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PARTIAL REGULARIZATION METHOD FOR EQUILIBRIUM PROBLEMS

IGOR V. KONNOV

Abstract. We consider a general equilibrium problem with a monotone cost bifunction in a reflexive Banach space setting and investigate partial Browder-Tikhonov regularization techniques for this problem. We establish several convergence results of solutions of perturbed auxiliary problems to a solution of the initial problem which generalize the convergence results for the full regularization method.

1. Introduction

Let U be a nonempty subset of a Banach space E and let $f : U \times U \to R$ be an equilibrium bifunction, i.e. $f(u, u) = 0$ for every $u \in U$. Then one can define the general equilibrium problem (EP for short) that is to find a point $u^* \in U$ such that

$$f(u^*, v) \geq 0 \quad \forall v \in U. \tag{1}$$

This problem represents a very common format for investigation and solution of various applied problems and is closely related with other general problems in Nonlinear Analysis, such as fixed point, game equilibrium, and variational inequality problems; see e.g. [1, 2] and references therein. It is well known that most approaches to obtaining joint existence and uniqueness results and to constructing effective solution methods require strengthened monotonicity type conditions on the bifunction f, whereas these conditions seem too restrictive for applications. One of the most popular approaches to overcome this drawback consists in applying regularization techniques. Usually, the standard regularization involves an addition of a regularization term with respect to all the variables (see e.g. [3, 4]), however, the partial regularization may be also sufficient for convergence of perturbed problems and it also often leads to simpler auxiliary problems. Some instances of such problems can be found in [5]. Motivated by these facts, we intend to investigate convergence properties of the partial regularization method applied to EP(1) with the monotone bifunction f. We suppose that the initial space E admits the partition $E = X \times Y$, i.e. it can be represented as a Cartesian product of spaces. Then, the partial regularization involves an addition of a regularization term with strengthened monotonicity properties only with respect to X. We consider conditions which provide convergence properties for such a method.

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2. Definitions and Preliminary Results

We now recall several definitions. The equilibrium bifunction $f : U \times U \to R$ is said to be

(i) *monotone*, if for all $u, v \in U$, we have

$$f(u,v) + f(v,u) \leq 0;$$

(ii) *strictly monotone*, if for all $u, v \in U, u \neq v$, we have

$$f(u,v) + f(v,u) < 0;$$

(see [1]). Also, we say that $f : U \times U \to R$ is a *uniformly monotone* bifunction, if for all $u, v \in U$, we have

$$f(u,v) + f(v,u) \leq -\theta(\|u - v\|)\|u - v\|,$$

where $\theta : R \to R$ is a continuous increasing function such that $\theta(0) = 0$ and $\theta(\tau) \to +\infty$ as $\tau \to +\infty$. If $\theta(\tau) = \mu\tau$ for some $\mu > 0$, then f is a strongly monotone bifunction. We recall that a function $\varphi : U \to R$ is said to be *hemicontinuous* if its restriction on linear segments of U is continuous.

In what follows, we shall use the following basic assumptions on the EP(1).

(A1) *U is a nonempty, convex, and closed subset of a reflexive Banach space E which can be represented as a Cartesian product of reflexive Banach spaces, i.e. $E = X \times Y$.*

(A2) *$f : U \times U \to R$ is a monotone equilibrium bifunction such that $f(\cdot, v)$ is hemicontinuous for each $v \in U$ and that $f(u, \cdot)$ is convex and lower semicontinuous for each $u \in U$.*

(A3) *The solutions set U^* of EP(1) is nonempty.*

We now recall several useful properties of EPs.

Proposition 2.1. [1, Theorem 10.1] *If* **(A1)** *and* **(A2)** *are fulfilled, then U^* coincides with the solutions set of the dual (Minty) EP: Find $v^* \in U$ such that*

$$f(u, v^*) \leq 0 \quad \forall u \in U, \tag{2}$$

and it is convex and closed.

Observe that the second part of **(A1)** means that each element u of E admits the corresponding partition, i.e. $u = (x, y)$ where $x \in X$ and $y \in Y$. Then we can introduce bifunctions on subspaces and investigate existence and uniqueness results for corresponding EPs. Namely, let us consider the perturbed EP: Find $u^\varepsilon \in U$ such that

$$f(u^\varepsilon, u) + \varepsilon h(x^\varepsilon, x) \geq 0 \quad \forall u \in U, \tag{3}$$

where $u^\varepsilon = (x^\varepsilon, y^\varepsilon)$, $h : X \times X \to R$ is an equilibrium bifunction such that $h(\cdot, x)$ is hemicontinuous for each $x \in X$ and that $h(x, \cdot)$ is convex and lower semicontinuous for each $x \in X$, $\varepsilon > 0$. For brevity, set

$$f_\varepsilon(u,v) = f(u,v) + \varepsilon h(x,x')$$

where $u = (x,y), v = (x',y')$. If **(A1)** and **(A2)** are fulfilled, then $f_\varepsilon(\cdot, v)$ is hemicontinuous for each $v \in U$ and $f_\varepsilon(u, \cdot)$ is convex and lower semicontinuous for each $u \in U$. We denote by U^ε the solutions set of EP(3).

Lemma 2.1. *Suppose* **(A1)** *and* **(A2)** *are fulfilled and $h : X \times X \to R$ is strictly monotone. Then the solutions set U^ε is of the form*

$$U^\varepsilon = \{x^\varepsilon\} \times L, \quad \textit{where } L \subseteq Y. \tag{4}$$

Proof. Take arbitrary elements $u = (x, y) \in U^\varepsilon$ and $v = (x', y') \in U^\varepsilon$. Then, by definition, we have

$$f_\varepsilon(u, v) \geq 0 \quad \text{and } f_\varepsilon(v, u) \geq 0.$$

Adding these inequalities and taking into account the monotonicity of f and h, we obtain

$$\varepsilon[h(x, x') + h(x', x)] \geq f(u, v) + f(v, u) + \varepsilon[h(x, x') + h(x', x)] \geq 0.$$

If $x \neq x'$, this inequality leads to the contradiction:

$$0 > h(x, x') + h(x', x) \geq 0,$$

and the result follows. □

Set

$$X_U = \{x \mid \exists y \in Y,\ (x, y) \in U\} \quad \text{and } Y_U = \{y \mid \exists x \in X,\ (x, y) \in U\}.$$

We now present existence and uniqueness results for EPs (1) and (3). First we specialize the existence result from [2] for EP(1).

Proposition 2.2. *Suppose that* **(A1)** *and* **(A2)** *are fulfilled and the monotonicity of f is replaced with its uniform monotonicity. Then U^* is a nonempty singleton.*

Proof. We apply Theorem 1 in [2] with coercivity condition d) where $h \equiv 0$, i.e. we have to show that

$$[f(u, \tilde{v}) + f(\tilde{v}, u)]/\|u - \tilde{v}\| \to -\infty \quad \text{if } \|u - \tilde{v}\| \to +\infty, u \in U$$

for a fixed $\tilde{v} \in U$. However, this property follows from the uniform monotonicity of f. Therefore, EP(1) is solvable. The uniqueness follows from the fact that f is also strictly monotone. □

A modification of this result for EP(3) can be formulated as follows.

Proposition 2.3. *Suppose that* **(A1)** *and* **(A2)** *are fulfilled, h is uniformly monotone, and that Y_U is bounded. Then EP(3) is solvable and (4) holds.*

Proof. Clearly, (4) follows from Lemma 2.1, since h is now strictly monotone. Again, we apply Theorem 1 in [2] with coercivity condition d) and $h \equiv 0$. It means that we have to only show that

$$[f_\varepsilon(u, \tilde{v}) + f_\varepsilon(\tilde{v}, u)] \ / \ \|u - \tilde{v}\| \to -\infty \quad \text{if } \|u - \tilde{v}\| \to +\infty, \quad u \in U \tag{5}$$

for a fixed $\tilde{v}$ in U. Note that

$$\begin{aligned} f_\varepsilon(u, \tilde{v}) + f_\varepsilon(\tilde{v}, u) &= f(u, \tilde{v}) + f(\tilde{v}, u) + \varepsilon[h(x, \tilde{x}) + h(\tilde{x}, x)] \\ &\leq -\varepsilon\theta(\|x - \tilde{x}\|)\|x - \tilde{x}\|, \end{aligned}$$

where $u = (x, y)$, $\tilde{v} = (\tilde{x}, \tilde{y})$. However, $\|u - \tilde{v}\| = \sqrt{\|x - \tilde{x}\|^2 + \|y - \tilde{y}\|^2}$ and $\|y - \tilde{y}\| \leq C < \infty$ for all $y \in Y_U$. It means that $\|u - \tilde{v}\| \to +\infty$ is equivalent to $\|x - \tilde{x}\| \to +\infty$ and it implies that (5) holds. The proof is complete. □

3. Convergence Results

In this section, we establish the basic approximation properties of the partial regularization method represented by the perturbed problem (3). We say that the equilibrium bifunction $h : X \times X \to R$ is *uniformly bounded* if there exists a nondecreasing function $\sigma : R \to R$ with $\sigma(0) = 0$, and $\sigma(\tau) > 0$ for every $\tau > 0$ such that for all $x, x' \in X$ we have

$$|h(x, x')| \leq \sigma(\|x\|)\|x - x'\|.$$

We now collect all the assumptions on the perturbation bifunction h.

(H1) *$h : X \times X \to R$ is an equilibrium bifunction which is uniformly monotone with function θ and uniformly bounded with function σ, $h(\cdot, x)$ is hemicontinuous for each $x \in X$ and $h(x, \cdot)$ is convex and lower semicontinuous for each $x \in X$.*

We first establish a convergence result under the boundedness condition on Y_U.

Theorem 3.1. *Suppose that assumptions* **(A1)** – **(A3)** *and* **(H1)** *are fulfilled, Y_U is bounded, and that the sequence $\{u^{\varepsilon_k}\}$, $u^{\varepsilon_k} = (x^{\varepsilon_k}, y^{\varepsilon_k})$, is constructed in conformity with (3), where $\{\varepsilon_k\} \searrow 0$. Then $\{x^{\varepsilon_k}\}$ converges strongly to the point x_n^* and $\{y^{\varepsilon_k}\}$ has weak limit points and all these points belong to Y^* such that $\{x_n^*\} \times Y^* \subseteq U^*$ and*

$$h(x_n^*, x) \geq 0 \quad \textit{for every} \quad x \quad \textit{such that}\ \exists y \in Y,\ (x, y) \in U^*. \tag{6}$$

Proof. First we note that the set U^* is nonempty, convex and closed due to Proposition 2.1. Also, along the lines of the proof of Proposition 2.3, we see that problem (6) is solvable and its solutions set U^n is of the form

$$U^n = \{x_n^*\} \times L \subseteq U^*, \quad \text{where} \ \ L \subseteq Y.$$

Next, due to Proposition 2.3, EP(3) is solvable for each $\varepsilon > 0$ and its solutions set is of the form

$$\{x^\varepsilon\} \times L_\varepsilon \subseteq U \quad \text{and} \quad L_\varepsilon \subseteq Y.$$

Take any $u^* = (x^*, y^*) \in U^*$, then

$$f(u^*, u^\varepsilon) \geq 0 \quad \text{and } f(u^\varepsilon, u^*) + \varepsilon h(x^\varepsilon, x^*) \geq 0.$$

Adding these inequalities gives

$$\varepsilon h(x^\varepsilon, x^*) \geq -[f(u^*, u^\varepsilon) + f(u^\varepsilon, u^*)] \geq 0.$$

Since h is uniformly monotone, it follows that

$$\begin{aligned} -\varepsilon h(x^*, x^\varepsilon) &= -\varepsilon[h(x^*, x^\varepsilon) + h(x^\varepsilon, x^*)] + \varepsilon h(x^\varepsilon, x^*) \\ &\geq \varepsilon\theta(\|x^* - x^\varepsilon\|)\|x^* - x^\varepsilon\|, \end{aligned} \tag{7}$$

and, by the uniform boundedness of h, we have

$$\sigma(\|x^*\|) \geq \theta(\|x^\varepsilon - x^*\|).$$

It means that the sequence $\{x^{\varepsilon_k}\}$ is bounded, but so is $\{y^{\varepsilon_k}\}$, hence these sequences have weak limit points. Note that f_ε is clearly monotone, and, in view of Proposition 2.1, we have

$$f_\varepsilon(u, u^\varepsilon) \leq 0 \quad \forall u \in U.$$

Since $f_\varepsilon(u,\cdot)$ is also convex and lower semicontinuons, it is weakly lower semicontinuons, and, for each pair of weak limit points (x',y') of $\{(x^{\varepsilon_k},y^{\varepsilon_k})\}$, we obtain

$$0 \geq \limsup_{k\to\infty}[f(u,u^{\varepsilon_k}) + \varepsilon_k h(x,x^{\varepsilon_k})] = \limsup_{k\to\infty} f(u,u^{\varepsilon_k}) \geq f(u,u')$$

for every $u \in U$, i.e. $u' = (x',y')$ solves EP(1) due to Proposition 2.1. Therefore, all the weak limit points of $\{u^{\varepsilon_k}\}$ belong to U^*. Using (7) with $\varepsilon = \varepsilon_k$ and $x^* = x_n^*$, we have

$$-h(x_n^*, x^{\varepsilon_k}) \geq \theta(\|x_n^* - x^{\varepsilon_k}\|)\|x_n^* - x^{\varepsilon_k}\|,$$

Setting $k \to \infty$ in this inequality gives

$$0 \geq -h(x_n^*, x') \geq \theta(\|x_n^* - x'\|)\|x_n^* - x'\|,$$

where x' is an arbitrary weak limit of $\{x^{\varepsilon_k}\}$. It means that x' coincides with x_n^* and that $\{x^{\varepsilon_k}\}$ converges strongly to x_n^*. The proof is complete. □

This result extends the known convergence properties of the full regularization methods (see [6, 4]).

We can somewhat strengthen the above assertion under the additional properties of f.

Corollary 3.1. *Suppose that all the assumptions of Theorem 3.1 are fulfilled and that f is strictly monotone with respect to y, i.e. for each pair of points $u = (x,y)$ and $v = (x',y')$ in U, we have*

$$f(u,v) + f(v,u) < 0 \quad \textit{if } y \neq y'.$$

Then, $\{x^{\varepsilon_k}\}$ converges strongly to the point x_n^* and $\{y^{\varepsilon_k}\}$ converges weakly to the point y^* such that $(x_n^*, y^*) \in U^*$ and (6) holds.

Proof. In this case, using the argument as that in the proof of Lemma 2.1, we see that

$$U^* = K \times \{y^*\},$$

i.e. $Y^* = \{y^*\}$. Hence, all the weak limit points of $\{y^{\varepsilon_k}\}$ coincide with y^* and the result follows. □

We now establish the convergence result for the unbounded case, which utilizes partial uniform monotonicity properties of f.

Theorem 3.2. *Suppose that assumptions* **(A1)** – **(A3)** *and* **(H1)** *are fulfilled and that f is uniformly monotone with respect to y, i.e. for each pair of points $u = (x,y)$ and $v = (x',y')$ in U, it holds that*

$$f(u,v) + f(v,u) \leq -\beta(\|y - y'\|)\|y - y'\|,$$

where $\beta : R \to R$ is a continuous increasing function such that $\beta(0) = 0$ and $\beta(\tau) \to +\infty$ as $\tau \to \infty$. Then the sequence $\{u^{\varepsilon_k}\}$, $u^{\varepsilon_k} = (x^{\varepsilon_k}, y^{\varepsilon_k})$, constructed in conformity with (3), where $\{\varepsilon_k\} \searrow 0$, converges strongly to the point $u^* = (x_n^*, y^*)$ such that $u^* \in U^*$ and (6) holds with $y = y^*$.

Proof. Since f is now strictly monotone in y, using the argument as that in the proof of Lemma 2.1, we conclude that

$$U^* = K \times \{y^*\},$$

whereas Proposition 2.1 implies that U^* is convex and closed. Next, along the lines of the proof of Proposition 2.3, we obtain that (6) has the unique solution $x_n^* \in K$ where $y = y^*$. We now show that each auxiliary EP (3) has a unique solution if $\varepsilon > 0$. We apply Theorem 1 in [2] with coercivity condition d) and $h \equiv 0$. In other words, we proceed to show that (5) holds. Note that, for a fixed $\tilde{v} = (\tilde{x}, \tilde{y}) \in U$, we have

$$f_\varepsilon(u, \tilde{v}) + f_\varepsilon(\tilde{v}, u) \leq -\varepsilon\theta(\|x - \tilde{x}\|)\|x - \tilde{x}\| - \beta(\|y - \tilde{y}\|)\|y - \tilde{y}\|,$$

where $u = (x, y)$. Let us consider two cases.

Case 1. Let $\|u - \tilde{v}\| \to +\infty$ and $\|y - \tilde{y}\| \leq C < \infty$. Then, setting $\tau = \|x - \tilde{x}\|$ and $a = \|y - \tilde{y}\|$, we have

$$\frac{f_\varepsilon(u, \tilde{v}) + f_\varepsilon(\tilde{v}, u)}{\|u - \tilde{v}\|} \leq -\frac{\varepsilon\theta(\tau)\tau + \beta(a)a}{\sqrt{\tau^2 + a^2}} = -\frac{\varepsilon\theta(\tau) + \beta(a)a\ /\tau}{\sqrt{1 + a^2\ /\ \tau^2}} \to -\infty,$$

i.e. (5) holds. The case when $\|u - \tilde{v}\| \to +\infty$ and $\|x - \tilde{x}\| \leq C < \infty$ can be considered similarly.

Case 2. Let $\|u - \tilde{v}\| \to +\infty$, but $\|x - \tilde{x}\| \to +\infty$ and $\|y - \tilde{y}\| \to +\infty$. Then, setting $\tau = \|x - \tilde{x}\|$ and $\sigma = \|y - \tilde{y}\|$, we have

$$\begin{aligned}\frac{f_\varepsilon(u, \tilde{v}) + f_\varepsilon(\tilde{v}, u)}{\|u - \tilde{v}\|} &\leq -\frac{\varepsilon\theta(\tau)\tau + \beta(\sigma)\sigma}{\sqrt{\tau^2 + \sigma^2}} \leq -\min\{\varepsilon\theta(\tau), \beta(\sigma)\}\frac{\tau + \sigma}{\sqrt{\tau^2 + \sigma^2}} \\ &\leq -\min\{\varepsilon\theta(\tau), \beta(\sigma)\} \to -\infty,\end{aligned}$$

i.e. (5) also holds. It means that EP (3) is solvable. Using the argument as that in the proof of Lemma 2.1, we conclude that EP (3) has the unique solution $u^\varepsilon = (x^\varepsilon, y^\varepsilon)$, i.e. the regularization method is also well defined.

Take any point $\tilde{u} = (\tilde{x}, y^*) \in U^*$, then $f(\tilde{u}, u^\varepsilon) \geq 0$ and $f(u^\varepsilon, \tilde{u}) + \varepsilon h(x^\varepsilon, \tilde{x}) \geq 0$. Adding these inequalities gives

$$\varepsilon h(x^\varepsilon, \tilde{x}) \geq -[f(\tilde{u}, u^\varepsilon) + f(u^\varepsilon, \tilde{u})] \geq \beta(\|y^\varepsilon - y^*\|)\|y^\varepsilon - y^*\|,$$

therefore, for each $\varepsilon \leq 1$, we obtain

$$h(x^\varepsilon, \tilde{x}) \geq \beta(\|y^\varepsilon - y^*\|)\|y^\varepsilon - y^*\|.$$

Since h is uniformly monotone, we have

$$\begin{aligned}-h(\tilde{x}, x^\varepsilon) &= -[h(\tilde{x}, x^\varepsilon) + h(x^\varepsilon, \tilde{x})] + h(x^\varepsilon, \tilde{x}) \\ &\geq \theta(\|x^\varepsilon - \tilde{x}\|)\|x^\varepsilon - \tilde{x}\| + \beta(\|y^\varepsilon - y^*\|)\|y^\varepsilon - y^*\|.\end{aligned}$$

Using the fact that h is also uniformly bounded, we obtain

$$\begin{aligned}\sigma(\|\tilde{x}\|)\|\tilde{x} - x^\varepsilon\| &\geq \theta(\|x^\varepsilon - \tilde{x}\|)\|x^\varepsilon - \tilde{x}\| + \beta(\|y^\varepsilon - y^*\|)\|y^\varepsilon - y^*\|. \\ &\geq \theta(\|x^\varepsilon - \tilde{x}\|)\|x^\varepsilon - \tilde{x}\|.\end{aligned}$$

It follows that both $\{x^{\varepsilon_k}\}$ and $\{y^{\varepsilon_k}\}$ are bounded, i.e. they have weak limit points. Next, similar to the proof of Theorem 3.1, we obtain that all the weak limit points of $\{u^{\varepsilon_k}\}$ coincide with $u^* = (x_n^*, y^*)$, i.e. $\{u^{\varepsilon_k}\}$ converges weakly to u^*, and that $\{x^{\varepsilon_k}\}$ converges strongly to x_n^*. By the above, we conclude that $\{y^{\varepsilon_k}\}$ also converges strongly to y^*. The proof is complete. □

It is well known that the regularization method may be treated as a penalty method for solving sequential (lexicographic) problems in optimization; see [3, 4] and references therein. Mathematical programs with equilibrium constraints (see e.g. [7]) and bi-level variational inequalities (see e.g. [8]) having a great number of various applications represent further extensions of such problems. Nevertheless, they also fall into format (6) so that the above results reveal new opportunities in solving partial sequential (lexicographic) problems.

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Igor V. Konnov
Department of Applied Mathematics, Kazan University, ul. Kremlevskaya, 18, Kazan 420008, Russia
E-mail address: ikonnov@ksu.ru