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A NOTE ON EXPLICIT ITERATIVE CONSTRUCTIONS OF SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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Abstract. An explicit, general algorithmic framework for the iterative construction of the unique sunny nonexpansive retraction onto the common fixed point set of a commuting family of nonexpansive mappings in a Banach space is proposed and a proof of convergence is given. It is shown that the proposed framework leads to an improvement of our own recent result regarding the same problem.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and let X^* be its dual. The value of $y \in X^*$ at $x \in X$ will be denoted by $\langle x, y\rangle$. Let $J : X \to 2^{X^*}$ be the normalized duality map from X into the family of nonempty (by the Hahn-Banach theorem) weak-star compact convex subsets of X^*, defined by $Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for all $x \in X$. We also denote by $\mathbb{N}$ and $\mathbb{R}_+$ the sets of nonnegative integers and nonnegative real numbers, respectively. Recall that the Banach space X is said to be smooth or to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for each $x, y \in X$ with $\|x\| = \|y\| = 1$. The space X is said to have a uniformly Gâteaux differentiable norm if, for each $y \in X$ with $\|y\| = 1$, the limit (1) is attained uniformly in $x \in X$ with $\|x\| = 1$. It is known [21, Lemma 2.2] that if the norm of X is uniformly Gâteaux differentiable, then the normalized duality map J is single-valued and norm to weak-star uniformly continuous on each bounded subset of X. Let C be a nonempty, closed and convex subset of X and let E be a nonempty subset of C. A mapping $T : C \to X$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. It is called *firmly nonexpansive* if $\|Tx-Ty\| \le \|r(x-y)+(1-r)(Tx-Ty)\|$ for all $r > 0$ and all $x, y \in C$. It is said to be *averaged* if it is of the form $(1-c)I+cS$, where $0 < c < 1$ and $S : C \to X$ is nonexpansive. A mapping $Q : C \to E$ is called a *retraction* from C onto E if $Qx = x$ for all $x \in E$. A retraction Q from C onto E is called *sunny* if Q has the following property: $Q(Qx+t(x-Qx)) = Qx$ for all $x \in C$ and $t \ge 0$ with $Qx + t(x - Qx) \in C$. It is known ([6, 18] and [12, Lemma 13.1]) that in a smooth Banach space X, a retraction Q from C onto E is both sunny and nonexpansive if and only if

$$\langle x - Qx, J(y - Qx)\rangle \le 0 \tag{2}$$

for all $x \in C$ and $y \in E$. Hence there is at most one sunny nonexpansive retraction from C onto E. For example, if E is a nonempty, closed and convex subset of a

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Hilbert space H, then the nearest point projection P_E from H onto E is the unique sunny nonexpansive retraction of H onto E. This is not true for all Banach spaces, since outside Hilbert space, nearest point projections, although sunny, are no longer nonexpansive. On the other hand, sunny nonexpansive retractions do sometimes play a similar role in Banach spaces to that of nearest point projections in a Hilbert space. So an interesting problem arises: for which subsets of a Banach space does a sunny nonexpansive retraction exist? If it does exist, how can one find it? It is known [12, Theorem 13.2] that if C is a closed, convex subset of a uniformly smooth Banach space and $T : C \to C$ is nonexpansive, then the fixed point set of T is a sunny nonexpansive retract of C. More generally, Bruck [5, Theorem 2] proves that if C is a closed, convex subset of a reflexive Banach space every bounded, closed and convex subset of which has the fixed point property for nonexpansive mappings and $T : C \to C$ is nonexpansive, then its fixed point set is a nonexpansive retract of C. (It is still an open question whether all bounded, closed and convex subsets of reflexive Banach spaces have this fixed point property.) In this connection, see also [7, Theorem 1]. If X is a reflexive Banach space with a uniformly Gâteaux differentiable norm, then any nonexpansive retract of a closed, convex $C \subset X$ is, in fact, a sunny nonexpansive retracts [19, Theorem 4.1]. In the present paper we show that in certain Banach spaces X, if F is the nonempty common fixed point set of a commuting family of nonexpansive self-mappings of a closed, convex subset C of X, satisfying an asymptotic regularity condition, then it is possible to construct the sunny nonexpansive retraction Q of C onto F in an explicit, iterative way. Our work is rooted in a recent publication by Domínguez Benavides, López Acedo and Xu [11], who attempted to construct sunny nonexpansive retractions using both implicit and explicit iterative schemes (cf. the discussion in [1]). Our work improves, corrects and generalizes some of the results obtained in the above paper. We also improve upon our own recent work [2] concerning this problem. We show, in particular, that several restrictive conditions imposed on the family of mappings and the sequence of parameters in our previous work are unnecessary and can be omitted. Our paper is also related to a paper of Reich [20], where the case of a single mapping is considered. In this connection, we would also like to refer the interested reader to the results obtained by Suzuki [27], who deals with an implicit scheme for constructing the sunny nonexpansive retraction onto the common fixed point set of certain one-parameter continuous semigroups of nonexpansive mappings. For more related results in Hilbert and Banach spaces see, for instance, the papers by Aleyner and Censor [1], Bauschke [4], Deutsch and Yamada [10], Halpern [13], Lions [15], O'Hara, Pillay and Xu [16], Shimizu and Takahashi [24], Shioji and Takahashi [25], Suzuki [26], and Wittmann [28]. We state our result in the next section and provide a self-contained proof of it in Section 3. Several examples are briefly discussed in Section 4.

2. Convergence Theorem

Let X be a real Banach space, C a nonempty, closed and convex subset of X, and G_Γ an unbounded subset of $\mathbb{R}_+$. Let $\Gamma = \{T_t : t \in G_\Gamma\}$ be a commuting family of nonexpansive self-mappings of C such that the set $F = \cap_{t \in G_\Gamma} Fix(T_t)$ of the common fixed points of Γ is nonempty. We make the following assumptions.

Assumptions on the space. X is a reflexive Banach space with a uniformly Gâteaux differentiable norm such that each nonempty, bounded, closed and convex subset K of X has the common fixed point property for nonexpansive mappings; that is, any commuting family of nonexpansive self-mappings of K has a common fixed point. Note that all these assumptions are fulfilled whenever X is uniformly smooth (see [14, Theorem 1] and [7, Theorem 1]).

Assumptions on the mappings. For each bounded subset K of C and each $s \in G_\Gamma$, there holds

$$\lim_{r\to\infty} \sup_K \|T_r x - T_s T_r x\| = 0, \tag{3}$$

where $r \in G_\Gamma$. In other words, Γ is a family of mappings which is asymptotically regular, uniformly on bounded subsets of C [9]. Note that, in contrast with [2, 11], we assume neither that Γ is a semigroup nor that condition (3) is satisfied uniformly in $s \in G_\Gamma$.

Assumptions on the parameters. $\{\lambda_n\}$ is a sequence of numbers in $[0,1)$ with the following properties:

$$\lambda_n \to 0 \tag{4}$$

and

$$\prod_{n=0}^{\infty}(1-\lambda_n) = 0; \text{ equivalently, } \sum_{n=0}^{\infty}\lambda_n = \infty. \tag{5}$$

Given points $u, x_0 \in C$ and $\{r_n\} \subseteq G_\Gamma$ such that $\lim_{n\to\infty} r_n = \infty$, we define the sequence $\{x_n\}$ by

$$x_{n+1} = \lambda_n u + (1-\lambda_n) T_{r_n} x_n, \tag{6}$$

where $n \in \mathbb{N}$; we say that $\{x_n\}$ has anchor u and initial point x_0.

Theorem 2.1. *If the above assumptions on the space, mappings and parameters hold, then the sequence generated by (6) converges in norm to Qu, where Q is the unique sunny nonexpansive retraction from C onto $\cap_{t\in G_\Gamma} Fix(T_t)$.*

3. Proof

We first prove our theorem for the special case where $x_0 = u$ and then extend it to the general case. We divide our proof into a sequence of separate claims.

Claim 1. *For all $n \geq 0$ and every $f \in F$,*

$$\|x_n - f\| \leq \|u - f\|. \tag{7}$$

We proceed by induction on n. Fix $f \in F$. Clearly, (7) holds for $n = 0$. If $\|x_n - f\| \leq \|u - f\|$, then

$$\begin{aligned}
\|x_{n+1} - f\| &\leq \lambda_n \|u - f\| + (1-\lambda_n)\|T_{r_n} x_n - f\| \\
&\leq \lambda_n \|u - f\| + (1-\lambda_n)\|x_n - f\| \\
&\leq \|u - f\|,
\end{aligned}$$

as required.

Claim 2. *The following strong convergence holds:*

$$x_{n+1} - T_{r_n} x_n \to 0. \tag{8}$$

This is true because (7) guarantees that $\{x_n\}$ is bounded, which, in turn, implies that $\{T_{r_n} x_n\}$ is also bounded. The boundedness of $\{T_{r_n} x_n\}$ together with (4) imply, in view of (6), our assertion.

Claim 3. *For each fixed $s \in G_\Gamma$,*

$$T_s x_n - x_n \to 0. \tag{9}$$

Indeed, let $K = \{x_n\}$ and let $s \in G_\Gamma$. Then from (3) and (8) it follows that

$$\begin{aligned}\lim_{n\to\infty} \|T_s x_n - x_n\| &= \lim_{n\to\infty} \|T_s T_{r_{n-1}} x_{n-1} - T_{r_{n-1}} x_{n-1}\| \\ &\le \lim_{r\to\infty} \sup_K \|T_s T_r x - T_r x\| = 0,\end{aligned}$$

as asserted.

Let LIM be a Banach limit and let $\{\alpha_t\}_{t\in G_\Gamma}$ be a net in the interval $(0,1)$ such that $\lim_{t\to\infty} \alpha_t = 0$. By Banach's fixed point theorem, for each $t \in G_\Gamma$, there exists a unique point $z_t \in C$ satisfying the equation $z_t = \alpha_t u + (1-\alpha_t) T_t z_t$. The following claim is analogous to those in [2, 11].

Claim 4.

$$z_t \to Qu, \tag{10}$$

as $t \to \infty$, where $Q : C \to F$ is the unique sunny nonexpansive retraction from C onto $F = \cap_{s\in G_\Gamma} Fix(T_s)$.

To prove (10), we use a variant of the optimization method [22].

Let $\{t_n\}$ be a subsequence of G_Γ such that $\lim_{n\to\infty} t_n = \infty$. Since $\{z_{t_n}\}$ is bounded, we can define a functional g on C by

$$g(x) = LIM(\{\|z_{t_n} - x\|^2\}).$$

By (3) and the definition of $\{z_t\}$, we have for each $s \in G_\Gamma$,

$$\begin{aligned}g(T_s x) = LIM(\{\|z_{t_n} - T_s x\|^2\}) &= LIM(\{\|T_s T_{t_n} z_{t_n} - T_s x\|^2\}) \\ &\le LIM(\{\|T_{t_n} z_{t_n} - x\|^2\}) \\ &= LIM(\{\|z_{t_n} - x\|^2\}).\end{aligned}$$

In other words,

$$g(T_s x) \le g(x) \tag{11}$$

for all $s \in G_\Gamma$ and $x \in C$. Let $K = \{x \in C : g(x) = \min_C g\}$. Since g is convex and continuous, $\lim_{\|x\|\to\infty} g(x) = \infty$, and X is reflexive, K is a nonempty, closed, bounded and convex subset of C. From (11) we see that K is invariant under each T_s; that is, $T_s(K) \subset K$, $s \in G_\Gamma$. Hence K contains a common fixed point of Γ. Let $q \in K \bigcap F$ be such a common fixed point. Since q is a minimizer of g over C, it follows that for each $x \in C$,

$$0 \le \lim_{\lambda\to 0^+} \frac{1}{\lambda}(g(q + \lambda(x-q)) - g(q))$$

$$= LIM(\{\lim_{\lambda\to 0^+}\frac{1}{\lambda}(\|(z_{t_n}-q)+\lambda(q-x)\|^2-\|z_{t_n}-q\|^2)\})$$
$$= LIM(\{2\langle q-x, J(z_{t_n}-q)\rangle\}).$$

Thus,

$$LIM(\{\langle x-q, J(z_{t_n}-q)\rangle\})\le 0 \tag{12}$$

for all $x\in C$. On the other hand, for any $f\in F$,

$$z_{t_n}-f=(1-\alpha_{t_n})(T_{t_n}z_{t_n}-f)+\alpha_{t_n}(u-f).$$

It follows that

$$\|z_{t_n}-f\|^2=(1-\alpha_{t_n})\langle T_{t_n}z_{t_n}-f, J(z_{t_n}-f)\rangle+\alpha_{t_n}\langle u-f, J(z_{t_n}-f)\rangle$$
$$\le (1-\alpha_{t_n})\|z_{t_n}-f\|^2+\alpha_{t_n}\langle u-f, J(z_{t_n}-f)\rangle.$$

Hence

$$\|z_{t_n}-f\|^2\le\langle u-f, J(z_{t_n}-f)\rangle. \tag{13}$$

Combining (12) and (13) with $x=u$ and $f=q$, we get

$$LIM(\{\|z_{t_n}-q\|^2\})\le 0.$$

Hence there is a subsequence $\{z_{r_j}\}$ of $\{z_{t_n}\}$ such that $\lim_{j\to\infty}\|z_{r_j}-q\|=0$. Assume that there exists another subsequence $\{z_{p_k}\}$ of $\{z_{t_n}\}$ such that $\lim_{k\to\infty}\|z_{p_k}-\tilde{q}\|=0$, where $\tilde{q}\in K\bigcap F$. Then (13) implies that

$$\|q-\tilde{q}\|^2\le\langle u-\tilde{q}, J(q-\tilde{q})\rangle. \tag{14}$$

Similarly, we have

$$\|\tilde{q}-q\|^2\le\langle u-q, J(\tilde{q}-q)\rangle. \tag{15}$$

Adding up (14) and (15), we obtain $q=\tilde{q}$. It follows that $\{z_t\}$ converges in norm, as $t\to\infty$, to a point in F. Now we define $Q: C\to F$ by $Qu=\lim_{t\to\infty}z_t$. Then Q is a retraction from C onto F. Moreover, by (13) we get for all $f\in F$,

$$\|Qu-f\|^2\le\langle u-f, J(Qu-f)\rangle.$$

That is,

$$\langle u-Qu, J(f-Qu)\rangle\le 0$$

for all $f\in F$ and $u\in C$. Therefore Q is the unique sunny nonexpansive retraction from C onto $F=\bigcap_{t\in G_\Gamma}Fix(T_t)$ (see (2)).

Claim 5.

$$\limsup_{n\to\infty}\langle u-Qu, J(x_n-Qu)\rangle\le 0. \tag{16}$$

We first note that since the normalized duality map J is the subdifferential of $\frac{1}{2}\|\cdot\|^2$,

$$\|x\|^2+2\langle y, Jx\rangle\le\|x+y\|^2\le\|x\|^2+2\langle y, J(x+y)\rangle,$$

for all $x,y\in X$. Now let $t\in G_\Gamma$. Writing

$$z_t-x_n=\alpha_t(u-x_n)+(1-\alpha_t)(T_tz_t-x_n),$$

we obtain

$$\|z_t - x_n\|^2 \le (1-\alpha_t)^2\|T_t z_t - x_n\|^2 + 2\alpha_t\langle u - x_n, J(z_t - x_n)\rangle.$$

Since

$$\begin{aligned}\|T_t z_t - x_n\| &= \|T_t z_t - T_t x_n + T_t x_n - x_n\|\\ &\le \|T_t z_t - T_t x_n\| + \|T_t x_n - x_n\|\\ &\le \|z_t - x_n\| + \|T_t x_n - x_n\|,\end{aligned}$$

we also obtain

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-\alpha_t)^2(\|z_t - x_n\| + \|T_t x_n - x_n\|)^2\\ &\quad + 2\alpha_t\|z_t - x_n\|^2 + 2\alpha_t\langle u - z_t, J(z_t - x_n)\rangle\\ &= (1+\alpha_t^2)\|z_t - x_n\|^2 + (1-\alpha_t)^2\cdot 2\|z_t - x_n\|\cdot\|T_t x_n - x_n\|\\ &\quad + (1-\alpha_t)^2\|T_t x_n - x_n\|^2 + 2\alpha_t\langle u - z_t, J(z_t - x_n)\rangle\\ &\le (1+\alpha_t^2)\|z_t - x_n\|^2 + 2\|z_t - x_n\|\cdot\|T_t x_n - x_n\|\\ &\quad + \|T_t x_n - x_n\|^2 + 2\alpha_t\langle u - z_t, J(z_t - x_n)\rangle\\ &= (1+\alpha_t^2)\|z_t - x_n\|^2 + \|T_t x_n - x_n\|(2\|z_t - x_n\| + \|T_t x_n - x_n\|)\\ &\quad + 2\alpha_t\langle u - z_t, J(z_t - x_n)\rangle\\ &\le (1+\alpha_t^2)\|z_t - x_n\|^2 + \|T_t x_n - x_n\|M + 2\alpha_t\langle u - z_t, J(z_t - x_n)\rangle,\end{aligned}$$

where $M \ge 2\|z_t - x_n\| + \|T_t x_n - x_n\|$ for all $n \in I\!N$. Thus

$$\langle u - z_t, J(x_n - z_t)\rangle \le \frac{\alpha_t}{2}\|z_t - x_n\|^2 + \frac{M}{2\alpha_t}\|T_t x_n - x_n\|.$$

By (9), there exists n_0 such that for all $n \ge n_0$ there holds

$$\|T_t x_n - x_n\| \le \frac{2\alpha_t^2}{M}.$$

Consequently,

$$\limsup_{n\to\infty}\langle u - z_t, J(x_n - z_t)\rangle \le \frac{\alpha_t M^2}{8} + \alpha_t.$$

Since $z_t \to Qu$ and the duality map J is norm to weak-star uniformly continuous on bounded subsets of X, there exists, for each $\varepsilon > 0$, a number $t(\varepsilon) \in G_\Gamma$ such that for all $t > t(\varepsilon)$ and all $n \in N$,

$$|\langle u - z_t, J(x_n - z_t)\rangle - \langle u - Qu, J(x_n - Qu)\rangle| < \varepsilon.$$

Letting $t \to \infty$, we obtain

$$\limsup_{n\to\infty}\langle u - Qu, J(x_n - Qu)\rangle \le \varepsilon.$$

Since ε is an arbitrary positive number, this implies (16).

Now we can conclude the proof for the special case $x_0 = u$, that is, where the initial point coincides with the anchor.

Claim 6.

$$x_n \to Qu.$$

Indeed, since

$$(1-\lambda_n)(T_{r_n}x_n - Qu) = (x_{n+1} - Qu) - \lambda_n(u - Qu),$$

we have

$$\|(1-\lambda_n)(T_{r_n}x_n - Qu)\|^2 \geq \|x_{n+1} - Qu\|^2 - 2\lambda_n\langle u - Qu, J(x_{n+1} - Qu)\rangle.$$

Hence

$$\|x_{n+1} - Qu\|^2 \leq (1-\lambda_n)\|x_n - Qu\|^2 + 2(1-(1-\lambda_n))\langle u - Qu, J(x_{n+1} - Qu)\rangle$$

for each $n \geq 0$. Let $\varepsilon > 0$ be given. By (16), there exists $m \geq 0$ such that

$$\langle u - Qu, J(x_n - Qu)\rangle \leq \frac{\varepsilon}{2}$$

for all $n \geq m$. Therefore

$$\|x_{n+m} - Qu\|^2 \leq \left(\prod_{k=m}^{n+m-1}(1-\lambda_k)\right)\|x_m - Qu\|^2 + \left(1 - \prod_{k=m}^{n+m-1}(1-\lambda_k)\right)\varepsilon$$

for all $n \geq 1$. It now follows from (5) that

$$\limsup_{n\to\infty}\|x_n - Qu\|^2 = \limsup_{n\to\infty}\|x_{n+m} - Qu\|^2 \leq \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that $\{x_n\}$ converges strongly to Qu; that is, the special case is verified.

Finally, we extend the proof to the general case. Let $\{x_n\}$ be the sequence generated by (6) with an initial point x_0 (possibly different from the anchor u), and let $\{y_n\}$ be another sequence generated by (6) with an initial point $y_0 = u$. On the one hand, by the special case we have just verified,

$$y_n \to Qu.$$

On the other hand, it is easily checked that

$$\|x_n - y_n\| \leq \|x_0 - y_0\|\prod_{k=0}^{n-1}(1-\lambda_k)$$

for all $n \geq 1$. Thus, $x_n - y_n \to 0$ and, therefore, $x_n \to Qu$. This completes the proof of our theorem.

4. Examples

Let X be a general Banach space, C a nonempty, closed and convex subset of X, $G_\Gamma = \mathbb{N}$, $T : C \to C$ either an averaged or a firmly nonexpansive mapping, and let $\Gamma = \{T_t : t \in \mathbb{N}\} = \{T^n\}$ comprise the iterates of $T = T_1$. Then it follows from the triple equalities established in [3, Theorem 2.1] and [23, Theorem 1], respectively, that condition (3) is satisfied. This also holds when T is *strongly nonexpansive* in the sense of [8]. Condition (3) is also satisfied when Γ is either the semigroup generated by $-a(I-T)$, where $a > 0$ and $T : C \to C$ is nonexpansive [3, Theorem 4.3] or the semigroup generated by the (negative) subdifferential of a proper, lower semicontinuous and convex function $f : H \to (-\infty, \infty]$ defined on a Hilbert space H (cf. [17]).

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