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PERIODIC SOLUTIONS OF NONLINEAR EVOLUTION INCLUSIONS IN BANACH SPACES

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Abstract. In this paper we study the existence of integral solutions to abstract periodic problems of the form

$$\begin{cases} -x'(t) \in Ax(t) + F(t, x(t)), & t \in T := [0, b], \\ x(0) = x(b), \end{cases}$$

where A is an m-accretive operator in a reflexive Banach space X and $F : T \times X \to 2^X$ is a multivalued map (perturbation). We prove three existence results: one when the multivalued nonlinearity $F(t, x)$ is convex-valued, the other for the case when $F(t, x)$ is nonconvex valued, and finally an existence result for the case when $F(t, x)$ is replaced by *ext* $F(t, x)$, the set of extreme points of $F(t, x)$.

1. Introduction

In this paper we study the existence of integral solutions for the following nonlinear periodic evolution inclusion

$$\begin{cases} -x'(t) \in Ax(t) + F(t, x(t)), & t \in T := [0, b], \\ x(0) = x(b), \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator in a reflexive Banach space X and $F : T \times X \to 2^X$ is a multivalued perturbation.

The case when the perturbation is a single valued map has been studied by many authors. A common approach is to impose conditions on the perturbation term strong enough in order to guarantee the uniqueness of solutions of a related Cauchy problem and then to apply some of the classical fixed point theorems to the corresponding Poincaré map. The first result in this direction is due to Browder [11], who considered the case when A is a linear, time-dependent, monotone operator in a Hilbert space and the perturbation term is assumed to be monotone with respect to x. The next major result on periodic solutions for semilinear evolution equations can be traced in the work of Pruss [26], who considered the case when the linear unbounded operator $-A$ generates a compact semigroup and $F : T \times D \to X$ is continuous and such that $A + F$ satisfies a Nagumo type tangential condition with respect to D, where D is a closed, convex, bounded subset of X. Subsequently, Becker [8] considered the case in which a closed densely defined linear operator A, which acts in a Hilbert space, is such that $-A$ generates a compact semigroup, and satisfies an extra condition (which amounts to saying that $A - \lambda I$ is m-accretive for

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some $\lambda > 0$). For a single-valued perturbation F of a special form, he proved the existence of a unique periodic solution.

The first fully nonlinear existence results for the periodic problem (1.1) with F single-valued were obtained by Vrabie [31] and Hirano [17]. Vrabie's result can be viewed as an extension of Becker's result to general Banach spaces and to fully nonlinear operators A and F. He proved the existence of periodic solutions for the case when the nonlinear operator A is such that $A-\lambda I$ is m-accretive for some $\lambda > 0$ and $-A$ generates a compact semigroup, while the perturbation F is a Carathéodory single valued map which satisfies an asymptotic growth condition. The main result is obtained by looking for fixed points for a suitably defined mapping which is single valued and continuous.

Hirano, on the other hand, improved the main result in [31] (which is valid in general Banach spaces) to the specific case in which A is the subdifferential of a lower semicontinuous convex and proper function acting on a real Hilbert space. He shows that if $-A$ generates a compact semigroup and the perturbation F is a Carathéodory function with sublinear growth which satisfies a unilateral coercivity type condition, then the periodic problem admits at least one (strong) solution.

Cascaval and Vrabie [12] extended Hirano's result to the case when A is m-accretive and $-A$ generates a compact semigroup in a Hilbert space. Further generalizations to the case when the perturbation is a single valued Carathéodory function and A is an m-accretive operator in a Banach space (resp., within the framework of evolution triples, A is a maximal monotone operator from a Banach space into its dual), were obtained by Shioji in [29] (resp., [28]).

A usual assumption to get existence of solutions for the periodic problem is a kind of coercivity condition relating A and F. Such a condition was replaced by a saddle-point condition in Hirano-Shioji [18].

Nonlinear periodic problems with a multivalued perturbation were studied by Avgerinos [2], Hu-Papageorgiou [19], Kandilakis-Papageorgiou [23], Lakshmikantham-Papageorgiou [24] and Papageorgiou-Yannakakis [25], within the framework of evolution triples. The multivalued perturbation is typically assumed to be convex valued, except in [25], where $F(t,x)$ is replaced by $ext\ F(t,x)$, the set of extreme points of $F(t,x)$, which is generally neither convex nor closed. The periodic problem for the case of convex valued perturbations has been investigated using a Nagumo type tangential condition. Bader [3] considered semilinear problems and used the semigroup theory and the Hausdorff measure of noncompactness. Hu and Papageorgiou [19] considered nonlinear problems governed by time-dependent maximal monotone coercive operators, in the context of evolution triples, and used Galerkin approximations. Bader's paper [3] extended to evolution inclusions some of the results of Pruss [26], while the work of Hu and Papageorgiou [19] is related to the papers of Vrabie [31] and Hirano [17].

Recently, Bader and Papageorgiou [5], and Hu and Papageorgiou [22] studied the existence of strong solutions for the periodic problem for nonlinear evolution inclusions of subdifferential type in Hilbert spaces. Bader and Papageorgiou [5] examined both the *convex* and the *nonconvex* periodic problems, that is problems of the form (1.1) with the perturbation F taking convex and nonconvex values, respectively, thereby extending to a multivalued setting the work of Hirano [17].

Hu and Papageorgiou [22] proved the existence of extremal periodic solutions, that is solutions of the periodic problem (1.1), where the perturbation $F(t,x)$ is replaced by *ext* $F(t,x)$, the set of extreme points of $F(t,x)$.

In this paper, working in the broader framework of reflexive Banach spaces and m-accretive operators and using the notion of integral solution, we prove three existence theorems. The first one deals with the case when the multivalued nonlinearity $F(t,x)$ is convex-valued ("*convex problem*"), the second one with the case when $F(t,x)$ is nonconvex valued ("*nonconvex problem*") and finally the third existence result is for the case when $F(t,x)$ is replaced by *ext* $F(t,x)$, the set of extreme points of $F(t,x)$, ("*extremal solutions*"). This last case is of special interest in control theory in connection with the maximum principle. We emphasize that with the exception of the third theorem, we do not impose any strong accretivity restriction on A. Also, as compared to earlier works, we do not need any condition relating A and F.

The plan of the paper is as follows. In Section 2 we review some background material on accretive operators and multivalued mappings. The main results are stated in Section 3 and the corresponding proofs are given in Section 4. Finally, in Section 5, we discuss an example that illustrates the applicability of one of our abstract results.

2. Preliminaries

For easy reference, in this section we present some notations, basic definitions and facts from nonlinear operator theory and multivalued analysis, which we will need in the sequel. Our basic references are the books [1], [7], [20], [21] and [30].

Throughout this paper, X is a real reflexive, separable Banach space with norm $\|.\|$ and 2^X denotes the collection of all subsets of X. Let X^* be the dual space of X, with norm $\|.\|_*$, $\sigma(X,X^*)$ be the weak toplogy on X, and denote by X_w the space X endowed with the toplogy $\sigma(X,X^*)$. The duality pairing between X and X^* will be denoted by $\langle .,. \rangle$. The duality mapping $J : X \to 2^{X^*}$ is given by

$$J(x) = \left\{ x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2 \right\},\ \forall x \in X.$$

The so called upper semi-inner product on X is then defined by

$$\langle y, x \rangle_+ = \sup \{ x^*(y) : x^* \in J(x) \}.$$

Recall that if X^* is uniformly convex, then J is single-valued and uniformly continuous on bounded subsets of X.

Let $A : X \to 2^X$ be a multivalued operator in X. The *domain* and respectively, the *range* of A, are defined by

$$D(A) := \{ x \in X : Ax \neq \varnothing \},\ R(A) := \bigcup_{x \in D(A)} Ax.$$

The operator A is called *m-accretive* if the following conditions are satisfied:

$$\langle y' - y, x' - x \rangle_+ \geq 0,\ \forall x, x' \in D(A), \forall y \in Ax, \forall y' \in Ax',$$

and

$$R(I + \lambda A) = X, \forall \lambda > 0,$$

where I is the identity map on X.

By a celebrated result of Crandall and Liggett [14], if A is m-accretive, then $-A$ generates a semigroup of contractions $\{S(t) : t \geq 0\}$ on $\overline{D(A)}$. If $S(t)$ maps bounded subsets of $\overline{D(A)}$ into precompact subsets of $\overline{D(A)}$, for each $t > 0$, then the semigroup $\{S(t) : t \geq 0\}$ is called a *compact semigroup.*

Let $T = [0, b]$, with $0 < b < \infty$. We denote by $C(T, X)$ the Banach space of all continuous functions $u : T \to X$ with norm

$$\|u\|_\infty = \sup_{t \in T} \|u(t)\| .$$

and for $1 \leq p < \infty$, we denote by $L^p(T, X)$ the Banach space of (equivalence classes of) measurable functions $u : T \to X$ such that $\|u\|^p$ is Lebesgue integrable, endowed with the norm

$$\|u\|_p = \left(\int_T \|u(t)\|^p \, dt \right)^{1/p} .$$

In the space $L^1(T, X)$ we also consider the following *weak norm* defined by

$$\|u\|_w = \sup \left\{ \left\| \int_s^t u(\tau) \, d\tau \right\| : 0 \leq s \leq t \leq b \right\}, \ \forall u \in L^1(T, X) .$$

The norm $\|.\|_w$ is weaker than the usual norm $\|.\|_1$ and for a broad class of subsets of $L^1(T, X)$, the topology defined by the weak norm coincides with the usual weak topology (see Proposition 4.14 in [20], p.195). The space $L^1(T, X)$, equipped with the weak norm, will be denoted by $L^1_w(T, X)$. This notation is to be distinguished from $L^1(T, X)_w$, which designates the space $L^1(T, X)$ with the $\sigma(L^1(T, X), L^\infty(T, X^*))$ topology.

Let A be m-accretive in X. For $f \in L^1(T, X)$ we consider the evolution equation

$$(E_f) \qquad -u'(t) \in Au(t) + f(t), \ t \in T,$$

whose solutions are meant in the sense of the following definition, that is due to Bénilan [9]:

Definition 1. A continuous function $u : T \to \overline{D(A)}$ is called an integral solution of (E_f) if for all $x \in D(A)$, $y \in Ax$ and all $0 \leq s \leq t \leq T$,

$$(2.1) \qquad \|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2 \int_s^t \langle -f(\tau) - y, u(\tau) - x \rangle_+ \, d\tau.$$

It is well-known that for each $u_0 \in \overline{D(A)}$ and $f \in L^1(T, X)$ the equation (E_f) admits a unique integral solution satisfying the initial condition $u(0) = u_0$. The following proposition summarizes an important property of integral solutions (Bénilan's inequality):

Proposition 2. *Let u and v be integral solutions of (E_f) and (E_g), respectively, where f, $g \in L^1(T, X)$). Then*

$$(2.2) \qquad \|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(s) - g(s)\| \, ds$$

for all $0 \leq s \leq t \leq b$.

We denote by $\mathcal{P}_f(X)$ (resp. $\mathcal{P}_{(w)k(c)}(X)$) the collection of all nonempty closed (resp. (weakly-) compact (convex)) subsets of X. We also denote by $\mathcal{B}(X)$ the Borel $\sigma-$ algebra on X.

Let (Ω, Σ) be a measurable space. We are particularly interested in the case when $(\Omega, \Sigma) = (T, \mathcal{L})$, with $T = [0, b]$, $\mathcal{L}$ the $\sigma-$ algebra of Lebesgue measurable subsets, as well as in the case when $(\Omega, \Sigma) = (T \times X, \mathcal{L} \otimes \mathcal{B}(X))$, where $\mathcal{L} \otimes \mathcal{B}(X)$ is the product $\sigma-$ algebra on $T \times X$ generated by sets of the form $A \times B$ with $A \in \mathcal{L}$ and $B \in \mathcal{B}(X)$.

Let $\Phi : \Omega \to \mathcal{P}_f(X)$. We say that Φ is measurable if for all $x \in X$, the function

$$\omega \to d(x, \Phi(\omega)) = \inf\{\|x - z\| : z \in \Phi(\omega)\}$$

is measurable. Φ is measurable iff it is graph measurable, that is

$$Gr\ \Phi := \{(\omega, x) \in \Omega \times X : x \in \Phi(\omega)\} \in \Sigma \otimes \mathcal{B}(X).$$

By $\mathcal{S}_\Phi^p$ $(1 \leq p < \infty)$ we denote the set of all measurable selections of Φ that belong to the Bochner-Lebesgue space $L^p(\Omega, X)$, that is,

$$\mathcal{S}_\Phi^p = \{\varphi \in L^p(\Omega, X) : \varphi(t) \in \Phi(t), \text{ a.e. on } \Omega\}.$$

By the Kuratowski-Ryll Nardzewski Theorem (see, e.g. [20], p.154) one has that for a measurable multifunction $\Phi : \Omega \to \mathcal{P}_f(X)$, the set $\mathcal{S}_\Phi^p$ is nonempty if and only if the function $\omega \to \inf\{\|z\| : z \in \Phi(\omega)\}$ belongs to $L_+^p(\Omega) := L^p(\Omega, \mathbb{R}^+)$.

Recall that a set $K \subseteq L^p(T, X)$ is said to be *decomposable* if for all $u, v \in K$ and all $A \in \Sigma$ we have

$$u\chi_A + v\chi_{T \setminus A} \in K,$$

where χ_A denotes the characteristic function of A. Clearly $\mathcal{S}_\Phi^p$ is decomposable.

Let now Y be a Hausdorff topological space and let $\Psi : Y \to 2^X$. For $A \in 2^X$ we set

$$\Psi^-(A) := \{y \in Y : \Psi(y) \cap A \neq \varnothing\}, \quad \Psi^+(A) := \{y \in Y : \Psi(y) \subset A\}.$$

The multifunction Ψ is said to be *upper semi-continuous on* X (u.s.c., for short) if the set $\Psi^+(A)$ is open in Y for any open subset of A of Z.(Equivalently, Ψ is u.s.c. if $\Psi^-(C)$ is closed in Y for each closed subset C of Z). If Ψ is an upper semicontinuous, closed valued multifunction, then Ψ is closed, that is, its graph Gr Ψ is closed in $Y \times X$. Conversely, if $\Psi : Y \to \mathcal{P}(Z)$ is closed and locally compact (i.e., for each $y \in Y$, there exists a neighborhood U of y such that $\Psi(U)$ is precompact), then Ψ is u.s.c. (see ([20], Chapter 1, Proposition 2.23).

We say that $\Psi : Y \to 2^X$ is lower semicontinuous (l.s.c., for short) if $\Psi^+(C)$ is closed in Y for each closed subset C of Z.

We conclude this section by recalling the notion of Hausdorff continuity for multifunctions. Let $h(.,.)$ be the so-called Hausdorff-Pompeiu generalized metric on $\mathcal{P}_f(X)$, defined by

$$h(A, B) = \max\left\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\right\},\ \forall A, B \in \mathcal{P}_f(X).$$

A multifunction $\Psi : Y \to \mathcal{P}_f(X)$ is said to be Hausdorff continuous if it is a continuous map from Y into the space $(\mathcal{P}_f(X), h)$, that is, for every $y_0 \in Y$ and every $\varepsilon > 0$ there exists a neighborhood U_0 of y_0 such that for every $y \in U_0$, we have $h(F(y), F(y_0)) < \varepsilon$.

3. Main results

Throughout this section, X denotes a real separable Banach space with a uniformly convex dual X^*, and $T = [0, b]$. We first establish the existence of integral solutions to the boundary-value problem

$$\begin{cases} -x'(t) \in Ax(t) + F(t, x(t)), & t \in T, \\ x(0) = x(b), \end{cases} \tag{3.1}$$

where $F : T \times X \to 2^X$ is convex valued and strongly-weakly upper semicontinuous with respect to the second variable. Our specific assumptions are the following:

(H_A) A is an m-accretive operator in X, with $0 \in A0$, such that $-A$ generates a compact semigroup on $\overline{D(A)}$;

(H_F) $F : T \times X \to \mathcal{P}_{wkc}(X)$ satisfies:

- (i) $t \to F(t, x)$ is measurable, for each $x \in X$,
- (ii) the graph of $x \to F(t, x)$ is sequentially closed in $X \times X_w$, for a. a. $t \in T$,
- (iii) for each $\rho > 0$ there exists a function $a_\rho \in L^1_+(T)$ such that

$$|F(t, x)| := \sup\{\|w\| : w \in F(t, x)\} \le a_\rho(t), \tag{3.2}$$

  for a. a. $t \in T$ and all $x \in X$ with $\|x\| \le \rho$,
- (iv) there exists $r > 0$ such that $\langle v, Jx\rangle \ge 0$ for all $v \in F(t, x)$, all $t \in T$ and all $x \in X$ with $\|x\| = r$.

Remark 3. *Condition* $(H_F)(iv)$ *is known in the literature as Hartman's condition and was first used by Hartman in the context of second order Dirichlet systems in* $\mathbb{R}^n$ *(see* [16]*).*

Definition 4. By an integral solution of (3.1) we mean a continuous function $x : T \to \overline{D(A)}$ with the property that $x(0) = x(b)$ and there exists $f \in L^1(T, X)$ such that $f(t) \in F(t, x(t))$, a. e. on T, and x is an integral solution (in the sense of Definition 1) of equation (E_f).

Our result for the *convex problem* is the following.

Theorem 5. *Let assumptions* (H_A) *and* (H_F) *be satisfied. Then the problem* (3.1) *has at least one integral solution.*

Our next result is concerned with the problem (3.1) where F is no longer convex valued. We assume instead that F is closed valued and lower semicontinuous in its second argument. More precisely, assumption (H_F) changes as follows:

(H_F^1) $F : T \times X \to \mathcal{P}_f(X)$ satisfies:

- (i) $(t, x) \to F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}(X)$ measurable,
- (ii) $x \to F(t, x)$ is lower semicontinuous for a.a. $t \in T$,
- (iii) same as $(H_F)(iii)$,
- (iv) same as $(H_F)(iv)$.

Theorem 6. *Let assumptions* (H_A) *and* (H_F^1) *be satisfied. Then there exists an integral solution to problem* (3.1) *.*

Finally, we examine the existence of so-called extremal solutions to problem (3.1) . Specifically, we consider the evolution inclusion

$$\begin{cases} -x'(t) \in Ax(t) + ext\ F(t, x(t)), \quad t \in T, \\ x(0) = x(b), \end{cases} \tag{3.3}$$

where $ext\ F(t, x(t))$ denotes the set of extreme points of $F(t, x(t))$. See Dunford-Schwartz [15], Chapter 5, Section 8 for background material on extreme points.

We assume that F has nonempty, weakly compact values, which insures that ext $F(t,x) \neq \varnothing$ for all $(t,x) \in T \times X$. However, since in general, the multivalued map $(t,x) \to ext\ F(t,x)$ is neither convex nor closed valued, Theorems 5 and 6 are not applicable to (3.3) . We impose the following conditions on A and F :

(H_A^1) A satisfies (H_A) and in addition there exists $\omega > 0$ such that $A - \omega I$ is accretive,

(H_F^2) $F : T \times X \to \mathcal{P}_{wkc}(X)$ is such that :

- (i) $t \to F(t,x)$ is measurable, for each $x \in X$,
- (ii) $x \to F(t,x)$ is Hausdorff continuous for a.a. $t \in T$,
- (iii) $(H_F)(iii)$ holds with $a_\rho \in L_+^p(T)$, $1 < p < \infty$,
- (iv) $(H_F)(iv)$ is satisfied.

Our final result establishes the existence of integral solutions to (3.3) , in the sense of Definition 4 where $F(t, x(t))$ is replaced by $ext\ F(t, x(t))$.

Theorem 7. *If conditions* (H_A^1) *and* (H_F^2) *are satisfied, then the problem* (3.3) *has at least one integral solution.*

4. Proofs

Proof of Theorem 5. For $g \in L^1(T, X))$ and $x_0 \in \overline{D(A)}$ we denote by $x(g, x_0)$ the unique integral solution of

$$\begin{cases} -x'(t) \in Ax(t) + g(t), \quad t \in T, \\ x(0) = x_0. \end{cases} \tag{4.1}$$

Let $x := x(g, x_0)$ and $\widehat{x} := x(\widehat{g}, \widehat{x}_0)$ be the integral solutions of (4.1) corresponding to (g, x_0) and $(\widehat{g}, \widehat{x}_0) \in L^1(T, X) \times \overline{D(A)}$, respectively. If $A - \omega I$ is m-accretive for some $\omega > 0$ then (see, e.g., [13]):

$$\|x(t) - \widehat{x}(t)\| \leq e^{-\omega t} \|x_0 - \widehat{x}_0\| + \int_0^t e^{-\omega(t-s)} \|g(s) - \widehat{g}(s)\| \, ds, \ \forall t \in T. \tag{4.2}$$

In particular (let $g = \widehat{g}$, and $t = b$ in (4.2)) the Poincaré map $x_0 = x(b)$ is a strict contraction on $\overline{D(A)}$. As a consequence, the periodic problem

$$\begin{cases} -x'(t) \in Ax(t) + g(t), \quad t \in T, \\ x(0) = x(b) \end{cases} \tag{4.3}$$

has a unique integral solution $x^g \in C\left(T, \overline{D(A)}\right)$ for each $g \in L^1(T, X)$. The map $g \to x^g$ will be denoted by ψ.

If A satisfies (H_A) and $\varepsilon > 0$, then it is obvious that $A + \varepsilon I$ is m-accretive. Accordingly, by the above remark, for each $g \in L^1(T, X)$ there exists a unique integral solution $x_\varepsilon = x_\varepsilon^g \in C\left(T, \overline{D(A)}\right)$ of the problem

$$\begin{cases} -x_\varepsilon'(t) \in (A + \varepsilon I)\, x_\varepsilon(t) + g(t), & t \in T, \\ x_\varepsilon(0) = x_\varepsilon(b). \end{cases} \tag{4.4}$$

Therefore, for every $\varepsilon > 0$ we can define the solution map $\psi_\varepsilon : L^1(T, X) \to C\left(T, \overline{D(A)}\right)$ by

$$\psi_\varepsilon(g) = x_\varepsilon^g \tag{4.5}$$

where x_ε^g is the integral solution of (4.4). We claim that ψ_ε is weakly-strongly sequentially continuous. Indeed, let $g_n \to g$, weakly in $L^1(T, X)$, as $n \to \infty$, and set $x_n := \psi_\varepsilon(g_n)$. Using Bénilan's inequality (2.2) (see Proposition 2) and the fact that $0 \in A0$ (cf. (H_A)) we have

$$\|x_n(t)\| \le \|x_n(s)\| + \int_s^t \|g_n(\tau)\|\, d\tau,\ \forall\ 0 \le s \le t \le b. \tag{4.6}$$

Let

$$m_n := \min_{t \in T} \|x_n(t)\|,\ M_n := \max_{t \in T} \|x_n(t)\|. \tag{4.7}$$

Combining (4.6) and (4.7), we arrive at

$$M_n \le m_n + C,\ \text{where } C = \sup_n \int_0^b \|g_n(\tau)\|\, d\tau. \tag{4.8}$$

Recalling that the duality map J is single valued (because X^* is assumed uniformly convex), by Definition 1 we next obtain

$$\|x_n(t)\|^2 + 2\varepsilon \int_s^t \|x_n(\tau)\|^2\, d\tau \le \|x_n(s)\|^2 - 2\int_s^t \langle g_n(\tau), Jx_n(\tau)\rangle\, d\tau, \tag{4.9}$$

for all $0 \le s \le t \le b$. Letting $s = 0$ and $t = b$ in (4.9), we deduce that

$$\varepsilon \int_0^b \|x_n(\tau)\|^2\, d\tau \le \int_0^b \|x_n(\tau)\|\, \|g_n(\tau)\|\, d\tau$$

which, on account (4.7), yields

$$\varepsilon b m_n^2 \le M_n C. \tag{4.10}$$

From (4.8) and (4.10) it follows that both m_n and M_n are bounded, and consequently, $\{x_n\}_{n\in\mathbb{N}}$ is bounded in $C(T, X)$. In particular, $\{x_n(0)\}_{n\in\mathbb{N}}$ is bounded in X. We also note that the $g_n's$ vary in a uniformly integrable subset of $L^1(T, X)$. Therefore, by (H_A), we can invoke Baras [6] (see also Theorem 2 in [31]) to infer that $\{x_n(b)\}_{n\in\mathbb{N}}$ is relatively compact in X. Inasmuch as $x_n(0) = x_n(b)$, it follows that $x_n(0)$ varies in a relatively compact subset of X. Applying Theorem 2 in [31] again, we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $C(T, X)$; hence without loss of generality, we may assume that $x_n \to x$ in $C(T, X)$, as $n \to \infty$. Clearly,

$x \in C\left(T, \overline{D(A)}\right)$, and $x(0) = x(b)$. Moreover, since $x_n = \psi_\varepsilon(g_n)$, with ψ_ε defined by (4.5), we have (cf., (2.1))that

$$\|x_n(t) - y\|^2 \leq \|x_n(s) - y\|^2 - 2\int_s^t \langle g_n(\tau) + z, J(x_n(\tau) - y)\rangle\, d\tau, \tag{4.11}$$

for all $0 \leq s \leq t \leq b$ and all y, z with $z \in (A + \varepsilon I)y$. Since $x_n \to x$ strongly in $C(T, X)$, $g_n \to g$ weakly in $L^1(T, X)$, and J is uniformly continuous from compact subsets of X to X^*, we can pass to the limit in (4.11) as $n \to \infty$ to conclude that $x = \psi_\varepsilon(g)$. This proves that our claim is true.

Next, let $F_1 : T \times X \to \mathcal{P}_{wkc}(X)$ be defined by

$$F_1(t, x) = \begin{cases} F(t, x), & \text{if } \|x\| \leq r, \\ F(t, p_r(x)) & \text{if } \|x\| > r, \end{cases} \tag{4.12}$$

where

$$p_r(x) = \begin{cases} x, & \text{if } \|x\| \leq r, \\ r\frac{x}{\|x\|} & \text{if } \|x\| > r, \end{cases}$$

with r as in $(H_F)(iv)$. It is easily seen that F_1 satisfies (i) and (ii) of (H_F) (where F is replaced by F_1), as well as a "global" variant of $(H_F)(iii)$, namely

$$|F_1(t, x)| \leq a_r(t), \text{ a. e. on } T, \ \forall x \in X. \tag{4.13}$$

We introduce the set-valued Nemitsky operator $N : C(T, X) \to 2^{L^1(T,X)}$ by

$$N(x) = S^1_{F_1(.,x(.))}, \forall x \in C(T, X). \tag{4.14}$$

It is easy to verify that $N(.)$ has nonempty, convex, and weakly compact values. In addition, by $(H_F)(i)$, (ii), Proposition 2.23 in [20], p.43 and the Convergence Theorem [1], p.60, (cf. also [30], p.120) it follows that N is an upper semicontinuous multifunction from $C(T, X)$ into $L^1(T, X)_w$.

Consider now the approximating problem

$$\begin{cases} -x_\varepsilon'(t) \in (A + \varepsilon I)x_\varepsilon(t) + F_1(t, x_\varepsilon(t)), & t \in T, \\ x_\varepsilon(0) = x_\varepsilon(b). \end{cases} \tag{4.15}$$

In view of (4.5) and (4.14) (cf. also Definition 4) it is clear that the existence of an integral solution to (4.15) is equivalent to the existence of a fixed point for the map $\psi_\varepsilon \circ N$ in $C(T, X)$. Since, as we already have shown, N is upper semicontinuous from $C(T, X)$ to $L^1(T, X)_w$ and ψ_ε is sequentially continuous from $L^1(T, X)_w$ into $C(T, X)$, and, as is easily seen, the map $\psi_\varepsilon \circ N$ is compact, we can apply Bader's fixed point theorem ([4], Theorem 8). To this end, we consider the set

$$S := \{x \in C(T, X) : x \in \lambda(\psi_\varepsilon \circ N)(x) \text{ for some } \lambda \in (0, 1]\}, \tag{4.16}$$

and prove that

$$\|x\|_\infty \leq r, \ \forall x \in S. \tag{4.17}$$

We argue by contradiction. Suppose that (4.17) doesn't hold. Then either $\|x(t)\| > r$, $\forall t \in T$, or there exist η, $\theta \in T$, $\eta < \theta$ such that $\|x(\eta)\| = r$ and $\|x(t)\| > r$,

$\forall t \in (\eta, \theta]$. In the first case, by (4.5), (4.14) (4.15), (4.16) and Definition 1, we have

$$\|x(b)\|^2 + 2\varepsilon \int_0^b \|x(t)\|^2 dt \leq \|x(0)\|^2 - 2\lambda^2 \int_0^b \left\langle \widehat{f}(t), J\left(\lambda^{-1}x(t)\right)\right\rangle dt \tag{4.18}$$

where $\widehat{f} \in L^1(T,X)$, $\widehat{f}(t) \in F_1(t, x(t))$, a.e. on T. The homogeneity of J, (4.12) and $(H_F)(iv)$ lead to

$$\left\langle \widehat{f}(t), J\left(\lambda^{-1}x(t)\right)\right\rangle = \lambda^{-1}r^{-1}\|x(t)\| \left\langle \widehat{f}(t), J\left(p_r(x(t))\right)\right\rangle \geq 0, \ \forall t \in T. \tag{4.19}$$

(Since $\|x(t)\| > r$ on T, it follows that $\|p_r(x(t))\| = r\frac{x(t)}{\|x(t)\|}$, with $\|p_r(x(t))\| = r$ for all $t \in T$.)

Combining (4.18) and (4.19) yields

$$\|x(b)\| < \|x(0)\| \tag{4.20}$$

which is absurd. In the second case, (4.18) holds with η and θ in place of 0 and b, respectively, while (4.19) is satisfied on $[\eta, \theta]$. As a consequence, (4.20) changes to

$$\|x(\theta)\| < \|x(\eta)\|$$

which contradicts the choice of η and θ. Hence (4.17) has been proved. Applying Bader's result [4], we conclude that $\psi_\varepsilon \circ N$ has a fixed point x_ε, which solves (4.15). Since x_ε must satisfy (4.17), it follows that $F_1(t, x_\varepsilon(t)) = F(t, x_\varepsilon(t))$ (see (4.12)), so that x_ε is an integral solution of

$$\begin{cases} -x_\varepsilon'(t) \in (A + \varepsilon I) x_\varepsilon(t) + f_\varepsilon(t), & t \in T, \\ x_\varepsilon(0) = x_\varepsilon(b), \end{cases} \tag{4.21}$$

where $f_\varepsilon \in L^1(T,X)$, $f_\varepsilon(t) \in F_1(t, x(t))$, a.e. on T.

In view of (4.17), $\{x_\varepsilon\}_{\varepsilon>0}$ is bounded in $C(T,X)$. By $(H_F)(iii)$, we infer that $\{f_\varepsilon\}_{\varepsilon>0}$ is uniformly integrable in $L^1(T,X)$. On account of (H_A), we can reason as in the second part of the proof of the weak-strong sequential continuity of ψ_ε to deduce that (on a subsequence, as $\varepsilon \to 0$)

$$x_\varepsilon \to x \text{ in } C(T,X), \ f_\varepsilon \to f \text{ weakly in } L^1(T,X). \tag{4.22}$$

By $(H_F)(i)$, (ii) and ([30], p.120), it follows that $f(t) \in F(t, x(t))$, a.e. on T. Then, using the continuity of J and (4.22), we may pass to the limit in (4.21) as $\varepsilon \to 0$ and conclude that x is an integral solution to the problem (3.1) in the sense of Definition 4. This completes the proof of Theorem 5. □

Proof of Theorem 6. Let again $F_1 : T \times X \to \mathcal{P}_f(X)$ and $N : C(T,X) \to 2^{L^1(T,X)}$ be defined by (4.12) and (4.14), respectively. It is readily verified that N is well-defined, with closed decomposable values. Moreover, by a minor adaptation of the proof of Theorem 7.28 in [20], p.238 we conclude that N is l.s.c. Hence, we can apply the Bressan-Colombo selection theorem [10] to find a continuous function $u : C(T,X) \to L^1(T,X)$ such that

$$u(x) \in N(x), \ \forall x \in C(T,X). \tag{4.23}$$

For $\varepsilon > 0$ we now consider the approximating problem

$$\begin{cases} -x_\varepsilon'(t) \in (A+\varepsilon I)\, x_\varepsilon(t) + u(x_\varepsilon)(t), & t \in T, \\ x_\varepsilon(0) = x_\varepsilon(b). \end{cases} \tag{4.24}$$

Clearly, the existence of an integral solution x_ε of (4.24) is equivalent to finding a fixed point for the map $\psi_\varepsilon \circ u$ in $C(T,X)$, where ψ_ε is defined by (4.5). Since $\psi_\varepsilon \circ u$ is continuous and compact, we can use the classical Leray-Schauder principle (cf., e.g., Schaefer [27]) to prove the existence of a fixed point.

Let

$$\widehat{S} := \{x \in C(T,X) : x = \lambda(\psi_\varepsilon \circ u)(x) \text{ for some } \lambda \in (0,1]\}.$$

Arguing as in the proof of Theorem 5, we infer that (4.17) holds with $\widehat{S}$ in place of S. As a consequence, $\psi_\varepsilon \circ u$ has a fixed point x_ε, which is the desired integral solution of (4.24). In addition, since $\|x_\varepsilon\|_\infty \le r$, we conclude (cf. (4.12), (4.23)) that $u(x_\varepsilon)(t) \in F(t, x_\varepsilon(t))$, a.e. on T.

Using (H_A) and $\left(H_F^1\right)(iii)$ (see the last part of the proof of Theorem 5) we may assume that $x_\varepsilon \to x$ in $C(T,X)$ as $\varepsilon \to 0$. Passage to the limit in (4.24), as $\varepsilon \to 0$, is then immediate in view of the continuity of u and J. The proof is complete. □

Proof of Theorem 7. We replace F by F_1, as defined by (4.12) and recall that (4.13) is satisfied. Set

$$V := \left\{g \in L^1(T,X) : \|g(t)\| \le a_r(t) \text{ a. e. on } T\right\} \tag{4.25}$$

and remark that V is weakly compact in $L^1(T,X)$. In view of the strong accretivity of A (cf. $\left(H_A^1\right)$), for each $g \in V$ there exists a unique integral solution $x = \psi(g)$ of (4.3).

As shown in the proof of Theorem 5 (see the properties of ψ_ε), ψ is weakly-strongly continuous as a map from V into $C(T,X)$. (Note that by [15], Theorem V.6.3, V equipped with the relative weak $L^1(T,X)$ topology is a metric space). Let $K := \overline{conv}\ \psi(V)$ and remark that K is a convex, compact subset of $C(T,X)$. By Theorem 8.31 in [20], p.260 there exists a continuous map $u : K \to L_w^1(T,X)$ such that

$$u(x) \in ext\ S^1_{F_1(.,x(.))}, \forall x \in K.$$

Moreover, by Theorem 4.6 in [20], p.192, we know that

$$ext\ S^1_{F_1(.,x(.))} = S^1_{ext\ F_1(.,x(.))},\ \forall x \in K,$$

so that we have

$$u(x) \in S^1_{ext\ F_1(.,x(.))},\ \forall x \in K. \tag{4.26}$$

We next consider the function $\psi \circ u$ and observe (cf. (4.13), (4.25), (4.26)) that it maps K into itself. In addition, $\psi \circ u$ is continuous. Indeed, let $x_n \to x$ in $C(T,X)$ as $n \to \infty$, with $x_n,\ x \in K$. The continuity of u implies that $u(x_n) \to u(x)$ in $L_w^1(T,X)$, as $n \to \infty$. Inasmuch as $ext\ F_1(t, x_n(t)) \subseteq F_1(t, x_n(t))$, a. e. on T, $\forall n \in \mathbb{N}$, it follows by (4.13) and (4.26) that

$$\|u(x_n)(t)\| \le a_r(t) \text{ a. e. on } T,\ \forall n \in \mathbb{N}, \tag{4.27}$$

where (see $\left(H_F^2\right)(iii)$), $a_r \in L_+^p(T)$ with $1 < p < \infty$. Therefore, we may invoke Lemma 2.8 in ([21], p.24) to conclude that $u(x_n) \to u(x)$, weakly in $L^1(T, X)$, as $n \to \infty$. This in conjunction with the weak-strong continuity of ψ yields $(\psi \circ u)(x_n) \to (\psi \circ u)(x)$ in $C(T, X)$, and the continuity of $\psi \circ u$ has been established.

We can now apply Schauder's fixed point theorem to deduce that there exists $x \in K$ such that $x = (\psi \circ u)(x)$. Accordingly, x is an integral solution of

$$\begin{cases} -x'(t) \in Ax(t) + u(x)(t), & t \in T, \\ x(0) = x(b), \end{cases} \tag{4.28}$$

where $u(.)$ satisfies (4.26). Using $\left(H_F^2\right)(iv)$ one shows, exactly as in the proof of Theorem 5, that $\|x\|_\infty \leq r$, so that $F_1(t, x(t)) = F(t, x(t))$, a. e. on T. This, (4.26) and (4.28) lead to the conclusion that x is an integral solution of (3.3), as desired. The proof is thereby complete. □

5. An example

In this section, we discuss a boundary-value problem for a multivalued partial-differential equation to which Theorem 5 can be applied.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary Γ, and let $\beta : D(\beta) \subseteq \mathbb{R} \to 2^{\mathbb{R}}$ satisfy

(H_β) β is m-accretive, with $0 \in D(\beta)$.

Let $p \in [2, \infty)$ and $\lambda \geq 0$ be given, and set $X = L^2(\Omega)$, which is a separable Hilbert space. We define the operator $A : D(A) \subseteq \mathbb{R} \to 2^X$ by

$$\begin{cases} Ax = -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\left(\left|\frac{\partial x}{\partial z_i}\right|^{p-2} \frac{\partial x}{\partial z_i}\right) + \lambda x\,|x|^{p-2} \\ D(A) = \left\{x \in W^{1,p}(\Omega) : Ax \in L^2(\Omega), -\dfrac{\partial x}{\partial \nu_p}(z) \in \beta(x(z)), \text{ a.e. on } \Gamma\right\}, \end{cases} \tag{5.1}$$

where

$$\frac{\partial x}{\partial \nu_p} = \sum_{i=1}^{n} \left|\frac{\partial x}{\partial z_i}\right|^{p-2} \frac{\partial x}{\partial z_i} \cos(\overline{n}, \overline{e}_i). \tag{5.2}$$

Here $x = x(z)$, $z = (z_1, ..., z_n)$, $\overline{n}$ is the outward unit normal to Γ and $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$. According to [30], pp.22, 23, A is m-accretive on X, $0 \in A0$ and $-A$ generates a compact semigroup on $\overline{D(A)} = X$, so that (H_A) is satisfied.

Let now $f : T \times \mathbb{R} \to \mathbb{R}$ $(T = [0, b])$ safisfy

(H_f) (i) $(t, x) \to f(t, x)$ is measurable,

(ii) there exist α_1, $\alpha_2 \in L_+^1(T)$ such that

$$|f(t, x)| \leq \alpha_1(t)\,|x| + \alpha_2(t), \text{ a.e. on } T, \ \forall x \in \mathbb{R}, \tag{5.3}$$

(iii) $xf(t, x) \geq 0$ a.e. on T, $\forall x \in \mathbb{R}$.

Since $x \to f(t,x)$ is locally bounded for almost all $t \in T$ (cf. (5.3)), we can define $f_l,\ f_u : T \times \mathbb{R} \to \mathbb{R}$ by

$$f_l(t,x) = \liminf_{x' \to x} f(t,x'),\ f_u(t,x) = \limsup_{x' \to x} f(t,x'). \tag{5.4}$$

It is well known (see, e.g.,[21], p.97) that f_l is l. s. c., and f_u is u. s. c. with respect to x. Let now $\widehat{f} : T \times \mathbb{R} \to 2^{\mathbb{R}}$ be given by

$$\widehat{f}(t,x) = [f_l(t,x), f_u(t,x)], \tag{5.5}$$

and define $F : T \times X \to 2^X$ by

$$F(t,x)(z) = \left\{ v \in X : v(z) \in \widehat{f}(t,x(z)),\ \text{a.e. on } \Omega \right\}. \tag{5.6}$$

By $(H_f)(i)-(iii)$, (5.5) and (5.6), it is readily verified (see [21], p.96) that F satisies (H_F). In particular $(H_F)(iv)$ holds for any $r > 0$. (Note that $J = I$ (the identity) in this setting.)

Consider the boundary value problem

$$\begin{cases} -\frac{\partial x}{\partial t}(t,z) + \sum_{i=1}^{n} \frac{\partial}{\partial z_i}\left(\left| \frac{\partial x}{\partial z_i} \right|^{p-2} \frac{\partial x}{\partial z_i} \right) - \lambda x |x|^{p-2} x(t,z) \\ \qquad\qquad\qquad \in \widehat{f}(t,x(z)),\ \text{a.e. on } T \times \Omega \\ -\frac{\partial x}{\partial \nu_p}(t,z) \in \beta(x(t,z)),\ \text{a.e. on } (0,T) \times \Gamma, \\ x(0,z) = x(b,z),\ \text{a.e. on } \Omega. \end{cases} \tag{5.7}$$

In view of the above discussion, it is clear that (5.7) can be rewriten in the abstract form (3.1) in the Hilbert space X, with A and F given by (5.1) and (5.6), respectively. Since conditions (H_A) and (H_F) are verified, Theorem 5 is applicable and yields the following result:

Theorem 8. *Assume that (H_β) and (H_f) are satisfied. Let $p \in [2,\infty)$, $\lambda \geq 0$, and let $\frac{\partial}{\partial \nu_p}$ and $\widehat{f}$ be given by (5.2) and (5.5), respectively. Then problem (5.7) has at least one integral solution $x \in C(T; L^2(\Omega))$.*

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References

[1] J. P. Aubin and A. Cellina. *Differential Inclusions.* Springer, Berlin, 1984.

[2] E. P. Avgerinos. On periodic solutions of multivalued nonlinear evolution equations. *J. Math. Anal. Appl.*, 198:643–656, 1996.

[3] R. Bader. On the semilinear multivalued flow under constraints and the periodic problem. *Comment. Math. Univ. Carolin.*, 41:719–734, 2000.

[4] R. Bader. A topological fixed point index theory for evolution inclusions. *Z. Anal. Anwendungen*, 20:3–15, 2001.

[5] R. Bader and N. S. Papageorgiou. On the problem of periodic evolution inclusions of the subdifferential type. *Z. Anal. Anwendungen*, 21:963–984, 2002.

[6] P. Baras. Compacité de l'opérateur $f \to u$ solution d'une équation non-linéaire $(du/dt) + Au \ni f$. *C. R. Acad. Sci. Paris*, 286:1113–1116, 1978.

[7] V. Barbu. *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff International Publ., Leyden, 1976.
[8] R. I. Becker. Periodic solutions of semilinear equations of evolutions of compact type. *J. Math. Anal. Appl.*, 82:33–48, 1981.
[9] P. Bénilan. Solutions intégrales d'équations d'évolution dans un espace de Banach. *C. R. Acad. Sci. Paris*, 274:47–50, 1972.
[10] A. Bressan and G. Colombo. Extensions and selections of maps with decomposable values. *Studia Math.*, 102:209–216, 1992.
[11] F. E. Browder. Existence of periodic solutions for nonlinear equations of evolution. *Proc. Nat. Acad. Sci. USA*, 53:1100–1103, 1965.
[12] R. Cascaval and I. I. Vrabie. Existence of periodic solutions for a class of nonlinear evolution equations. *Rev. Mat. Complut.*, 7:325–338, 1994.
[13] M. G. Crandall. Nonlinear semigroups and evolution governed by accretive operators. In F. E. Browder, editor, *Nonlinear Functional Analysis and its Applications. Proceedings of Symposia in Pure Math.*, Vol. 45, Part 1, pp. 305–337, Amer. Math. Soc., Providence, R. I., 1986.
[14] M. G. Crandall and T. Liggett. Generation of semigroups of nonlinear transformations in Banach spaces. *Amer. J. Math.*, 93:265–298, 1971.
[15] N. Dunford and J. T. Schwartz. *Linear Operators, Part. I.* Interscience, New York, 1958.
[16] P. Hartman. On boundary value problems for systems of ordinary, nonlinear, second order differential equations. *Trans. Amer. Math. Soc.*, 96:493–509, 1960.
[17] N. Hirano. Existence of periodic solutions for nonlinear evolution equations in Hilbert spaces. *Proc. Amer. Math. Soc.*, 120:185–192, 1994.
[18] N. Hirano and N. Shioji. Existence of periodic solutions under saddle point type conditions. *J. Nonlinear Convex Anal.*, 1:115–128, 2000.
[19] S. Hu and N. S. Papageorgiou. On the existence of periodic solutions for a class of nonlinear evolution inclusions. *Boll. Unione Mat. Ital.*, 7B:591–605, 1993.
[20] S. Hu and N. S. Papageorgiou. *Handbook of Multivalued Analysis, Vol. I: Theory.* Kluwer, Dordrecht, 1997.
[21] S. Hu and N. S. Papageorgiou. *Handbook of Multivalued Analysis, Vol. II: Applications.* Kluwer, Dordrecht, 2000.
[22] S. Hu and N. S. Papageorgiou. Extremal periodic solutions for subdifferential evolution inclusions. *Differential Equations Dynam. Systems*, 10:277–304, 2002.
[23] D. Kandilakis and N. S. Papageorgiou. Periodic solutions for nonlinear evolution inclusions. *Arch. Math. (Brno)*, 32:195–209, 1996.
[24] V. Lakshmikantham and N. S. Papageorgiou. Periodic solutions for nonlinear evolution inclusions. *J. Comput. Appl. Math.*, 52:277–286, 1994.
[25] N. S. Papageorgiou and N. Yannakakis. Existence of extremal periodic solutions for nonlinear evolution inclusions. *Arch. Math. (Brno)*, 37:9–23, 2001.
[26] J. Pruss. Periodic solutions for semilinear evolution equations. *Nonlinear Anal.*, 3:221–235, 1979.
[27] H. Schaefer. Uber die methode der apriori-shranken. *Math. Ann.*, 129:415–416, 1955.
[28] N. Shioji. Existence of periodic solutions for nonlinear evolution equations with nonmonotone perturbations. *Proc. Amer. Math. Soc.*, 125:2921–2929, 1997.
[29] N. Shioji. Periodic solutions for nonlinear evolution equations in Banach spaces. *Funkcial. Ekvac.*, 42:157–164, 1999.
[30] I. I. Vrabie. *Compactness Methods for Nonlinear Evolutions.* Longman, Harlow, 1987.
[31] I. I. Vrabie. Periodic solutions for nonlinear evolution equations. *Proc. Amer. Math. Soc.*, 109:653–661, 1990.

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