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DENSELY RELATIVE PSEUDOMONOTONE VARIATIONAL INEQUALITIES OVER PRODUCT OF SETS

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Abstract. In this paper, we consider variational inequality problem over product of sets, which is equivalent to the system of variational inequalities. By introducing the concept of densely relative pseudomonotone operators, we establish the existence of a solution of our problem. As an application of our results, we prove the existence of a coincidence point of two families of nonlinear operators.

1. Introduction

In the recent past, variational inequality problem over product of sets, which is equivalent to the problem of system of variational inequalities, is used as a tool to solve various equilibrium-type problems from operations research, economics, game theory, mathematical physics amd other fields, see for example [1–3, 5, 10–11, 14–17] and references therein. Pang [16] uniformly modeled traffic equilibrium problem, spatial equilibrium problem, Nash equilibrium problem and general equilibrium programming problem in the form of a variational inequality defined on a product of sets. He decomposed the original variational inequality into a system of variational inequalities, which are easy to solve, to establish some solution methods for finding the approximate solutions of above mentioned equilibrium problems. He also studied the convergence of such solutions. The decomposition method is also used by many other authors, see for example [3, 5, 10, 11, 14, 17] and references therein.

Recently, Konnov [12] noticed that the solution sets of variational inequality problem over product of sets are invariant with respect to certain affine transformations of cost mappings. Taking these as a basis, he introduced new concept of (pseudo) monotonicity, called *relatively (pseudo) monotonicity*, which are adjusted for a decomposable structure of the initial problem. By using the famous Fan-KKM lemma [4], he proved some existence results for a solution of variational inequality problem over product of sets under these relatively monotonicities.

Inspired by the work of Luc [13], in this paper, we introduce the concept of relative quasimonotonicity and densely relative pseudomonotonicity, which are much weaker than the relatively pseudomonotonicity, considered by Konnov [12]. Under these assumptions, we establish some existence results for a solution of variational inequality problem over product of sets. As an application of our results, we derive the existence of a coincidence point of two families of nonlinear operators.

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2. Formulations and preliminaries

Let I be a finite index set, that is, $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let X_i be a Hausdorff topological vector space with its dual X_i^*, K_i a nonempty convex subset of X_i, $K = \prod_{i \in I} K_i$, $X = \prod_{i \in I} X_i$, and $X^* = \prod_{i \in I} X_i^*$. For each $i \in I$, when X_i is a normed space, its norm is denoted by $||\cdot||_i$ and the product norm on X will be denoted by $||\cdot||$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between X_i^* and X_i. For each $x \in X$, we write $x = (x_i)_{i \in I}$, where $x_i \in X_i$, that is, for each $x \in X$, $x_i \in X_i$ denotes the ith component of x. For each $i \in I$, let $F_i : K \to X_i^*$ be a nonlinear operator. We consider the following *variational inequality problem over product of sets* (for short, VIPPS): Find $\bar{x} \in K$ such that

$$\sum_{i \in I} \langle F_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i, \ i \in I. \tag{1}$$

Of course, if we define the mapping $F : K \to X^*$ by

$$F(x) = \big(F_i(x)\big)_{i \in I}, \tag{2}$$

then (VIPPS) can be equivalently re-written as the usual variational inequality problem of finding $\bar{x} \in K$ such that

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \text{for all } y \in K.$$

The (VIPPS) was first considered and studied by Konnov [12]. By introducing the concept of relatively pseudomonotonicity and strongly relatively pseudomonotonicity, he proved some existence results for a solution of (VIPPS) in the setting of Banach spaces. It is easy to see that (VIPPS) is equivalent to the following *problem of system of variational inequalities*, which is the model of various equilibrium-type problems from operations research, economics, game theory, mathematical physics and other areas, see for example, [1, 15, 16, 5] and references therein:

$$\text{(SVI)} \qquad \begin{cases} \text{Find } \bar{x} \in X \text{ such that for each } i \in I, \\ \langle F_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } y_i \in K_i, \end{cases}$$

For every nonempty set A, we denote by 2^A the family of all subsets of A. If A is a nonempty subset of a vector space, then co$A$ denotes the convex hull of A.

We shall use the following Fan-KKM lemma (see [4, 6]).

Theorem 2.1. *Let K be a compact and convex subset of a Hausdorff topological vector space X and $K_0 \subseteq K$ be nonempty. Assume that $G : K^0 \to 2^K \setminus \{\emptyset\}$ be a multivalued mapping satisfying the following conditions:*

(i) *For each $x \in K^0$, $G(x)$ is closed;*

(ii) *For every finite set $\{x^1, \ldots, x^m\}$ of K^0 one has co$\{x^1, \ldots, x^m\} \subseteq \bigcup_{k=1}^m G(x^k)$.*

Then $\bigcap_{x \in K^0} G(x) \neq \emptyset$.

3. Existence results

Definition 3.1. The map $F : K \to X^*$, defined by (2), is said to be

(i) *relative pseudomonotone at $y \in K$* [12] if for all $x \in K$, we have

$$\sum_{i \in I} \langle F_i(x), y_i - x_i \rangle \geq 0 \ \Rightarrow \ \sum_{i \in I} \langle F_i(y), y_i - x_i \rangle \geq 0,$$

and *relative strictly pseudomonotone at* $y \in K$ if the second inequality is strict for all $x \neq y$;

(ii) *relative quasimonotone at* $y \in K$ if for all $x \in K$, we have

$$\sum_{i\in I}\langle F_i(x), y_i - x_i\rangle > 0 \;\Rightarrow\; \sum_{i\in I}\langle F_i(y), y_i - x_i\rangle \geq 0.$$

If F is relative pseudomonotone (respectively, relative strictly pseudomonotone and relative quasimonotone) at each $y \in K$, then we say that it is relative pesudomonotone (respectively, relative strictly pseudomonotone and relative quasimonotone) on K.

Of course, if I is a singleton set, then Definition 3.1 (ii) reduces to the usual definition of quasimonotonicity, see for example, [8, 7].

Definition 3.2. The map $F : K \to X^*$, defined by (2), is said to be *hemicontinuous* if for all $x, y \in K$ and $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle F(x + \lambda(y - x)), y - x\rangle$ is continuous.

Lemma 3.1. *Let* F*, defined by (2), be hemicontinuous and relative quasimonotone on* K*. Then for every* $x, y \in K$ *with* $\sum_{i\in I}\langle F_i(x), y_i - x_i\rangle \geq 0$ *we have either*

$$\sum_{i\in I}\langle F_i(y), y_i - x_i\rangle \geq 0 \text{ or } \sum_{i\in I}\langle F_i(x), z_i - x_i\rangle \leq 0 \textit{ for all } z_i \in K_i \; i \in I.$$

Proof. It is sufficient to show that if for all $z_i \in K_i$, $i \in I$,

$$\sum_{i\in I}\langle F_i(x), z_i - x_i\rangle > 0,$$

then we will have

$$\sum_{i\in I}\langle F_i(y), y_i - x_i\rangle \geq 0.$$

Let us set $y^t = tz + (1 - t)y$ for $0 < t \leq 1$. Then, obviously, $y^t \in K$ and

$$\sum_{i\in I}\langle F_i(x), y_i^t - x_i\rangle > 0.$$

By relative quasimonotonicity of F, we get

$$\sum_{i\in I}\langle F_i(y^t), y_i^t - x_i\rangle \geq 0.$$

Now let $t \to 0$. Since $y^t \to y$ along a line segment, and by hemicontinuity of F, we have

$$\sum_{i\in I}\langle F_i(y), y_i - x_i\rangle \geq 0.$$

This completes the proof. □

Remark 3.1. If the index set I is singleton, then Lemma 3.1 reduces to Lemma 3.1 (ii) in [7].

Definition 3.3. [13] A subset K^0 of K is said to be *segment-dense in* K if for all $x \in K$, there can be found $x^0 \in K^0$ such that x is a cluster point of the set $[x, x^0] \cap K^0$, where $[x, x^0]$ denotes the line segment joining x and x^0 including end points.

Definition 3.4. [13] For each $i \in I$, let K_i be a nonempty convex subset of X_i. For each $i \in I$, we set

$$K_i^{\perp} := \{\xi_i \in X_i^* : \langle \xi_i, y_i - x_i \rangle = 0 \quad \text{for all } x_i, y_i \in K_i\}$$

and call it the *orthogonal complement of* K_i. Then

$$\begin{aligned} K^{\perp} &:= \prod_{i \in I} K_i^{\perp} = \prod_{i \in I} \{\xi_i \in X_i^* \ : \ \langle \xi_i, y_i - x_i \rangle = 0 \text{ for all } x_i, y_i \in K_i\} \\ &= \{\xi := (\xi_i)_{i \in I} \in X^* : \text{ for each } i \in I, \ \langle \xi_i, y_i - x_i \rangle = 0 \text{ for all } x_i, y_i \in K_i\} \end{aligned}$$

Remark 3.2. For a given $\xi_i \in X_i^*$, the following two statements are equivalent:

(a) For each $i \in I$, $\langle \xi_i, y_i - x_i \rangle = 0$ for all $x_i, y_i \in K_i$;
(b) $\sum_{i \in I} \langle \xi_i, y_i - x_i \rangle = 0$ for all $x_i, y_i \in K_i, \ i \in I$.

Indeed, (a) implies (b) is obvious. For (b) implies (a), let $y_j = x_j$ for $j \neq i$, in (b) then we obtain (a).

In view of above remark, we have

$$K^{\perp} = \{\xi = (\xi_i)_{i \in I} \in X^* : \sum_{i \in I} \langle \xi_i, y_i - x_i \rangle = 0 \quad \text{for all } x_i, y_i \in K_i, \ i \in I\}$$

and we call it the *orthogonal complement of* K.

Definition 3.5. Let F be a map from K to X^* defined by (2). We say that $x^0 \in K$ is a *positive point of* F *on* K if for all $x \in K$ one has either $F(x) \in K^{\perp}$, that is, for each $i \in I$, $F_i(x) \in K_i^{\perp}$ or there exists $y \in K$ such that

$$\sum_{i \in I} \langle F_i(x), y_i - x_i^0 \rangle > 0.$$

The set of all positive points of F on K is denoted by K_F.

We denote by $F(K)$ the image of K under F, that is, $F(K) = \{F(x) : x \in K\}$.

Proposition 3.1. *Let F, defined by (2), be hemicontinuous and relative quasi-monotone on K such that $F(K) \cap K^{\perp} = \emptyset$, that is, for each $i \in I$, $F_i(K) \cap K_i^{\perp} = \emptyset$. Then F is relative pseudomonotone at every positive point.*

Proof. Let $y \in K_F$ and $x \in K$ be any point such that $\sum_{i \in I} \langle F_i(x), y_i - x_i \rangle \geq 0$. Then by Lemma 3.1, we have either

$$\sum_{i \in I} \langle F_i(y), y_i - x_i \rangle \geq 0 \text{ or } \sum_{i \in I} \langle F_i(x), z_i - x_i \rangle \leq 0 \text{ for all } z_i \in K_i, \ i \in I. \tag{3}$$

To complete the proof, it is sufficient to show that the second inequality in (3) is impossible.

Indeed, since $y \in K_F$ and for each $i \in I$, $F_i(x) \notin K_i^{\perp}$, then there exists $z \in K$ such that

$$\sum_{i \in I} \langle F_i(x), z_i - y_i \rangle > 0.$$

Then

$$\sum_{i \in I} \langle F_i(x), z_i - x_i \rangle = \sum_{i \in I} \langle F_i(x), z_i - y_i \rangle + \sum_{i \in I} \langle F_i(x), y_i - x_i \rangle > 0$$

which shows that the second inequality in (3) is impossible, and the proof is completed. □

Proposition 3.2. *Let K be closed and convex subset of X and K^0 a segment-dense subset of K. If F, defined by (2), is relative quasimonotone at every point of K^0 and hemicontinuous on K, then it is relative quasimonotone on K.*

Proof. Let $x, y \in K$ with

$$\sum_{i\in I}\langle F_i(x), y_i - x_i\rangle > 0. \tag{4}$$

Since K^0 is a segment-dense subset of K, we can find $y^0 \in K^0$ and $y^m \in [y, y^0] \cap K^0$, for all $m \in \mathbb{N}$ such that $\lim y^m = y$. Then from (4), we obtain

$$\sum_{i\in I}\langle F_i(x), y_i^m - x_i\rangle > 0, \quad \text{for all } m \in \mathbb{N}.$$

Since F is relative quasimonotone at y^m, we get

$$\sum_{i\in I}\langle F_i(y^m), y_i^m - x_i\rangle \geq 0.$$

Since $\lim y^m = y$ and by hemicontinuity of F, we have

$$\sum_{i\in I}\langle F_i(y), y_i - x_i\rangle \geq 0.$$

Hence F is relative quasimonotone on K. □

Now we are ready to define a new concept of densely relative pseudomonotonicity, which generalize the notion of densely pseudomonotonicity considered by Luc [13].

Definition 3.6. The map $F : K \to X^*$, defined by (2), is said to be *densely relative pseudomonotone (*respectively*, densely relative strict pseudomonotone) on* K if there exists a segment-dense subset $K^0 \subseteq K$ such that F is relative pseudomonotone (respectively, relative strict pseudomonotone) on K^0.

The following lemma can be treated as a generalization of Minty lemma (see, [9], Chapter 3, Lemma 1.5) to (VIPPS).

Lemma 3.2. *Let K be a nonempty convex subset of X and K^0 be the same as in the definition of densely relative pseudomonotone map. If F, defined by (2), is hemicontinuous and densely relative pseudomonotone, then the following problem is equivalent to (VIPPS):*

$$(\text{DVIPPS})^0 \quad \begin{cases} \text{Find } \bar{x} \in K \textit{ such that} \\ \sum_{i\in I}\langle F_i(y), y_i - \bar{x}_i\rangle \geq 0, \quad \textit{for all } y_i \in K_i^0,\ i \in I. \end{cases}$$

The solution sets of (VIPPS) and $(\text{DVIPPS})^0$ are denoted by K_s and K_{sd}^0, respectively.

Proof. By the densely relative pseudomonotonicity of F, we have $K_s \subseteq K_{sd}^0$.

Conversely, let $\bar{x} \in K$ be a solution of $(\text{DVIPPS})^0$. Then

$$\sum_{i\in I}\langle F_i(y), y_i - \bar{x}_i\rangle \geq 0, \quad \text{for all } y_i \in K_i^0,\ i \in I. \tag{5}$$

Since K^0 is segment-dense, for all $z \in K$, we can find $z^0 \in K^0$ and $z^m \in [z, z^0] \cap K^0$ for all $m \in \mathbb{N}$ such that $\lim z^m = z$. Then from (5), we get

$$\sum_{i \in I} \langle F_i(z^m), z_i^m - \bar{x}_i \rangle \geq 0, \quad \text{for all } m \in \mathbb{N}.$$

Since $\lim z^m = z$ and F is hemicontinuous, we obtain

$$\sum_{i \in I} \langle F_i(z), z_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } z_i \in K_i,\ i \in I.$$

Again by hemicontinuity of F (see the proof of Lemma 2 in [12]), we have

$$\sum_{i \in I} \langle F_i(\bar{x}), z_i - \bar{x}_i \rangle \geq 0, \quad \text{for all } z_i \in K_i,\ i \in I.$$

Hence $\bar{x} \in K_s$ and thus $K_s = K_{sd}^0$. □

Theorem 3.1. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and F, defined by (2), be hemicontinuous and densely relative pseudomonotone on K. Then (VIPPS) has a solution.*

Proof. Let K^0 be the same as in the definition of a densely relative pseudomonotone map. For each $y \in K^0$, define two multivalued maps $S, T : K^0 \to 2^K$ by

$$S(y) = \{x \in K : \sum_{i \in I} \langle F_i(x), y_i - x_i \rangle \geq 0\}$$

and

$$T(y) = \{x \in K : \sum_{i \in I} \langle F_i(y), y_i - x_i \rangle \geq 0\}.$$

Then for each $y \in K^0$, $T(y)$ is closed, and also by relative pseudomonotonicity of F on K^0, we have $S(y) \subseteq T(y)$. By using the standard argument, it is easy to see that for every finite set $\{y^1, \ldots, y^m\}$ of K^0 one has $\mathrm{co}\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m S(y^k)$ (see for example, the proof of Theorem 1 in [12]). Since for all $y \in K^0$, $S(y) \subseteq T(y)$, we also have, $\mathrm{co}\{y^1, \ldots, y^m\} \subseteq \bigcup_{k=1}^m T(y^k)$. By applying Theorem 2.1, we have $\bigcap_{y \in K^0} T(y) \neq \emptyset$, that is, there exists $\bar{x} \in K$ such that

$$\sum_{i \in I} \langle F_i(y), y_i - x_i \rangle \geq 0, \quad \text{for all } y_i \in K_i^0,\ i \in I.$$

By Lemma 3.1, $\bar{x} \in K$ is a solution of (VIPPS). □

Corollary 3.1. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and F, defined by (2), be hemicontinuous and relative quasimonotone on K such that K_F is segment-dense in K. Then (VIPPS) has a solution.*

Proof. Let $\bar{x} \in K$ such that $F(\bar{x}) \in K^{\perp}$, then $\sum_{i \in I} \langle F_i(x), y_i - \bar{x}_i \rangle = 0$ for all $y_i \in K_i,\ i \in I$. Hence $\bar{x} \in K$ is a solution of (VIPPS). Therefore, we may assume that $F(K) \cap K^{\perp} = \emptyset$. Then by Proposition 3.1, F is relative pseudomonotone at every point of K_F. Since K_F is segment-dense in K, F is densely relative pseudomonotone on K. Thus by Theorem 3.1, (VIPPS) has a solution. □

Corollary 3.2. *For each $i \in I$, let K_i be a nonempty, compact and convex subset of X_i, and F, defined by (2), be hemicontinuous and densely relative strict pseudomonotone on K. Then (VIPPS) has a solution $\bar{x} \in K$, and it is unique if $\bar{x} \in K^0$, where K^0 is the same as in the definition of a densely relative pseudomonotone map.*

Proof. In view of Theorem 3.1, it is sufficient to show that (VIPPS) has at most one solution. Assume to the contrary that $x', x'' \in K^0$ are two solutions of (VIPPS) such that $x' \neq x''$. Then

$$\sum_{i \in I} \langle F_i(x'), x_i'' - x' \rangle \geq 0,$$

By densely relative strict pseudomonotonicity of F on K^0, we have

$$\sum_{i \in I} \langle F_i(x''), x_i'' - x' \rangle > 0, \text{ i.e. } \sum_{i \in I} \langle F_i(x''), x_i' - x'' \rangle < 0.$$

Thus x'' is not a solution of (VIPPS), which is a contradiction of our assumption. This completes the proof. □

Corollary 3.3. *For each $i \in I$, let X_i be a real reflexive Banach space and K_i a nonempty, closed and convex subset of X_i. Let F, defined by (2), be hemicontinuous and densely relative pseudomonotone on K. Then under each of the following conditions, (VIPPS) has a solution.*

For each sequence $\{x^m\} \subseteq K$ with $||x^m|| \to \infty$ as $m \to \infty$,

(h1) *there exists $m_0 > 0$ such that $\sum_{i \in I} \langle F_i(x^{m_0}), x_i^{m_0} \rangle \geq 0$;*
(h2) *there exist $m_0 > 0$ and $y \in K$ with $||y|| < ||x^{m_0}||$ such that $\sum_{i \in I} \langle F_i(x^{m_0}), y_i - x_i^{m_0} \rangle \leq 0$;*
(h3) *there exist $m_0 > 0$ and $y \in K$ such that $\sum_{i \in I} \langle F_i(y), x_i^m - y_i \rangle > 0$, for all $m \geq m_0$.*

Proof. For each $i \in I$, we denote by $B_i(m) = \{x_i \in K_i : ||x_i||_i \leq m\}$ the closed ball with center at 0 and radius $m \in \mathbb{N}$ in K_i, and $B(m) = \prod_{i \in I} B_i(m)$ for all $m \in \mathbb{N}$. Then for each $i \in I$ and for all $m \in \mathbb{N}$, $B_i(m)$ is nonempty, weakly compact and convex. By Theorem 3.1, there exists $x_i^m \in K_i$ for each $i \in I$ such that

$$\sum_{i \in I} \langle F_i(x^m), y_i - x_i^m \rangle \geq 0, \quad \text{for all } y_i \in B_i(m),\ i \in I \text{ and for every } m \in \mathbb{N}. \tag{6}$$

Set

$$g(y) = \sum_{i \in I} \langle F_i(x^m), y_i \rangle, \quad \text{for all } y \in K.$$

Then, clearly g is linear and hence convex. If $||x^m|| < m$ for some m, then by (6), we have

$$g(x^m) \leq g(y), \quad \text{for all } y \in B(m).$$

Thus x^m is a local minimum of a convex function g, hence it is a global minimum, that is,

$$g(x^m) \leq g(y), \quad \text{for all } y \in K,$$

that is,

$$\sum_{i \in I} \langle F_i(x^m), y_i - x_i^m \rangle \geq 0, \quad \text{for all } y \in K,$$

which means that x^m is a solution of (VIPPS).

If $||x^m|| = m$ for all $m \in \mathbb{N}$. Assume that condition (h1) holds. Then we show that x^{m_0} is a solution of (VIPPS).

Indeed, for all $y_i \in K_i$, there is a $t \in (0,1]$ for each $i \in I$ such that $ty_i \in B_i(m_0)$. From (6), we get

$$\begin{aligned} 0 &\leq \sum_{i \in I} \langle F_i(x^{m_0}), ty_i - x_i^{m_0} \rangle \\ &\leq t \sum_{i \in I} \langle F_i(x^{m_0}), y_i - x_i^{m_0} \rangle - (1-t) \sum_{i \in I} \langle F_i(x^{m_0}), x_i^{m_0} \rangle \end{aligned}$$

By condition (h1), we obtain

$$\sum_{i \in I} \langle F_i(x^{m_0}), y_i - x_i^{m_0} \rangle \geq 0, \quad \text{for all } y_i \in K_i, \ i \in I.$$

Hence x^{m_0} is a solution of (VIPPS).

Under condition (h2) and by using (6), we obtain

$$\sum_{i \in I} \langle F_i(x^{m_0}), y_i \rangle = \sum_{i \in I} \langle F_i(x^{m_0}), x_i^{m_0} \rangle,$$

that is,

$$g(y) = g(x^{m_0}), \quad \text{for all } x^{m_0} \text{ with } m_0 = ||x^{m_0}|| > ||y||.$$

It follows that y is a local minimum of g on $B(m_0)$. Consequently, it is a global minimum of g and we obtain $g(y) \leq g(z)$ for all $z \in K$, that is,

$$\sum_{i \in I} \langle F_i(x^{m_0}), z_i \rangle \geq \sum_{i \in I} \langle F_i(x^{m_0}), y_i \rangle = \sum_{i \in I} \langle F_i(x^{m_0}), x_i^{m_0} \rangle.$$

This implies that

$$\sum_{i \in I} \langle F_i(x^{m_0}), z_i - x_i^{m_0} \rangle \geq 0, \quad \text{for all } z_i \in K_i, \ i \in I.$$

Hence x^{m_0} is a solution of (VIPS).

Finally, condition (h3) and the relative quasimonotonicity of F (in view of Proposition 3.2) imply

$$\sum_{i \in I} \langle F_i(x^m), y_i - x_i^m \rangle \leq 0, \quad \text{for all } m \geq m_0.$$

For m sufficiently large, we have $||y|| < ||x^m||$, that is, condition (h2) holds. Hence (VIPPS) has a solution. □

4. A coincidence theorem

As an application of Corollary 3.3, we derive the following existence result for a coincidence point of two families of nonlinear operators.

Theorem 4.1. *For each $i \in I$, let X_i be a real reflexive Banach space. Let F, defined by (2), and $G = (G_i)_{i \in I}$ be two nonlinear operators from X to X^* such that $(F - G)$ is hemicontinuous and densely relative pseudomonotone on X, where* *for each $i \in I$, $G_i : X \to X_i$ is a nonlinear map. Assume that at least one of the following conditions holds:*

For each sequence $\{x^m\} \subseteq X$ with $||x^m|| \to \infty$ as $m \to \infty$,

(h11) *there exists $m_0 > 0$ such that*

$$\sum_{i \in I} \langle F_i(x^{m_0}), x_i^{m_0} \rangle \geq \sum_{i \in I} \langle G_i(x^{m_0}), x_i^{m_0} \rangle;$$

(h22) *there exist $m_0 > 0$ and $y \in K$ with $||y|| < ||x^{m_0}||$ such that*

$$\sum_{i \in I} \langle F_i(x^{m_0}), y_i - x_i^{m_0} \rangle \leq \sum_{i \in I} \langle G_i(x^{m_0}), y_i - x_i^{m_0} \rangle;$$

(h33) *there exist $m_0 > 0$ and $y \in K$ such that*

$$\sum_{i \in I} \langle F_i(y), x_i^m - y_i \rangle > \sum_{i \in I} \langle G_i(y), x_i^m - y_i \rangle, \quad \text{for all } m \geq m_0,\ i \in I.$$

Then there exists $\bar{x} \in K$ such that $F_i(\bar{x}) = G_i(\bar{x})$ for each $i \in I$.

Proof. From Corollary 3.3, there exists $\bar{x} \in X$ such that for each $i \in I$,

$$\langle F_i(\bar{x}), y_i - \bar{x}_i \rangle \geq \langle G_i(\bar{x}), y_i - \bar{x}_i \rangle, \quad \text{for all } y_i \in X_i.$$

Therefore we have, $F_i(\bar{x}) = G_i(\bar{x})$ for each $i \in I$. □

References

[1] Q. H. Ansari and J. C. Yao, *A fixed-point theorem and its applications to the system of variational inequalities*, Bull. Austral. Math. Soc. **59** (1999), 433-442.

[2] J. P. Aubin, Mathematical Methods of Game Theory and Economic, North-Holland, Amsterdam, (1982).

[3] C. Cohen and F. Chaplais, *Nested monotony for variational inequalities over product of spaces and convergence of iterative algorithms*, J. Optim. Theory Appl. **59** (1988), 360-390.

[4] K. Fan, *A generalization of Tychonoff's fixed point theorem*, *Math. Ann.* **142** (1961), 305-310.

[5] M. Ferris and J.-S. Pang, *Engineering and economic applications of complementarity problems*, SIAM Rev. **39** (1997), 669-713.

[6] A. Granas and M. Lassonde, *Sur un principe géométrique en analyse convexe*, Studia Math. **101** (1991), 1-18.

[7] N. Hadjisavvas and S. Schaible, *Quasimonotone variational inequalities in Banach spaces*, J. Optim. Theory Appl. **90** (1996), 95-111.

[8] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, J. Optim. Theory Appl. **66** (1990), 37-46.

[9] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, (1980).

[10] I. V. Konnov, *Combined relaxation methods for variational inequalities over product sets*, Lobachevskii J. Math. **2** (1999), 3-9.

[11] I. V. Konnov, *Combined relaxation methods for decomposable variational inequalities*, Optim. Methods and Software **10** (1999), 711-728.

[12] I. V. Konnov, *Relatively monotone variational inequalities over product sets*, Oper. Res. Lett. **28** (2001), 21-26.

[13] D. T. Luc, *Existence results for densely pseudomonotone variational inequalities*, J. Math. Anal. Appl. **254** (2001), 291-308.

[14] S. Makler-Scheimberg, V. H. Nguyen, and J. J. Strodiot, *Family of perturbation methods for variational inequalities*, J. Optim. Theory Appl. **89** (1996), 423-452.

[15] A. Nagurney, Network Economics: A Variational Inequality Approach, Kluwer Academic Publishers, Dordrecht/Boston/London, (1993).

[16] J.-S. Pang, *Asymmetric variational inequality problems over product sets: Applications and iterative methods*, Math. Prog. **31** (1985), 206-219.
[17] D. L. Zhu and P. Marcotte, *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optim. **6** (1996), 714-726.

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