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A CLASS OF MULTI-VALUED QUASI-VARIATIONAL INEQUALITIES

K. R. KAZMI, M. I. BHAT, AND F. A. KHAN

Abstract. In this paper, we prove the existence of solution and discuss the convergence criteria of iterative algorithms for a class of multi-valued quasi-variational inequalities in Banach spaces. The theorems presented in this paper generalize, improve and unify the known results in the literature.

1. Introduction

Iterative algorithms play a central role in the approximation-solvability, especially of nonlinear variational inequalities as well as of nonlinear equations in several fields such as applied mathematics, mathematical programming, mathematical finance, control theory and optimization, engineering sciences, see for example [4].

Recently, Noor *et al.* [9] considered a class of multi-valued quasi-variational inequalities in Banach spaces and suggested an iterative algorithm using retraction mapping. They proved the existence of solution and discussed the convergence criteria of an iterative algorithm for the class of multi-valued quasi-variational inequalities. It is remarked that the main result, Theorem 3.2, of Noor *et al.* [9] does not serve the purpose, which is to be discussed in Section 3.

Very recently, He [6] has shown that if a multi-valued mapping S defined on a Banach space is lower semicontinuous and ϕ-strongly accretive then the value of S at any point of its domain is a singleton set, see Lemma 2.3 below.

In view of above result of He [6], one can observe that the results of Noor [7,8] are, in reality, for single-valued variational inequalities inspite of involving multi-valued mappings. Therefore, the improvement of the methods developed by Noor [7,8] is needed to study the existence of solution and the convergence criteria of iterative algorithms for monotone multi-valued variational inequalities.

In this paper, we consider a class of multi-valued quasi-variational inequalities (in short, MQVI) in uniformly smooth Banach space and suggest iterative algorithms for MQVI. Further, we prove the existence of solution and discuss the convergence criteria for the iterative algorithms for MQVI. The theorems presented in this paper generalize, improve and unify the results given in [7,8,9].

2. Preliminaries

We assume that E is a real Banach space equipped with norm $\|\cdot\|$; $E^\star$ is the topological dual space of E; $\langle\cdot,\cdot\rangle$ is the dual pair between E and $E^\star$; 2^E is the family of all nonempty subsets of E. $CB(E)$ is the family of all nonempty closed

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and bounded subsets of E; $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A,B)=\max\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\},\ A,B\in CB(E);$$

and $J : E \to 2^{E^\star}$ is the normalized duality mapping defined by

$$J(u)=\{f\in E^\star \mid \langle u,f\rangle=\|u\|^2=\|f\|^2_{E^\star}\},\quad \forall u\in E.$$

We note that if E is a smooth Banach space then the J becomes single-valued mapping, and if $E\equiv H$, a Hilbert space, then J becomes identity mapping.

From now onwards, unless or otherwise stated, we assume that E is a *real uniformly smooth Banach space.* Let $T,A,S : E\to CB(E)$ be three multi-valued mappings; let $g : E\to E$ be a single-valued mapping and let $N(\cdot,\cdot,\cdot) : E\times E\times E\to E$ be a nonlinear single-valued mapping. Let $K : E\to 2^E$ be a multi-valued mapping such that for any $x\in E,\ K(x)$ is a closed convex set in E, then MQVI problem is to find $x\in E,\ u\in T(x),\ v\in A(x),\ w\in S(x)$ such that $g(x)\in K(x)$ and

$$\langle N(u,v,w), J(y-g(x))\rangle \ \geq\ 0,\ \forall y\in K(x). \tag{1}$$

Some special cases.

(1) If S is single-valued mapping, $S\equiv g$ and $N(u,v,g(x))\equiv g(x)+M(u,v)$, $\forall u,v,x\in E$, where $M : E\times E\to E$, MQVI (1) reduces to the problem of finding $x\in E,\ u\in T(x),\ v\in A(x)$ such that $g(x)\in K(x)$ and

$$\langle g(x)+M(u,v), J(y-g(x))\rangle\geq 0,\quad \forall y\in K(x). \tag{2}$$

Problem (2) in the setting of *Hilbert space* has been considered by Noor [7].

(2) If $N(u,v,w)=M(u,v)$, $\forall u,v,w\in E$, where $M : E\times E\to E$, MQVI (1) reduces to the problem of finding $x\in E,\ u\in T(x),\ v\in A(x)$ such that $g(x)\in K(x)$ and

$$\langle M(u,v), J(y-g(x))\rangle\geq 0,\ \forall y\in K(x). \tag{3}$$

Problem (3) in the setting of *Hilbert space* has been considered by Noor [8].

(3) If $K(x)\subseteq Range(g)$, $\forall x\in E$ then for each $z\in K(x)$, $\exists y\in E$ such that $z=g(y)$ and if $N(u,v,w)=M(u,v)$ $\forall u,v,w\in E$, where $M : E\times E\to E$, MQVI (1) reduces to the problem of finding $x\in E,\ u\in T(x),\ v\in A(x)$ such that $g(x)\in K(x)$ and

$$\langle M(u,v), J(g(y)-g(x))\rangle\geq 0,\ \forall g(y)\in K(x). \tag{4}$$

Problem (4) has been considered by Noor *et al.* [9].

(4) If $E\equiv H$, a real Hilbert space, then MQVI (1) reduces to the problem of finding $x\in H,\ u\in T(x),\ v\in A(x),\ w\in S(x)$ such that $g(x)\in K(x)$ and

$$\langle N(u,v,w), y-g(x)\rangle \ \geq\ 0,\ \forall y\in K(x). \tag{5}$$

For the applications and numerical methods of special cases of MQVI(1), see [7,8,9,3,4] and the references therein.

Now, we give the following concepts and results which are needed in the sequel:

Definition 2.1. A single-valued mapping $g : E\to E$ is said to be

(a) *k-strongly accretive*, if there exists $k>0$ such that

$$\langle g(x)-g(y), J(x-y)\rangle\geq k\|x-y\|^2,\ \forall x,y\in E\ ;$$

(b) *δ-Lipschitz continuous*, if there exists $\delta > 0$ such that

$$\|g(x) - g(y)\| \leq \delta\|x - y\|, \ \forall x, y \in E.$$

It is remarked that if g is *k-strongly accretive*, then g satisfies

$$\|g(x) - g(y)\| \geq k\|x - y\|, \ \forall x, y \in E. \tag{6}$$

The mapping g with condition (6) is called *k-expanding.*

Definition 2.2. A multi-valued mapping $T : E \to CB(E)$ is said to be

(a) *σ-strongly accretive*, if there exists $\sigma > 0$ such that

$$\langle u - v, J(x - y)\rangle \geq \sigma\|x - y\|^2, \ \forall x, y \in E, u \in T(x), v \in T(y);$$

(b) *η-H-Lipschitz continuous*, if there exists $\eta > 0$ such that

$$H(T(x), T(y)) \leq \eta\|x - y\|, \ \forall x, y \in E.$$

Definition 2.3. Let $g : E \to E$; $T, A, S : E \to CB(E)$. A mapping $N(\cdot, \cdot, \cdot) : E \times E \times E \to E$ is said to be

(a) *α-strongly g-accretive*, with respect to T, A and S, if there exists $\alpha > 0$ such that

$$\langle N(u_1, v_1, w_1) - N(u_2, v_2, w_2), J(g(x) - g(y))\rangle \geq \alpha\|x - y\|^2, \ \forall x, y \in E,$$
$$u_1 \in T(x), \ u_2 \in T(y); v_1 \in A(x), v_2 \in A(y), w_1 \in S(x), w_2 \in S(y);$$

(b) *(β, γ, ξ)-Lipschitz continuous*, if there exist $\beta, \gamma, \xi > 0$ such that

$$\|N(x_1, y_1, z_1) - N(x_2, y_2, z_2)\| \leq \beta\|x_1 - x_2\| + \gamma\|y_1 - y_2\| + \xi\|z_1 - z_2\|,$$
$$\forall x_1, x_2, y_1, y_2, z_1, z_2 \in E.$$

Remark 2.1 ([9]).

(a) Let E be a real Banach space. Let $G : E \to CB(E)$ and let $\epsilon > 0$ be any real number, then for every $x, y \in E$ and $u_1 \in G(x)$, there exists $u_2 \in G(y)$ such that

$$\|u_1 - u_2\| \leq H(G(x), G(y)) + \epsilon\|x - y\|.$$

(b) Let $G : E \to CB(E)$ and let $\delta > 1$ be any real number, then for every $x, y \in E$ and $u_1 \in G(x)$, there exists $u_2 \in G(y)$ such that

$$\|u_1 - u_2\| \leq \delta H(G(x), G(y)).$$

We note that if $G : E \to C(E)$, where $C(E)$ denotes the family of all nonempty compact subsets of E, then Remark 2.1(a)-(b) is true for $\epsilon = 0$ and $\delta = 1$, respectively.

Definition 2.4 ([1,2,5])**.** Let $K \subset E$ be a nonempty closed convex set. A mapping $P_K : E \to K$ is said to be

(a) *retraction*, if

$$P_K^2 = P_K;$$

(b) *nonexpansive*, if

$$\|P_K x - P_K y\| \leq \|x - y\|, \ \forall x, y \in E;$$

(c) *sunny*, if

$$P_K(P_Kx - t(x - P_Kx)) = P_Kx, \ \forall x \in E, t \in R.$$

Lemma 2.1 ([5]). *A retraction P_K is sunny and nonexpansive if and only if*

$$\langle x - P_K(x), J(P_K(x) - y)\rangle \geq 0, \ \forall x, y \in E.$$

Lemma 2.2 ([1,2,5]). *Let $J : E \to E^\star$ be a normalized duality mapping. Then for all $x, y \in E$, we have*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle$,

(ii) $\langle x-y, Jx-Jy\rangle \leq 2d^2\rho_E(4\|x-y\|/d)$, *where* $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$ $\rho_E(t) = \sup\{\frac{\|x\|+\|y\|}{2} - 1 : \|x\| = 1, \|y\| = t\}$ *is called the modulus of smoothness of* E.

Definition 2.5. A mapping $F : E \to 2^E$ is said to be ϕ-strongly accretive if there exists a strictly increasing continuous function. Let $\phi : R_+ \to R_+$ with $\phi(0) = 0$ such that, for any $x, \ y \in E$,

$$\langle u_1 - u_2, J(x - y)\rangle \geq \phi(\|x - y\|)\|x - y\|, \ \forall u_1 \in F(x), \ u_2 \in F(y).$$

Lemma 2.3 ([6]). *Let E be a real Banach space and $F : E \to 2^E \setminus \{\emptyset\}$ be a lower semicontinuous and ϕ-strongly accretive mapping, then for any $x \in E, Fx$ is a one-point set.*

Lemma 2.4. *MQVI (1) has a solution $x \in E, u \in T(x), v \in A(x), w \in S(x), g(x) \in K(x)$ if and only if $x \in E, u \in T(x), v \in A(x), w \in S(x), g(x) \in K(x)$ satisfies the relation*

$$g(x) = P_{K(x)}[g(x) - \rho N(u, v, w)],$$

where $\rho > 0$ is a constant.

Proof. Proof is directly followed from Lemma 2.1. □

Assumption 2.1. For all $x, y, z \in E$, the operator $P_{K(x)}$ satisfies the condition:

$$\|P_{K(x)}(z) - P_{K(y)}(z)\| \leq \nu\|x - y\|,$$

where $\nu > 0$ is a constant.

3. Main results

Using Lemma 2.4 and Remark 2.1(a)–(b), we give the following iterative algorithms for MQVI (1) in Banach spaces.

Algorithm 3.1. For given $x_0 \in E, \ u_0 \in T(x_0), v_0 \in A(x_0), w_0 \in S(x_0)$, and given $\epsilon \in (0, 1)$, compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ defined by the iterative schemes:

$$g(x_{n+1}) = P_{K(x_n)}[g(x_n) - \rho N(u_n, v_n, w_n)], \ n = 0, 1, 2, ..., \tag{7}$$

$$u_n \in T(x_n) : \|u_{n+1} - u_n\| \leq H(T(x_{n+1}), T(x_n)) + \epsilon^{n+1}\|x_{n+1} - x_n\|, \tag{8}$$

$$v_n \in A(x_n) : \|v_{n+1} - v_n\| \leq H(A(x_{n+1}), A(x_n)) + \epsilon^{n+1}\|x_{n+1} - x_n\|, \tag{9}$$

$$w_n \in S(x_n) : \|w_{n+1} - w_n\| \leq H(S(x_{n+1}), S(x_n)) + \epsilon^{n+1}\|x_{n+1} - x_n\|, \tag{10}$$

where $\rho > 0$ is a constant.

Algorithm 3.2. For given $x_0 \in E,\ u_0 \in T(x_0), v_0 \in A(x_0), w_0 \in S(x_0)$, and given $\epsilon \in (0,1)$, compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}$ defined by the iterative schemes:

$$g(x_{n+1}) = P_{K(x_n)}[g(x_n) - \rho N(u_n, v_n, w_n)],\ n = 0, 1, 2, ...,$$

$$u_n \in T(x_n) : \|u_{n+1} - u_n\| \leq (1 + (1+n)^{-1})H(T(x_{n+1}), T(x_n)),$$

$$v_n \in A(x_n) : \|v_{n+1} - v_n\| \leq (1 + (1+n)^{-1})H(A(x_{n+1}), A(x_n)),$$

$$w_n \in S(x_n) : \|w_{n+1} - w_n\| \leq (1 + (1+n)^{-1})H(S(x_{n+1}), S(x_n)),$$

where $\rho > 0$ is a constant.

Special cases. If $N(u, v, w) \equiv M(u, v),\ \forall u, v, w \in E$, where $M : E \times E \to E$, then Algorithm 3.1 reduces to the following algorithm:

Algorithm 3.3 ([9])**.** For given $x_0 \in E,\ u_0 \in T(x_0), v_0 \in A(x_0)$, and given $\epsilon \in (0,1)$, compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}$ defined by the iterative schemes:

$$g(x_{n+1}) = P_{K(x_n)}[g(x_n) - \rho M(u_n, v_n)],\ n = 0, 1, 2, ...,$$

$$u_n \in T(x_n) : \|u_{n+1} - u_n\| \leq H(T(x_{n+1}), T(x_n)) + \epsilon^{n+1}\|x_{n+1} - x_n\|,$$

$$v_n \in A(x_n) : \|v_{n+1} - v_n\| \leq H(A(x_{n+1}), A(x_n)) + \epsilon^{n+1}\|x_{n+1} - x_n\|,$$

where $\rho > 0$ is a constant.

For appropriate choices of the mappings T, S, A, g, K, N, and space E, one can obtain many new and known algorithms as special case from Algorithms 3.1 & 3.2, see [2,7-9] and the references therein.

Now, we recall the main result (Theorem 3.2) of Noor *et al.* [9]:

Theorem 3.1 ([9])**.** *Let E be a real uniformly smooth Banach space. Let the operator $M(\cdot,\cdot)$ be β-Lipschitz continuous and γ-Lipschitz continuous in the first and second arguments, respectively. Let the operator g be δ-Lipschitz continuous and k-strongly accretive. Assume that the operators $T, A : E \to CB(E)$ are μ-H-Lipschitz continuous and η-H-Lipschitz continuous, respectively. If the Assumption 2.1 holds and*

$$0 < \rho < \frac{\sqrt{2k-1} - (\delta + \nu)}{\beta\mu + \gamma\eta};\ k > 1/2, \tag{11}$$

then there exist $x \in E,\ u \in T(x),\ v \in A(x)$, satisfying (4) and the iterative sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.3 converge to x, u, and v, respectively.

Remark 3.1. The condition (11) of Theorem 3.1 does not serve the convergence criteria for the sequences $\{x_n\}, \{u_n\}$, and $\{v_n\}$ generated by Algorithm 3.3. Evidently $k \leq \delta$. Next claim that $\sqrt{2k-1} < (\delta + \nu)$. On the contrary, we assume that $\sqrt{2k-1} \geq (\delta + \nu)$,

$$2k - 1 \geq (\delta + \nu)^2 \geq (k + \nu)^2$$

$$-(k-1)^2 \geq (2k + \nu)\nu$$

$$(k-1)^2 \leq -(2k + \nu)\nu,$$

which is not possible, since $(2k + \nu)\nu \geq 0$.

Hence, $\sqrt{2k-1} < (\delta + \nu)$. Thus (11) yields $0 < \rho < 0$ which is impossible.

Next, we prove the main result of the paper.

Theorem 3.2. *Let E be a real uniformly smooth Banach space with $\rho_E(t) \le ct^2$ for some $c > 0$. Let the mappings $T, A, S : E \to CB(E)$ be μ-H-Lipschitz, η-H-Lipschitz and σ-H-Lipschitz continuous, respectively and let the mapping $N(\cdot,\cdot,\cdot)$ be α-strongly g-accretive with respect to T, A and S, and (β,γ,ξ)-Lipschitz continuous. Let the mapping g be δ-Lipschitz continuous and let the mapping $(g - I) : E \to E$ be k-strongly accretive, where $I : E \to E$ is an identity mapping. If Assumption 2.1 holds and there exists a constant $\rho > 0$ such that*

$$\left|\rho - \frac{\alpha}{64c\lambda^2}\right| < \frac{\sqrt{\alpha^2 - (\delta^2 - t^2)64c\lambda^2}}{64c\lambda^2};\ \alpha^2 > (\delta^2 - t^2)64c\lambda^2, \tag{12}$$

where $\lambda = \beta\mu + \gamma\eta + \xi\sigma$; $t = \sqrt{2k+1} - \nu$, then there exist $x \in E$, $u \in T(x)$, $v \in A(x)$, $w \in S(x)$, satisfying (1) and the iterative sequences $\{x_n\}, \{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.1 converge strongly to x, u, v, w, respectively, in E.

Proof. From Algorithm 3.1, we estimate, using Assumption 2.1,

$$\begin{aligned} &\|g(x_{n+2}) - g(x_{n+1})\| \\ &\le \|P_{K(x_{n+1})}[g(x_{n+1}) - \rho N(u_{n+1}, v_{n+1}, w_{n+1})] \\ &\quad - P_{K(x_n)}[g(x_n) - \rho N(u_n, v_n, w_n)]\| \\ &\le \|g(x_{n+1}) - g(x_n) - \rho[N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)]\| \\ &\quad + \nu\|x_{n+1} - x_n\|. \end{aligned} \tag{13}$$

Next, using Lemma 2.2; α-strongly g-accretiveness and (β,γ,ξ)-Lipschitz continuity of $N(\cdot,\cdot,\cdot)$; δ-Lipschitz continuity of g; μ-H-Lipschitz continuity of T; η-H-Lipschitz continuity of A and σ-H-Lipschitz continuity of S, we have

$$\begin{aligned} &\|g(x_{n+1}) - g(x_n) - \rho[N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)]\|^2 \\ &\le \|g(x_{n+1}) - g(x_n)\|^2 - 2\rho\langle N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n), \\ &\quad J(g(x_{n+1}) - g(x_n) - \rho[N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)])\rangle \\ &\le \|g(x_{n+1}) - g(x_n)\|^2 \\ &\quad - 2\rho\langle N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n), J(g(x_{n+1}) - g(x_n))\rangle \\ &\quad - 2\rho\langle N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n), J(g(x_{n+1}) - g(x_n) \\ &\quad - \rho[N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)]) - J(g(x_{n+1}) - g(x_n))\rangle \\ &\le \|g(x_{n+1}) - g(x_n)\|^2 \\ &\quad - 2\rho\langle N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n), J(g(x_{n+1}) - g(x_n))\rangle \\ &\quad + 64c\rho^2\|N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)\|^2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\|N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)\| \\ &\le \beta\|u_{n+1} - u_n\| + \gamma\|v_{n+1} - v_n\| + \xi\|w_{n+1} - w_n\| \end{aligned} \tag{15}$$

$$\begin{aligned}&\leq \beta[H(T(x_{n+1}),T(x_n))+\epsilon^{n+1}\|x_{n+1}-x_n\|]\\&\quad+\gamma[H(A(x_{n+1}),A(x_n))+\epsilon^{n+1}\|x_{n+1}-x_n\|]\\&\quad+\xi[H(S(x_{n+1}),S(x_n))+\epsilon^{n+1}\|x_{n+1}-x_n\|]\\&\leq\Big[\beta\mu+\gamma\eta+\xi\sigma+(\beta+\gamma+\xi)\epsilon^{n+1}\Big]\|x_{n+1}-x_n\|.\end{aligned}$$

From (13), (14) and (15), we have

$$\|g(x_{n+2})-g(x_{n+1})\|\leq\Big\{\Big(\delta^2-2\rho\alpha+64c\rho^2[\beta\mu+\gamma\eta+\xi\sigma+(\beta+\gamma+\xi)\epsilon^{n+1}]^2\Big)^{\frac{1}{2}}+\nu\Big\}\|x_{n+1}-x_n\|. \tag{16}$$

Since $(g-I)$ is k-strongly accretive, we have

$$\begin{aligned}&\|x_{n+2}-x_{n+1}\|^2\\&=\|g(x_{n+2})-g(x_{n+1})+x_{n+2}-x_{n+1}-(g(x_{n+2})-g(x_{n+1}))\|^2\\&\leq\|g(x_{n+2})-g(x_{n+1})\|^2-2\langle(g-I)x_{n+2}-(g-I)x_{n+1},J(x_{n+2}-x_{n+1})\rangle\\&\leq\|g(x_{n+2})-g(x_{n+1})\|^2-2k\|x_{n+2}-x_{n+1}\|^2,\end{aligned}$$

which implies

$$\|x_{n+2}-x_{n+1}\|\leq\frac{1}{\sqrt{2k+1}}\|g(x_{n+2})-g(x_{n+1})\|.$$

The preceding inequality with inequality (16) gives

$$\|x_{n+2}-x_{n+1}\|\leq\theta_n\|x_{n+1}-x_n\|, \tag{17}$$

where

$$\theta_n:=\frac{1}{\sqrt{2k+1}}\Big\{\Big(\delta^2-2\rho\alpha+64c\rho^2[\beta\mu+\gamma\eta+\xi\sigma+(\beta+\gamma+\xi)\epsilon^{n+1}]^2\Big)^{\frac{1}{2}}+\nu\Big\}.$$

Since $\epsilon\in(0,1)$, it follows that $\epsilon^{n+1}\to 0$ and $\theta_n\to\theta$ as $n\to\infty$, where

$$\theta:=\frac{1}{\sqrt{2k+1}}\Big\{\Big(\delta^2-2\rho\alpha+64c\rho^2(\beta\mu+\gamma\eta+\xi\sigma)^2\Big)^{\frac{1}{2}}+\nu\Big\}.$$

Since $\theta<1$ by condition (12). Hence $\theta_n<1$ for sufficiently n large. Therefore, (17) implies that $\{x_n\}$ is a Cauchy sequence in E. Hence there exists $x\in E$ such that $x_n\to x$ as $n\to\infty$. By H-Lipschitz continuity of T and (8), we have

$$\begin{aligned}\|u_{n+1}-u_n\|&\leq H(T(x_{n+1}),T(x_n))+\epsilon^{n+1}\|x_{n+1}-x_n\|\\&\leq(\mu+\epsilon^{n+1})\|x_{n+1}-x_n\|.\end{aligned}\tag{18}$$

Since $\{x_n\}$ is a Cauchy sequence in E, (18) implies that $\{u_n\}$ is a Cauchy sequence in E. Hence there exists $u\in E$ such that $u_n\to u$ as $n\to\infty$. Similarly, we can show that $\{v_n\}$, $\{w_n\}$ are Cauchy sequences and hence there exist $v,w\in E$ such that $v_n\to v$ and $w_n\to w$ as $n\to\infty$. Further, since $u_n\in T(x_n)$, we have

$$\begin{aligned}d(u,T(x))&\leq\|u-u_n\|+d(u_n,T(x_n))\\&\leq\|u-u_n\|+H(T(x_n),T(x))\\&\leq\|u-u_n\|+\mu\|x_n-x\|\to 0,\end{aligned}$$

as $n \to \infty$ and hence $u \in T(x)$. Similarly, we can show that $v \in A(x)$ and $w \in S(x)$.

Furthermore, continuity of the mappings $N(\cdot,\cdot,\cdot), T, A, S, P_{K(x)}$ and g, Assumption 2.1 and (7) give that

$$g(x) = P_{K(x)}[g(x) - \rho N(u,v,w)]$$

and hence, from Lemma 2.4, it follows that $x \in E, u \in T(x), v \in A(x), w \in S(x)$ is a solution of MQVI (1). This completes the proof. □

Remark 3.2.

(a) It is clear that $\alpha \leq \delta\lambda$. Further condition $\theta \in (0,1)$ and (12) hold for some suitable values of cofficients, for example, $\alpha = 3,\ \beta = 1,\ \gamma = 2,\ \xi = 2,\ \delta = 1,\ k = 1,\ \mu = 1,\ \eta = 1,\ \sigma = 1,\ \nu = 1,\ \rho \in \left[\frac{1}{10}, \frac{5}{4}\right],\ c = \frac{1}{320}$.

(b) Theorem 3.2 also serves the convergence criteria for Algorithm 3.2.

If we take $N(u,v,w) \equiv M(u,v),\ \forall u,v,w \in E$, where $M : E \times E \to E$, Theorem 3.2 reduces to the following result which is improved form of Theorem 3.2[9]:

Corollary 3.1. *Let E be a real uniformly smooth Banach space with $\rho_E(t) \leq ct^2$ for some $c > 0$. Let the mappings $T, A : E \to CB(E)$ be μ-H-Lipschitz and η-H-Lipschitz continuous, respectively and let the mapping $M(\cdot,\cdot)$ be α-strongly g-accretive with respect to T, A and (β,γ)-Lipschitz continuous. Let the mapping g be δ-Lipschitz continuous and let the mapping $(g - I) : E \to E$ be k-strongly accretive, where $I : E \to E$ is an identity mapping. If Assumption 2.1 holds and there exists a constant $\rho > 0$ such that (12) holds with $\lambda = \beta\mu + \gamma\eta$, then there exist $x \in E,\ u \in T(x),\ v \in A(x)$, satisfying (3) and the iterative sequences $\{x_n\}, \{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.3 converge strongly to x, u, v, respectively, in E.*

If $E \equiv H$, a real Hilbert space, then Theorem 3.2 reduces to the following result:

Corollary 3.2. *Let H be a real Hilbert space. Let the mappings $T, A, S : E \to CB(E)$ be μ-H-Lipschitz, η-H-Lipschitz and σ-H-Lipschitz continuous, respectively and let the mapping $N(\cdot,\cdot,\cdot)$ be α-strongly g-accretive with respect to T, A and S, and (β,γ,ξ)-Lipschitz continuous. Let the mapping g be δ-Lipschitz continuous and let the mapping $(g - I) : E \to E$ be k-strongly accretive, where $I : E \to E$ is an identity mapping. If Assumption 2.1 holds and there exists a constant $\rho > 0$ such that (12) holds, then there exist $x \in H,\ u \in T(x),\ v \in A(x)\ w \in S(x)$ satisfying (5) and the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}$ and $\{w_n\}$ generated by Algorithm 3.2 converge strongly to x, u, v, w, respectively, in H.*

Remark 3.3. Corollary 3.2 generalizes and improves the results given in [2,7-9].

References

[1] Ya. I. Alber, 'Metric and generalized projection operators in Banach spaces: Properties and applications', In Theory and Applications of Nonlinear Operators of Monotone and Accretive Type, (Edited by A. Kartsatos), pp. 15-50, Marcel Dekker, New York 1996.

[2] J.Y. Chen, N.C. Wong and J.C. Yao, 'Algorithms for generalized co-complementarity problems in Banach spaces', *Computers Math. Applic.* **43** (2002), 49-54.

[3] V.E. Demyanov, G.E. Stavroutakis, L.N. Polyakov and P.D. Panagiotopolous, Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics, Kluwer Academic, Dordrecht 1996.

[4] F. Giannessi and A. Mougeri, Variational Inequalities and Network Equillibrium Problems, Plenum Press, New York 1995.

[5] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker, New York 1984.

[6] X. He, 'On ϕ-strongly accretive mappings and some set-valued variational problems', *J. Math. Anal. Appl.* **277**(2) (2003), 504-511.

[7] M.A. Noor, 'Generalized quasivariational inequalities and implicit Wiener-Hopf equations', *Optimization* **45** (1999), 197-222.

[8] M.A. Noor, 'Generalized multivalued quasivariational inequalities (II)', *Computers Math. Applic.* **35** (1998), 63-78.

[9] M.A. Noor, A. Moudafi, B. Xu, 'Multi-valued quasi variational inequalities in Banach spaces', *J. Inequal. Pure Appl. Math.* **3**(3) (2002), 1-6.

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K. R. Kazmi
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: kr.kazmi@lycos.com

M. I. Bhat
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
Current address: Department of Applied Mathematics, Baba Ghulam Shah Badshah University, Rajouri, Jammu and Kashmir, India

F. A. Khan
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India