)-\left(y^1\right)^T z^1-\left(y^2\right)^T z^2\\
&\qquad-\frac{1}{2}\left(p^1\right)^T\nabla_{y^1y^1}f\left(x^1,y^1\right)p^1-\frac{1}{2}\left(p^2\right)^T\nabla_{y^2y^2}g\left(x^2,y^2\right)p^2\\
&\quad\geq f\left(u^1,v^1\right)+g\left(u^2,v^2\right)-s\left(v^1\left|D^1\right.\right)-s\left(v^2\left|D^2\right.\right)+\left(u^1\right)^T w^1+\left(u^2\right)^T w^2\\
&\qquad\qquad-\frac{1}{2}\left(q^1\right)^T\nabla_{x^1x^1}f\left(u^1,v^1\right)q^1-\frac{1}{2}\left(q^2\right)^T\nabla_{x^2x^2}g\left(u^2,v^2\right)q^2.
\end{aligned}$$

That is, $\inf(\mathrm{SNP})\geq\sup(\mathrm{SND})$. □

Theorem 3 (Strong duality)**.** *Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2}\right)$ *be an optimal solution for (NMP). Suppose that the Hessian matrix* $\nabla^2_{x^1}f\left(\overline{x^1},\overline{y^1}\right)$ *is positive definite and* $\nabla_{y^1}f-\bar{z}^1\geq 0$; *and* $\nabla^2_{y^2}g\left(\overline{x^2},\overline{y^2}\right)$ *is positive definite and* $\nabla_{y^2}g-\bar{z}^2\geq 0$; *or* $\nabla^2_{x^1}f\left(\overline{x^1},\overline{y^1}\right)$ *is negative definite and* $\nabla_{y^1}f-\bar{z}^1\leq 0$; *and* $\nabla^2_{y^2}g\left(\overline{x^2},\overline{y^2}\right)$ *is negative definite and* $\nabla_{y^2}g-\bar{z}^2\leq 0$. *If the generalized convexity hypotheses and conditions* (i)–(iv) *of Theorem 1 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2}\right)$ *is an optimal solution for (NMD).*

Proof. The proof of this theorem can be established on the lines of the proof of strong duality Theorem 4 given in [18] in light of the above Theorem 1. □

Theorem 4 (Strong duality)**.** *Let* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{p^1},\overline{p^2}\right)$ *be an optimal solution for (SNP) such that*

$$\nabla_{y^1}f\left(\overline{x^1},\overline{y^1}\right)+\nabla_{y^1y^1}f\left(\overline{x^1},\overline{y^1}\right)\overline{p^1}\neq\overline{z^1};$$

and $\nabla_{y^2}g\left(\overline{x^2},\overline{y^2}\right)+\nabla_{y^2y^2}g\left(\overline{x^2},\overline{y^2}\right)\overline{p^2}\neq\overline{z^2}$. *Suppose that the Hessian matrix* $\nabla^2_{x^1}f\left(\overline{x^1},\overline{y^1}\right)$ *is positive definite and* $\left(\overline{p^1}\right)^T\left[\nabla_{y^1}f-\bar{z}^1\right]\geq 0$; *and* $\nabla^2_{y^2}g\left(\overline{x^2},\overline{y^2}\right)$ *is positive definite and* $\left(\overline{p^2}\right)^T\left[\nabla_{y^2}g-\bar{z}^2\right]\geq 0$; *or* $\nabla^2_{x^1}f\left(\overline{x^1},\overline{y^1}\right)$ *is negative definite and* $\left(\overline{p^1}\right)^T\left[\nabla_{y^1}f-\bar{z}^1\right]\leq 0$; *and* $\nabla^2_{y^2}g\left(\overline{x^2},\overline{y^2}\right)$ *is negative definite and* $\left(\overline{p^2}\right)^T\left[\nabla_{y^2}g-\bar{z}^2\right]\leq 0$. *If the generalized convexity hypotheses and conditions* (i)–(iv) *of Theorem 2 are satisfied, then* $\left(\overline{x^1},\overline{x^2},\overline{y^1},\overline{y^2},\overline{z^1},\overline{z^2},\overline{p^1},\overline{p^2}\right)$ *is an optimal solution for (SND).*

Proof. The proof of this theorem can be established on the lines of the proof of strong duality Theorem 3.2 given by Hou and Yang [8] in light of the above Theorem 2. □

5. Special cases

In this section, we consider some special cases of our problems (NMP) and (NMD) as well as (SNP) and (SND) by choosing particular forms of the compact sets involved in the problems.

If $C^1 = C^2 = D^1 = D^2 = \{0\}$,then (NMP) and (NMD) reduce to the pair of problems studied in Chandra *et al.* [1].

If $|J_2| = 0, |K_2| = 0$, then (NMP) and (NMD) reduce to the pair of problems (P1) and (D1) of Mond and Schechter [18].

If $|J_2| = 0, |K_2| = 0$, then the second order dual pairs (SNP) and (SND) reduce to the pair of problems studied by Hou and Yang [8].

If $|J_2| = 0, |K_2| = 0$, and $C^1 = C^2 = D^1 = D^2 = \{0\}$,then (SNP) and (SND) become pair of problems (MP) and (MD) studied by Mishra [13].

6. Conclusion

The results discussed in this paper can be extended to the *higher order* case as well as to other generalized convexity assumptions. These results can also be extended to the case of multi-objective problems. These results can be extended to the case of continuous-time problems as well. In light of the results established by Mishra and Rueda [15], the results of this paper can further be extended to the case of complex spaces also.

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