Journal of Nonlinear and Convex Analysis
Volume 6, Number 3, 2005, 465–485

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SOME LIMIT RESULTS FOR INTEGRANDS AND HAMILTONIANS WITH APPLICATION TO VISCOSITY

C. CASTAING, A. JOFRE, AND A. SYAM

1. Introduction

The present work is essentially a continuation of [9], [6], [7] dealing with Control problems where the dynamics are driven by ordinary differential equations (ODE) [9] and evolution inclusions (EI) governed by nonconvex sweeping process and m-accretive operators [6], [7] via the fiber product of Young measures [9], associated with the Hamilton-Jacobi-Bellman equation

$$\frac{\partial u}{dt} + H(t, x, D_x u) = 0$$

where the Hamiltonian is given by

$$H(t, x, y) = \inf_{\mu \in \mathcal{M}^1_+(Y)} \sup_{\nu \in \mathcal{M}^1_+(Z)} \{\langle y, g(t, x, \mu, \nu)\rangle\} + G(t, x, y),$$

$\mathcal{M}^1_+(Y)$ (resp. $\mathcal{M}^1_+(Z)$) is the compact metrizable space of the set of all probability Radon measures on a compact metric space Y (resp. Z) endowed with the vague topology, $g : [0, 1] \times \mathbf{R}^d \times \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z) \to \mathbf{R}^d$ is a bounded continuous mapping, uniformly Lipschitzean with respect to the variable $x \in \mathbf{R}^d$, and $G : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ is an upper semicontinuous integrand. Given a bounded continuous cost function $J : [0, 1] \times \mathbf{R}^d \times Y \times Z \to \mathbf{R}$ and the lower value function

$$V_J(\tau, x) = \max_{\nu \in \mathcal{K}} \min_{\mu \in \mathcal{H}} \{\int_\tau^1 [\int_Y [\int_Z J(t, u_{x,\mu,\nu}(t), y, z)\mu_t(dy)]\nu_t(dz)]dt\}$$

where $\tau \in [0, 1]$, $\mathcal{H}$ (resp. $\mathcal{K}$) is the space of Young measures $\mathcal{Y}([0, 1], Y)$, (resp. $\mathcal{Y}([0, 1], Z)$), $u_{x,\mu,\nu}$ denotes the unique absolutely continuous solution associated with the control $(\mu, \nu) \in \mathcal{H} \times \mathcal{K}$ of the dynamic control

$$\dot{u}_{x,\mu,\nu}(t) = g(t, u_{x,\mu,\nu}(t), \mu_t, \nu_t); u_{x,\mu,\nu}(\tau) = x.$$

Under some suitable conditions, V_J is a viscosity solution of the associated HJB under consideration, we refer to [9], [7] for details. In the present paper, we provide some limit results for both normal integrands and Hamitonians. In particular, we show some viscosity properties of the value function according to the lower semicontinuity (resp. upper semicontinuity) in time of the dynamic. In Section 2, we prove two existence theorems of viscosity solutions in a control problem with Young measures for an ODE in $\mathbf{R}$ where the family $(g(., ., \mu, \nu))_{(\mu,\nu)\in \mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ is equi-lower (resp. equi-upper) semicontinuous with respect to the variables (t, x). In Section 3, we present some limit results for normal integrands via the Lebesgue derivation theorem. In Section 4, we provide a lim inf (resp. lim sup) result for the Hamiltonians when the dynamic is globally lower (resp. upper) semicontinuous by using the results obtained in Section 3. We refer to [17], [3], [20] for the study of time-measurable Hamiltonians, to [2], [3], [15], [16] for viscosity solutions in ordinary differential equations (ODE), to [6], [7] for viscosity solutions in evolution inclusions (EI) with Young measure controls, to [12], [21], [22], for convex sweeping process, to [4], [13], [11], [25] for nonconvex sweeping process, to [1], [10], [26], [27] for Young measures.

Throughout E is the finite dimensional space $\mathbf{R}^d$, $\mathcal{L}([0,1])$ is the σ-algebra of all Lebesgue-measurable sets in $[0,1]$.

2. Control problem governed by ordinary differential equations

We present a study of viscosity solutions to ODE where the controls are Young measures. We recall and summarize some results given in [9]. We assume that
(K_1) $f : [0,1] \times E \times Y \times Z \to E$ is bounded, say, $||f(t,x,y,z)|| \le M$ for some $M > 0$, for all $(t,x,y,z) \in [0,1]\times E\times Y\times Z$; for all $t \in [0,1]$, $f(t,.,.,.)$ is continuous on $E \times Y \times Z$; for all $(x,y,z) \in E \times Y \times Z$, $f(.,x,y,z)$ is Lebesgue-measurable on $[0,1]$ and uniformly Lipschitzean in $x \in E$, that is,

$$||f(t,x_1,y,z) - f(t,x_2,y,z)|| \le \eta||x_1 - x_2||$$

for some $\eta > 0$, for all $(t,x_1,y,z),(t,x_2,y,z) \in [0,1] \times E \times Y \times Z$.
(K_2) $J : [0,1] \times E \times Y \times Z \to \mathbf{R}$ is bounded, say, $|J(t,x,y,z)| \le N$, for some $N > 0$, for all $(t,x,y,z) \in [0,1]\times E\times Y\times Z$; for all $t \in [0,1]$, $J(t,.,.,.)$ is continuous on $E \times Y \times Z$; for all $(x,y,z) \in E \times Y \times Z$, $J(.,x,y,z)$ is Lebesgue-measurable on $[0,1]$.

We will need three technical results which are borrowed from [6], [7], [9].

Lemma 2.1. *Let $f : [0,1]\times E\times Y\times Z \to E$ satisfying (K_1). Let $(t_0,x_0) \in [0,1]\times E$. Assume that $\Lambda : [0,1] \times E \times \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z) \to \mathbf{R}$ is a mapping satisfying*

(i) *the family $(\Lambda(.,.,\mu,\nu))_{(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ is equi-lower semicontinuous on $[0,1] \times E$,*
(ii) *for every fixed $(t,x) \in [0,1] \times E$, $\Lambda(t,x,.,.)$ is upper semicontinuous on $\mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z)$,*
(iii) *for any bounded subset B of E, the restriction $\Lambda|_{[0,1]\times B\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ is bounded.*

If $\inf_{\mu\in\mathcal{M}^1_+(Y)} \max_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0,x_0,\mu,\nu) > \eta > 0$ for some $\eta > 0$, then there is $\sigma > 0$ such that, for each $\mu \in \mathcal{H}$, we have

$$\sup_{\nu\in\mathcal{K}} \int_{t_0}^{t_0+\sigma} \Lambda(t,u_{x_0,\mu,\nu}(t),\mu_t,\nu_t)dt > \sigma\eta/2,$$

where $u_{x_0,\mu,\nu}$ denotes the unique trajectory solution of

$$\begin{cases} \dot u_{x_0,\mu,\nu}(t) = \int_Z[\int_Y f(t,u_{x_0,\mu,\nu}(t),y,z)\,\mu_t(dy)]\,\nu_t(dz) \\ u_{x_0,\mu,\nu}(t_0) = x_0 \in E, \end{cases}$$

associated with the controls $(\mu,\nu) \in \mathcal{H} \times \mathcal{K}$.

Proof. By the equi-lower semicontinuity hypothesis (i), there is $\zeta > 0$ such that, for all $(\mu,\nu) \in \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z)$, if $0 \le t - t_0 \le \zeta$ and $||x - x_0|| \le \zeta$, then

$$\Lambda(t,x,\mu,\nu) > \Lambda(t_0,x_0,\mu,\nu) - \frac{\eta}{2}.$$

It is worthy to mention that, from (i)–(ii), Λ is Borel on $[0,1]\times E\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)$. Let $\mu\in\mathcal{H}$. By (ii) there exists a Lebesgue-measurable mapping $\nu^\mu:[0,1]\to\mathcal{M}^1_+(Z)$ such that

$$\Lambda(t_0,x_0,\mu_t,\nu^\mu_t)=\max_{\nu'\in\mathcal{M}^1_+(Z)}\Lambda(t_0,x_0,\mu_t,\nu')$$

for all $t\in[0,1]$, because the nonempty Borelian-valued multifunction

$$t\to\{\nu\in\mathcal{M}^1_+(Z):\Lambda(t_0,x_0,\mu_t,\nu)=\max_{\nu'\in\mathcal{M}^1_+(Z)}\Lambda(t_0,x_0,\mu_t,\nu')$$

has its graph in $\mathcal{L}([0,1])\otimes\mathcal{B}(\mathcal{M}^1_+(Z))$. Take $\sigma>0$ such that $\sigma\le\min\{\frac{\zeta}{M},\zeta\}$, we get

$$||u_{x_0,\mu,\nu}(t)-u_{x_0,\mu,\nu}(t_0)||\le\zeta,$$

for all $t\in[t_0,t_0+\sigma]$ and for all $\nu\in\mathcal{R}$. Furthermore, thank to the above remark, the function $t\mapsto\Lambda(t,u_{x_0,\mu,\nu^\mu}(t),\mu_t,\nu^\mu_t)$ is Lebesgue-mesurable and integrable by using (iii). By integrating,

$$\begin{aligned}\int_{t_0}^{t_0+\sigma}\Lambda(t,u_{x_0,\mu,\nu^\mu}(t),\mu_t,\nu^\mu_t)dt&\ge\int_{t_0}^{t_0+\sigma}[\Lambda(t_0,x_0,\mu_t,\nu^\mu_t)-\frac{\eta}{2}]dt\\&>\int_{t_0}^{t_0+\sigma}\frac{\eta}{2}dt=\frac{\sigma\eta}{2}.\end{aligned}$$

□

Lemma 2.2. *Let $f:[0,1]\times E\times Y\times Z\to E$ satisfying (K_1). Let $(t_0,x_0)\in[0,1]\times E$. Assume that $\Lambda:[0,1]\times E\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)\to\mathbf{R}$ is a mapping satisfying*

(i) *the family $(\Lambda(.,.,\mu,\nu))_{(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ is equi-upper semicontinuous on $[0,1]\times E$,*

(ii) *for every fixed $(t,x)\in[0,1]\times E$, $\Lambda(t,x,.,.)$ is lower semicontinuous on $\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)$,*

(iii) *for any bounded subset B of E, the restriction $\Lambda|_{[0,1]\times B\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ is bounded.*

If $\min_{\mu\in\mathcal{M}^1_+(Y)}\max_{\nu\in\mathcal{M}^1_+(Z)}\Lambda(t_0,x_0,\mu,\nu)<-\eta<0$ for some $\eta>0$, then there is $\overline{\mu}\in\mathcal{H}$ and $\sigma>0$ such that,

$$\sup_{\nu\in\mathcal{K}}\int_{t_0}^{t_0+\sigma}\Lambda(t,u_{x_0,\overline{\mu},\nu}(t),\overline{\mu}_t,\nu_t)dt<-\sigma\eta/2,$$

where $u_{x_0,\overline{\mu},\nu}$ denotes the unique trajectory solution of

$$\begin{cases}\dot{u}_{x_0,\overline{\mu},\nu}(t)=\int_Z[\int_Y f(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\,\overline{\mu}_t(dy)]\,\nu_t(dz)\\u_{x_0,\overline{\mu},\nu}(t_0)=x_0\in E,\end{cases}$$

associated with the controls $(\overline{\mu},\nu)\in\mathcal{H}\times\mathcal{K}$.

Proof. By hypothesis, there is $\overline{\mu}\in\mathcal{M}^1_+(Y)$ such that

$$\max_{\nu\in\mathcal{M}^1_+(Z)}\Lambda(t_0,x_0,\overline{\mu},\nu)<-\eta<0.$$

By the equi-upper semicontinuous hypothesis (i), there is $\zeta>0$ such that

$$\max_{\nu\in\mathcal{M}^1_+(Z)}\Lambda(t,x,\overline{\mu},\nu)<-\eta/2,$$

for $0 < t - t_0 \leq \zeta$ and $||x - x_0|| \leq \zeta$. Thus for $\sigma > 0$ such that $\sigma \leq \min\{\zeta, \frac{\zeta}{M}\}$, we get

$$||u_{x_0,\overline{\mu},\nu}(t) - u_{x_0,\overline{\mu},\nu}(t_0)|| \leq \zeta,$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathcal{R}$. Denote by $\overline{\mu}$ the constant Young measure $t \mapsto \overline{\mu}_t = \overline{\mu}$. Using $(i) - -(iii)$ we see that the functions $t \mapsto \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}_t, \nu_t)$ are Lebesgue-measurable and integrable on $[t_0, t_0 + \sigma]$. Then by integrating

$$\begin{aligned}\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}_t, \nu_t)dt &\leq \int_{t_0}^{t_0+\sigma} [\max_{\nu' \in \mathcal{M}_+^1(Z)} \Lambda(t, u_{x_0,\overline{\mu},\nu'}(t), \overline{\mu}, \nu')]dt \\ &< -\sigma\eta/2,\end{aligned}$$

for all $\nu \in \mathcal{K}$ and the result follows. □

Remarks. In Lemma 2.1-2.2, we have used the fact that any function defined on the product of two separable metric spaces which is separately upper semicontinuous in one variable and lower semicontinuous in the other variable is globally Borel. This was already proved in [18]. This property is no more true for a function which is separately upper (or lower) semicontinuous in both variables: see a counterexample by Sierpinski [23], where a non Borel function f is defined on the plane $\mathbf{R} \times \mathbf{R}$ such that the restriction of f to any line of $\mathbf{R} \times \mathbf{R}$ is the indicator function of two points.

We will need a variant of Lemma 2.1-2.2 dealing with globally upper semicontinuous integrands. The following is borrowed from ([7], Lemma 4.1).

Lemma 2.3. *Let $f : [0,1] \times E \times Y \times Z \to E$ satisfying (K_1). Let $(t_0, x_0) \in [0,1] \times E$. Assume that $\Lambda_1 : [0,1] \times E \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z) \to \mathbf{R}$ is continuous and $\Lambda_2 : [0,1] \times E \times \mathcal{M}_+^1(Z) \to \mathbf{R}$ is upper semicontinuous such that, for any bounded subset B of E, $\Lambda_2|_{[0,1]\times B \times \mathcal{M}_+^1(Z)}$ is bounded, and assume that $\Lambda := \Lambda_1 + \Lambda_2$ satisfies the following condition*

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0 \text{ for some } \eta > 0.$$

Then there is $\overline{\mu} \in \mathcal{M}_+^1(Y)$ and $\sigma > 0$ such that

$$\sup_{\nu \in \mathcal{K}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)dt < -\sigma\eta/2,$$

where $u_{x_0,\overline{\mu},\nu}$ denotes the unique trajectory solution of

$$\begin{cases} \dot{u}_{x_0,\overline{\mu},\nu}(t) = \int_Z [\int_Y f(t, u_{x_0,\overline{\mu},\nu}(t), y, z)\overline{\mu}(dy)]\, \nu_t(dz) \\ u_{x_0,\overline{\mu},\nu}(t_0) = x_0 \in E, \end{cases}$$

associated with the controls $(\overline{\mu}, \nu) \in \mathcal{H} \times \mathcal{K}$.

Proof. By hypothesis,

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0,$$

that is,

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)] < -\eta < 0.$$

As the function Λ_1 is continuous, so is the function

$$\mu \mapsto \max_{\nu \in \mathcal{M}^1_+(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)].$$

Hence there exists $\overline{\mu} \in \mathcal{M}^1_+(Y)$ such that

$$\max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \overline{\mu}, \nu) = \min_{\mu \in \mathcal{M}^1_+(Y)} \max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu).$$

As the function $(t, x, \nu) \mapsto \Lambda_1(t, x, \overline{\mu}, \nu)$ is continuous and the function $(t, x, \nu) \mapsto \Lambda_2(t, x, \nu)$ is upper semicontinuous, $(t, x, \nu) \mapsto \Lambda_1(t, x, \overline{\mu}, \nu) + \Lambda_2(t, x, \nu)$ is upper semicontinuous, so is the function

$$(t, x) \mapsto \max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t, x, \overline{\mu}, \nu).$$

Hence there is $\zeta > 0$ such that

$$\max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t, x, \overline{\mu}, \nu) < -\eta/2,$$

for $0 < t - t_0 \leq \zeta$ and $||x - x_0|| \leq \zeta$. Thus for $\sigma > 0$ such that $\sigma \leq \min\{\zeta, \frac{\zeta}{M}\}$, we get

$$||u_{x_0,\overline{\mu},\nu}(t) - u_{x_0,\overline{\mu},\nu}(t_0)|| \leq \zeta,$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathcal{R}$. Hence the functions $\Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)$ bounded and Lebesgue-measurable on $[t_0, t_0 + \sigma]$. Then by integrating

$$\begin{aligned}\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)dt &\leq \int_{t_0}^{t_0+\sigma} [\max_{\nu' \in \mathcal{M}^1_+(Z)} \Lambda(t, u_{x_0,\overline{\mu},\nu'}(t), \overline{\mu}, \nu')]dt \\ &< -\sigma\eta/2 < 0,\end{aligned}$$

for all $\nu \in \mathcal{R}$ and the result follows. □

Let us recall the following dynamic programming principle theorem [9], using the fiber product lemma for Young measures.

Theorem 2.1. *Assume that* (K_1)–(K_2) *are satisfied. Let us consider the upper value function*

$$U_J(\tau, x) := \min_{\mu \in \mathcal{H}} \max_{\nu \in \mathcal{K}} \{ \int_\tau^1 [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\, \mu_t(dy)] \nu_t(dz)]\, dt \},$$

where $u_{x,\mu,\nu}$ *is the unique trajectory solution of*

$$\begin{cases} \dot{u}_{x,\mu,\nu}(t) = \int_Z [\int_Y f(t, u_{x,\mu,\nu}(t), y, z)\mu_t(dy)]\, \nu_t(dz), \ \ a.e.\ t \in [\tau, 1], \\ u_{x,\mu,\nu}(\tau) = x \in E. \end{cases}$$

Then for any $\sigma \in]0, 1[$ *with* $\tau + \sigma < 1$,

$$\begin{aligned}U_J(\tau, x) = \min_{\mu \in \mathcal{H}} \max_{\nu \in \mathcal{K}} \{ \int_\tau^{\tau+\sigma} [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\mu_t(dy)]\, \nu_t(dz)]\, dt \\ + U_J(\tau + \sigma, u_{x,\mu,\nu}(\tau + \sigma)) \},\end{aligned}$$

where

$$U_J(\tau + \sigma, u_{x,\mu,\nu}(\tau + \sigma))$$

$$= \min_{\beta\in\mathcal{H}} \max_{\gamma\in\mathcal{K}} \int_{\tau+\sigma}^{1} [\int_Z [\int_Y J(t, v_{x,\beta,\gamma}(t), y, z)\beta_t(dy)]\, \gamma_t(dz)]\, dt,$$

where $v_{x,\beta,\gamma}$ *denotes the trajectory solution of the above dynamic associated with the controls* $(\beta, \gamma) \in \mathcal{H} \times \mathcal{K}$ *with initial condition* $v_{x,\beta,\gamma}(\tau + \sigma) = u_{x,\mu,\nu}(\tau + \sigma)$.

For simplicity we present some viscosity properties for the value function in the particular case of one space of Young measure controls $\mathcal{K}$. We would like to mention that, even in this case, new details of proofs are necessary. We will consider the value function

$$V_J(\tau, x) := \max_{\nu\in\mathcal{K}} \{\int_{\tau}^{1} [\int_Z J(t, u_{x,\nu}(t), z)\, \nu_t(dz)]\, dt\},$$

where $u_{x,\nu}$ is the unique trajectory solution of

$$\begin{cases} \dot{u}_{x,\nu}(t) = \int_Z f(t, u_{x,\nu}(t), z)\, \nu_t(dz), \ \textit{a.e. } t \in [\tau, 1], \\ u_{x,\nu}(\tau) = x \in E, \end{cases}$$

in the particular case where E is $\mathbf{R}$, and f and J satisfy the following assumptions.
$(H_1)(a)$ $f : [0, 1] \times E \times Z \to \mathbf{R}^+$ is bounded, say, $f(t, x, z) \leq M$, for some $M > 0$, for all $(t, x, z) \in [0, 1] \times E \times Z$; uniformly Lipschitz with respect to the variable $x \in E$, $f(t, ., .)$ is continuous on $E \times Z$ for every $t \in [0, 1]$ and the family

$$(\hat{f}(., ., \nu))_{\nu\in\mathcal{M}^1_+(Z)}$$

is equi-lower semicontinuous on $[0, 1] \times E$, where

$$\hat{f}(t, x, \nu) := \int_Z f(t, x, z)\nu(dz)$$

$\forall (t, x, \nu) \in [0, 1] \times E \times \mathcal{M}^1_+(Z)$.
$(H_2)(a)$ $J : [0, 1] \times E \times Z \to R$ is bounded, say, $|J(t, x, z)| \leq N$,for some $N > 0$, for all $(t, x, z) \in [0, 1] \times E \times Z$; $J(t, ., .)$ is continuous on $E \times Z$ for every $t \in [0, 1]$ and the family

$$(\hat{J}(., ., \nu))_{\nu\in\mathcal{M}^1_+(Z)}$$

is equi-lower semicontinuous on $[0, 1] \times E$, where

$$\hat{J}(t, x, \nu) := \int_Z J(t, x, z)\nu(dz),$$

$\forall (t, x, \nu) \in [0, 1] \times E \times \mathcal{M}^1_+(Z)$.
Similarly we consider the assumptions.
$(H_1)(b)$ $f : [0, 1] \times E \times Z \to \mathbf{R}^+$ is bounded, say, $f(t, x, z) \leq M$, for some $M > 0$, for all $(t, x, z) \in [0, 1] \times E \times Z$, uniformly Lipschitz with respect to the variable $x \in E$, $f(t, ., .)$ is continuous on $E \times Z$ for every $t \in [0, 1]$ and the family

$$(\hat{f}(., ., \nu))_{\nu\in\mathcal{M}^1_+(Z)}$$

is equi-upper semicontinuous on $[0, 1] \times E$.
$(H_2)(b)$ $J : [0, 1] \times E \times Z \to R$ is bounded, say, $|J(t, x, z)| \leq N$, for some $N > 0$, for all $(t, x, z) \in [0, 1] \times E \times Z$; $J(t, ., .)$ is continuous on $E \times Z$ for every $t \in [0, 1]$ and the family

$$(\hat{J}(., ., \nu))_{\nu\in\mathcal{M}^1_+(Z)}$$

is equi-upper semicontinuous on $[0,1] \times E$.

Theorem 2.2 (Existence of viscosity supersolutions). *Assume that $(H_1)(a)$ and $(H_2)(a)$ are satisfied. Let us consider the value function V_J*

$$V_J(\tau, x) := \max_{\nu \in \mathcal{K}} \{ \int_\tau^1 [\int_Z J(t, u_{x,\nu}(t), z)\, \nu_t(dz)]\, dt \},$$

where $u_{x,\nu}$ is the unique trajectory solution of

$$\begin{cases} \dot{u}_{x,\nu}(t) = \int_Z f(t, u_{x,\nu}(t), z)\, \nu_t(dz), \ \textit{a.e. } t \in [\tau, 1], \\ u_{x,\nu}(\tau) = x \in E. \end{cases}$$

Let us consider the upper Hamiltonian

$$H^+(t, x, y) = \max_{\nu \in \mathcal{M}^1_+(Z)} \{ y.\hat{f}(t, x, \nu) + \hat{J}(t, x, \nu) \}.$$

Then V_J is a semi-viscosity supersolution *of the Hamilton-Jacobi-Bellman equation*

$$U_t + H^+(t, x, \nabla U) = 0,$$

that is, if for any $\varphi \in \mathcal{C}^1_{\mathbf{R}}([0,1] \times E)$ for which $\nabla\varphi$ is ≥ 0 and bounded and $V_J - \varphi$ reaches a local minimum at $(t_0, x_0) \in [0,1] \times E$, then

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H^+(t_0, x_0, \nabla\varphi(t_0, x_0)) \leq 0.$$

Proof. We will make use of some arguments developed in [16], [15], [9],[6], [7]. This needs a careful look because f and J are not globally continuous. Assume by contradiction that there exists a $\varphi \in \mathcal{C}^1_{\mathbf{R}}([0,1] \times E)$ for which $\nabla\varphi$ is ≥ 0 and bounded and a point $(t_0, x_0) \in [0,1] \times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H^+(t_0, x_0, \nabla\varphi(t_0, x_0)) > \eta,$$

for some $\eta > 0$. As both $\hat{f}$ and $\nabla\varphi$ are ≥ 0 and bounded, by using $(H_1)(a)$ and $(H_2)(a)$ we may apply Lemma 2.1 to the integrand Λ defined by on $[0,1] \times E \times \mathcal{M}^1_+(Z)$ by

$$\Lambda(t, x, \nu) = \hat{J}(t, x, \nu) + \nabla\varphi(t, x).\hat{f}(t, x, \nu) + \frac{\partial \varphi}{\partial t}(t, x),$$

for all $(t, x, \nu) \in [0,1] \times E \times \mathcal{M}^1_+(Z)$ because the family

$$(\nabla\varphi(.,.).\hat{f}(.,.,\nu))_{\nu \in \mathcal{M}^1_+(Z)}$$

inherits the equi-lower semicontinuity property of the family

$$(\hat{f}(.,.,\nu))_{\nu \in \mathcal{M}^1_+(Z)}.$$

This provides $\sigma > 0$ such that,

$$\begin{aligned} (2.2.1) \quad \max_{\nu \in \mathcal{K}} \{ & \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu}(t), z)\nu_t(dz)]\, dt \\ & + \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\nu}(t)).f(t, u_{x_0,\nu}(t), z)]\nu_t(dz)]\, dt \end{aligned}$$

$$+ \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{x_0,\nu}(t))dt\} > \sigma\eta/2,$$

where $u_{x_0,\nu}$ is the trajectory solution associated with the control $\nu \in \mathcal{K}$ of

$$\begin{cases} \dot{u}_{x_0,\nu}(t) = \int_Z f(t, u_{x_0,\nu}(t), z)\nu_t(dz), \ \ a.e. \ t \in [0,1], \\ u_{x_0,\nu}(t_0) = x_0 \in E. \end{cases}$$

From Theorem 2.1 (of dynamic programming) (see e.g [9], Theorem 3.2.1) we deduce

$$(2.2.2) \quad V_J(t_0, x_0) = \max_{\nu \in \mathcal{K}}\{\int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu}(t), z)\nu_t(dz)]\, dt + V_J(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma))\}.$$

Since $V_J - \varphi$ has a local minimum at (t_0, x_0), so for σ small enough

$$(2.2.3) \quad V_J(t_0, x_0) - \varphi(t_0, x_0) \leq V_J(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma)) - \varphi(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma))$$

for all $\nu \in \mathcal{K}$. It follows that

$$(2.2.4) \quad \max_{\nu \in \mathcal{K}}\{\int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu}(t), z)\nu_t(dz)]\, dt + V_J(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma))\} + \max_{\nu \in \mathcal{K}}\{\varphi(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma)) - \varphi(t_0, x_0) - V_J(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma))\} \leq 0.$$

From (2.2.4) we get

$$(2.2.5) \quad \max_{\nu \in \mathcal{K}}\{\int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu}(t), z)\ \nu_t(dz)]dt + \varphi(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma)) - \varphi(t_0, x_0)\} \leq 0.$$

As φ is $\mathcal{C}^1$ and $u_{x_0,\nu}$ is the trajectory solution of our dynamic

$$(2.2.6) \quad \varphi(t_0+\sigma, u_{x_0,\nu}(t_0+\sigma)) - \varphi(t_0, x_0) = \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\nu}(t)).f(t, u_{x_0,\nu}(t), z)]\nu_t(dz)]\, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{x_0,\nu}(t))\, dt.$$

Using (2.2.6) and coming back to (2.2.5), we have a contradiction to (2.2.1). Therefore we must have

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H^+(t_0, x_0, \nabla\varphi(t_0, x_0)) \leq 0. \qquad \square$$

Similarly repeating the techniques of the preceding proof and using Lemma 2.2, we have

Theorem 2.3 (Existence of viscosity subsolutions). *Assume that $(H_1)(b)$ and $(H_2)(b)$ are satisfied. Let us consider the value function V_J*

$$V_J(\tau, x) := \max_{\nu \in \mathcal{K}}\{\int_\tau^1 [\int_Z J(t, u_{x,\nu}(t), z)\, \nu_t(dz)]\, dt\},$$

where $u_{x,\nu}$ is the unique trajectory solution of

$$\begin{cases} \dot{u}_{x,\nu}(t) = \int_Z f(t, u_{x,\nu}(t), z)\, \nu_t(dz), \ \textit{a.e. } t \in [\tau, 1], \\ u_{x,\nu}(\tau) = x \in E. \end{cases}$$

Let us consider the upper Hamiltonian

$$H^+(t,x,y) = \max_{\nu \in \mathcal{M}^1_+(Z)} \{ y.\hat{f}(t,x,\nu) + \hat{J}(t,x,\nu) \}.$$

Then V_J is a semiviscosity subsolution *of the HJB equation*

$$U_t + H^+(t,x,\nabla U) = 0,$$

that is, if for any $\varphi \in \mathcal{C}^1_{\mathbf{R}}([0,1] \times E)$ for which $\nabla\varphi$ is ≥ 0 and bounded and $V_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0,1] \times E$, then

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H^+(t_0, x_0, \nabla\varphi(t_0, x_0)) \geq 0.$$

Proof. The proof is somewhat different from the ones of Theorem 2.2. Assume by contradiction that there exists a $\varphi \in \mathcal{C}^1_{\mathbf{R}}([0,1] \times E)$ for which $\nabla\varphi$ is ≥ 0 and bounded and a point $(t_0, x_0) \in [0,1] \times E$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H^+(t_0, x_0, \nabla\varphi(t_0, x_0)) < -\eta,$$

for some $\eta > 0$. As both $\hat{f}$ and $\nabla\varphi$ are ≥ 0 and bounded, by using $(H_1)(b)$ and $(H_2)(b)$ we may apply Lemma 2.2 to the integrand Λ defined by on $[0,1] \times E \times \mathcal{M}^1_+(Z)$ by

$$\Lambda(t,x,\nu) = \hat{J}(t,x,\nu) + \nabla\varphi(t,x).\hat{f}(t,x,\nu) + \frac{\partial \varphi}{\partial t}(t,x),$$

for all $(t,x,\nu) \in [0,1] \times E \times \mathcal{M}^1_+(Z)$ because the family

$$(\nabla\varphi(.,.).\hat{f}(.,.,\nu))_{\nu \in \mathcal{M}^1_+(Z)}$$

inherits the equi-upper semicontinuity property of the family

$$(\hat{f}(.,.,\nu))_{\nu \in \mathcal{M}^1_+(Z)}.$$

This provides $\sigma > 0$ such that,

$$\begin{aligned} (2.3.1) \quad \max_{\nu \in \mathcal{K}} \{ \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu}(t), z) \nu_t(dz)]\, dt \\ + \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\nu}(t)).f(t, u_{x_0,\nu}(t), z)] \nu_t(dz)]\, dt \\ + \int_{t_0}^{t_0+\sigma} \frac{\partial \varphi}{\partial t}(t, u_{x_0,\nu}(t)) dt \} < -\sigma\eta/2, \end{aligned}$$

where $u_{x_0,\nu}$ is the trajectory solution associated with the control $\nu \in \mathcal{K}$ of

$$\begin{cases} \dot{u}_{x_0,\nu}(t) = \int_Z f(t, u_{x_0,\nu}(t), z) \nu_t(dz), \ \textit{a.e. } t \in [0,1], \\ u_{x,\nu}(t_0) = x_0 \in E. \end{cases}$$

From Theorem 2.1 (of dynamic programming) (see e.g [9], Theorem 3.2.1) we deduce

$$(2.3.2)\quad V_J(t_0,x_0)=\max_{\nu\in\mathcal{K}}\{\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{x_0,\nu}(t),z)\nu_t(dz)]\,dt + V_J(t_0+\sigma,u_{x_0,\nu}(t_0+\sigma))\}.$$

Since $V_J-\varphi$ has a local maximum at (t_0,x_0), so for σ small enough

$$(2.3.3)\quad V_J(t_0,x_0)-\varphi(t_0,x_0)\geq V_J(t_0+\sigma,u_{x_0,\nu}(t_0+\sigma))-\varphi(t_0+\sigma,u_{x_0,\nu}(t_0+\sigma))$$

for all $\nu\in\mathcal{K}$. For each $n\in\mathbf{N}$, there is $\nu^n\in\mathcal{K}$ such that

$$(2.3.4)\quad V_J(t_0,x_0)\leq\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{x_0,\nu^n}(t),z)\nu_t^n(dz)]\,dt + V_J(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma))+1/n.$$

From (2.3.3) and (2.3.4) we deduce that

$$\begin{aligned}V_J(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma))&-\varphi(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma))\\ &\leq\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{x_0,\nu^n}(t),z)\nu_t^n(dz)]\,dt+1/n\\ &\quad-\varphi(t_0,x_0)+V_J(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma)).\end{aligned}$$

Whence we have

$$(2.3.4)\quad 0\leq\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{x_0,\nu^n}(t),z)\nu_t^n(dz)]\,dt + \varphi(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma))-\varphi(t_0,x_0)+1/n.$$

As φ is $\mathcal{C}^1$ and u_{x_0,ν^n} is the trajectory solution of our dynamic

$$\begin{aligned}(2.3.5)\quad &\varphi(t_0+\sigma,u_{x_0,\nu^n}(t_0+\sigma))-\varphi(t_0,x_0)\\ &=\int_{t_0}^{t_0+\sigma}[\int_Z[\nabla\varphi(t,u_{x_0,\nu^n}(t)).f(t,u_{x_0,\nu^n}(t),z)]\nu_t^n(dz)]\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\nu^n}(t))\,dt.\end{aligned}$$

By (2.3.4) and (2.3.5) we have, for each n,

$$\begin{aligned}(2.3.6)\quad 0&\leq\int_{t_0}^{t_0+\sigma}[\int_Z J(t,u_{x_0,\nu^n}(t),z)\nu_t^n(dz)]\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}[\int_Z[\nabla\varphi(t,u_{x_0,\nu^n}(t)).f(t,u_{x_0,\nu^n}(t),z)]\nu_t^n(dz)]\,dt\\ &\quad+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\nu^n}(t))dt+1/n.\end{aligned}$$

As $\mathcal{K}$ is compact metrisable for the stable topology, we may assume that (ν^n) stably converges to a Young measure $\overline{\nu}\in\mathcal{K}$. This implies that u_{x_0,ν^n} converges uniformly to $u_{x_0,\overline{\nu}}$ that is a trajectory solution of our dynamic associated to the control $\overline{\nu}$ and

$\delta_{u_{x_0,\nu^n}} \otimes \nu^n$ stably converges to $\delta_{u_{x_0,\overline{\nu}}} \otimes \overline{\nu}$ (see [9] and [10] for details). It follows that

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\nu^n}(t), z)\nu_t^n(dz)]\, dt = \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\overline{\nu}}(t), z)\overline{\nu}_t(dz)]\, dt$$

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\nu^n}(t)).f(t, u_{x_0,\nu^n}(t), z)]\nu_t^n(dz)]\, dt = \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\overline{\nu}}(t)).f(t, u_{x_0,\overline{\nu}}(t), z)]\overline{\nu}_t(dz)]\, dt,$$

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\nu^n}(t))dt = \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\nu}}(t))dt.$$

Finally by passing to the limit in (2.3.6) when $n \to \infty$ we get

$$0 \le \int_{t_0}^{t_0+\sigma} [\int_Z J(t, u_{x_0,\overline{\nu}}(t), z)\overline{\nu}_t(dz)]\, dt + \int_{t_0}^{t_0+\sigma} [\int_Z [\nabla\varphi(t, u_{x_0,\overline{\nu}}(t)).f(t, u_{x_0,\overline{\nu}}(t), z)]\overline{\nu}_t(dz)]\, dt + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\nu}}(t))dt. \tag{2.3.7}$$

This contradicts (2.3.1) and the proof is therefore complete. □

3. Some limit results using the Lebesgue derivation theorem

Now we are going to discuss other variants of Theorem 2.2-2.3 regarding the viscosity solutions when the dynamics are NOT globally continuous. For this purpose, we will exploit some ideas given in [8] concerning the use of Lebesgue derivation theorem for normal integrands. Before going further we present two lemmas which allow to study the limits for the Hamiltonians under consideration. In the sequel, (r^n) denotes a sequence of positive numbers such that $\lim_{n\to\infty} r^n = 0$.

Lemma 3.1. *Assume that (X, d_X) and (Y, d_Y) are two Polish spaces and $f : [0,1]\times X \times Y \to \mathbf{R}$ be a mapping satisfying:*

1) *$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le \eta[d_X(x_1, x_2) + d_Y(y_1, y_2)]$, for some $\eta > 0$, for all $(t, x_1, y_1), (t, x_2, y_2) \in [0,1] \times X \times Y$,*
2) *$f(., x, y)$ is Lebesgue-measurable for all $(x, y) \in X \times Y$,*
3) *f is bounded, say, $|f(t, x, y)| \le M$ for all $(t, x, y) \in [0,1] \times X \times Y$, for some positive constant M.*

If (μ^n) (resp. (ν^n)) is a sequence of Young measures in $\mathcal{Y}([0,1], X)$ (resp. $\mathcal{Y}([0,1], Y)$) which pointwisely converges on $[0,1]$ to a Young measure $\mu^\infty \in \mathcal{Y}([0,1], X)$ (resp. $\nu^\infty \in \mathcal{Y}([0,1], Y)$) for the narrow topology on $\mathcal{M}_+^1(X)$ (resp. $\mathcal{M}_+^1(Y)$), then

$$\lim_{n\to\infty} \frac{1}{2r^n} \int_{I_{t,r^n}} \langle f_s, \mu_t^n \otimes \nu_t^n\rangle ds = \langle f_t, \mu_t^\infty \otimes \nu_t^\infty\rangle$$

for almost all $t \in [0,1]$; here $I_{t,r^n} = [t - r^n, t + r^n] \cap [0,1]$.

Proof. Let us recall that $\mathcal{M}^1_+(X)$ and $\mathcal{M}^1_+(Y)$ endowed with the narrow topology are Polish spaces. Let $\mathcal{D}_X$ and $\mathcal{D}_Y$ be a countable dense subset of $\mathcal{M}^1_+(X)$ and $\mathcal{M}^1_+(Y)$ respectively. By 1)– 3), it is clear that for each $(\mu,\nu) \in \mathcal{M}^1_+(X) \times \mathcal{M}^1_+(Y)$ the real-valued function

$$s \mapsto \langle f_s, \mu\otimes\nu\rangle := \int_Y [\int_X f(s,x,y)\mu(dx)]\nu(dy)$$

is bounded and Lebesgue-measurable on $[0,1]$. We begin to prove that

$$\lim_{n\to\infty} \frac{1}{2r^n}\int_{I_{t,r^n}} \langle f_s, \mu\otimes\nu\rangle ds = \langle f_t, \mu\otimes\nu\rangle \tag{3.1.1}$$

for almost all $t \in [0,1]$, namely, there is a Lebesgue-negligible set N in $[0,1]$ which does not depend on $(\mu,\nu) \in \mathcal{M}^1_+(X) \times \mathcal{M}^1_+(Y)$ such that (3.1.1) holds for all $t \in [0,1]\backslash N$. Let μ and ν be arbitrary but fixed in $\mathcal{M}^1_+(X)$ and $\mathcal{M}^1_+(Y)$ respectively and let (μ^k) and (ν^k) be a sequence in $\mathcal{D}_X$ and $\mathcal{D}_Y$ respectively which narrowly converges to μ and ν respectively. By Lebesgue derivation theorem, there is a Lebesgue-negligible set N in $[0,1]$ such that

$$\lim_{n\to\infty} \frac{1}{2r^n}\int_{I_{t,r^n}} \langle f_s, \mu^k\otimes\nu^k\rangle ds = \langle f_t, \mu^k\otimes\nu^k\rangle \tag{3.1.2}$$

for all k and for all $t \in [0,1] \setminus N$. As (μ^k) and (ν^k) narrowly converge to μ and ν respectively, the product $(\mu^k\otimes\nu^k)$ narrowly converges to $\mu\otimes\nu$. By virtue of Dudley embedding theorem [14], $(\mu^k\otimes\nu^k)$ converges to $\mu\otimes\nu$ in the strong dual $BLIP(X\times Y)'$ of the Banach space $BLIP(X\times Y)$ of all real-valued bounded Lipschitzean functions defined on $X \times Y$ endowed with the norm $||f||_{BLIP(X\times Y)}$ given by

$$||f||_{BLIP(X\times Y)} := ||f||_\infty + \sup\{\frac{|f(x_1,y_1) - f(x_2,y_2)|}{d_{X\times Y}((x_1,y_1),(x_2,y_2))} : (x_1,y_1) \neq (x_2,y_2\},$$

here $X\times Y$ is endowed with the usual distance

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

for $(x_1,y_1),(x_2,y_2) \in X\times Y$. Taking account into assumptions 1)– 3) and the preceding consideration we have

$$K := \sup_{s\in[0,1]} ||f_s||_{BLIP(X\times Y)} < +\infty.$$

Consequently we have

$$|\langle f_s, \mu^k\otimes\nu^k\rangle - \langle f_s, \mu\otimes\nu\rangle| \leq K||\mu^k\otimes\nu^k - \mu\otimes\nu||_{BLIP(X\times Y)'}$$

for all $s \in [0,1]$. Let us write

$$|\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds - \langle f_t, \mu\otimes\nu\rangle|$$
$$\leq |\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds - \frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu^k\otimes\nu^k\rangle ds|$$
$$+ |\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu^k\otimes\nu^k\rangle ds - \langle f_t, \mu^k\otimes\nu^k\rangle|$$
$$+ |\langle f_t, \mu^k\otimes\nu^k\rangle - \langle f_t, \mu\otimes\nu\rangle|.$$

It follows that

$$|\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds - \langle f_t, \mu\otimes\nu\rangle| \leq 2K||\mu^k\otimes\nu^k - \mu\otimes\nu||_{BLIP(X\times Y)'}$$
$$+ |\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu^k\otimes\nu^k\rangle ds - \langle f_t, \mu^k\otimes\nu^k\rangle|.$$

Let $\varepsilon > 0$. There is $N_\varepsilon \in \mathbf{N}$ such that

$$||\mu^k\otimes\nu^k - \mu\otimes\nu||_{BLIP(X\times Y)'} \leq \varepsilon$$

for all $k \geq N_\varepsilon$. By (3.1.2) there is a Lebesgue-negligible set N in $[0,1]$ such that

$$\lim_n \frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu^k\otimes\nu^k\rangle ds = \langle f_t, \mu^k\otimes\nu^k\rangle$$

for all $k \in \mathbf{N}$ and for all $t \in [0,1]\setminus N$, so that

$$\limsup_n \frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds \leq \langle f_t, \mu\otimes\nu\rangle + 2K\varepsilon$$

for all $t \in [0,1]\setminus N$. Hence

$$\limsup_n \frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds \leq \langle f_t, \mu\otimes\nu\rangle$$

for all $t \in [0,1]\setminus N$ and similarly,

$$\liminf_n \frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu\otimes\nu\rangle ds \geq \langle f_t, \mu\otimes\nu\rangle$$

for all $t \in [0,1]\setminus N$. Let us write

$$|\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu_t^n\otimes\nu_t^n\rangle ds - \langle f_t, \mu_t^\infty\otimes\nu_t^\infty\rangle|$$
$$\leq |\frac{1}{2r^n}\int_{I_{t,r^n}}[\langle f_s, \mu_t^n\otimes\nu_t^n\rangle - \langle f_s, \mu_t^\infty\otimes\nu_t^\infty\rangle]ds|$$
$$+ |\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s, \mu_t^\infty\otimes\nu_t^\infty\rangle ds - \langle f_t, \mu_t^\infty\otimes\nu_t^\infty\rangle|.$$

Repeating the preceding arguments, we see that the product $\mu_t^n\otimes\mu_t^n$ converges to $\mu_t^\infty\otimes\nu_t^\infty$ in the strong dual $BLIP(X\times Y)'$ of the Banach space $BLIP(X\times Y)$. Using the estimate

$$|\langle f_s, \mu_t^n\otimes\nu_t^n\rangle - \langle f_s, \mu_t^\infty\otimes\nu_t^\infty\rangle| \leq K||\mu_t^n\otimes\nu_t^n - \mu_t^\infty\otimes\nu_t^\infty||_{BLIP(X\times Y)'},$$

we get

$$\begin{aligned}|\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^n\otimes\nu_t^n\rangle ds-\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle| \\ \leq K||\mu_t^n\otimes\nu_t^n-\mu_t^\infty\otimes\nu_t^\infty||_{BLIP(X\times Y)'} \\ +|\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^\infty\otimes\nu_t^\infty\rangle ds-\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle|.\end{aligned}$$

As

$$\lim_n||\mu_t^n\otimes\nu_t^n-\mu_t^\infty\otimes\nu_t^\infty||_{BLIP(X\times Y)'}=0$$

for all $t\in[0,1]$ and

$$\lim_n\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^\infty\otimes\nu_t^\infty\rangle ds=\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle|$$

for almost all $t\in[0,1]$, we conclude that

$$\lim_n\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^n\otimes\nu_t^n\rangle ds=\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle$$

for almost all $t\in[0,1]$. □

Lemma 3.2. *Assume that (X,d_X) and (Y,d_Y) are two Polish spaces and $f:[0,1]\times X\times Y\to\mathbf{R}$ is a mapping satisfying:*

1) *f is $\mathcal{L}([0,1])\otimes\mathcal{B}(X)\otimes\mathcal{B}(Y)$-measurable,*
2) *$f(t,.,.)$ is upper semicontinuous (resp. lower semicontinuous) on $X\times Y$,*
3) *f is bounded, say, $|f(t,x,y)|\leq M$, for all $(t,x,y)\in[0,1]\times X\times Y$, for some positive constant M.*

If (μ^n) (resp. (ν^n)) is a sequence of Young measures in $\mathcal{Y}([0,1],X)$ (resp. $\mathcal{Y}([0,1],Y)$) which pointwisely converges on $[0,1]$ to a Young measure $\mu^\infty\in\mathcal{Y}([0,1],X)$ (resp. $\nu^\infty\in\mathcal{Y}([0,1],Y)$) for the narrow topology on $\mathcal{M}_+^1(X)$ (resp. $\mathcal{M}_+^1(Y)$), then

$$\limsup_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^n\otimes\nu_t^n\rangle ds\leq\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle$$

$$(\textit{resp.}\ \liminf_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu_t^n\otimes\nu_t^n\rangle ds\geq\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle)$$

for almost all $t\in[0,1]$.

Proof. We need only to prove the case when $f(t,.,.)$ is upper semicontinuous. For each positive integer k, set

$$f^k(t,x,y):=\inf_{(u,v)\in X\times Y}\{M-f(t,u,v)+kd_{X\times Y}((x,y),(u,v))\}$$

for all $(t,x,y)\in[0,1]\times X\times Y$. By measurable projection theorem, ([12], Theorem III.23), $f^k(.,x,y)$ is Lebesgue-measurable. Further $0\leq f^k\leq M-f$ for all k and (f^k) is nondecreasing with $\sup_k f^k(t,x,y)=M-f(t,x,y)$ for all $(t,x,y)\in[0,1]\times X\times Y$. And

$$|f^k(t,x,y)-f^k(t,u,v)|\leq kd_{X\times Y}((x,y),(u,v))$$

for all $(t,x,y),(t,u,v) \in [0,1]\times X\times Y$. By Lemma 3.1, we have

$$\lim_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f^k_s,\mu^n_t\otimes\nu^n_t\rangle ds = \langle f^k_t,\mu^\infty_t\otimes\nu^\infty_t\rangle \tag{3.2.1}$$

for all k and all $t\in[0,1]\setminus N$, where N is a Lebesgue-negligible set in $[0,1]$. For each $t\in[0,1]$, we have

$$\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f^k_s,\mu^n_t\otimes\nu^n_t\rangle ds \le M-\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds.$$

Hence

$$\liminf_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f^k_s,\mu^n_t\otimes\nu^n_t\rangle ds \le M-\limsup_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds.$$

According to (3.2.1) and the preceding inequality we get

$$\langle f^k_t,\mu^\infty_t\otimes\nu^\infty_t\rangle \le M-\limsup_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds \tag{3.2.2}$$

for all $t\in[0,1]\setminus N$. Taking the supremum over k in (3.2.2) we get

$$M-\langle f_t,\mu^\infty_t\otimes\nu^\infty_t\rangle \le M-\limsup_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds \tag{3.2.3}$$

for almost all $t\in[0,1]$. From (3.2.3) it is immediate that

$$\limsup_{n\to\infty}\frac{1}{2r^n}\int_{I_{t,r^n}}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds \le \langle f_t,\mu^\infty_t\otimes\nu^\infty_t\rangle$$

for almost all $t\in[0,1]$. □

Let us mention a useful corollary of the preceding results.

Corollary 3.1. *Assume that $f:\mathbf{R}\times X\times Y\to\mathbf{R}$ is bounded, upper semicontinuous (resp. lower semicontinuous) mapping and the hypotheses and notations of Lemma 4.2 are satisfied. If (μ^n) (resp. (ν^n)) is a sequence of Young measures in $\mathcal{Y}([0,1],X)$ (resp. $\mathcal{Y}([0,1],Y)$) which pointwisely converges on $[0,1]$ to a Young measure $\mu^\infty\in\mathcal{Y}([0,1],X)$ (resp. $\nu^\infty\in\mathcal{Y}([0,1],Y)$) for the narrow topology on $\mathcal{M}^1_+(X)$ (resp. $\mathcal{M}^1_+(Y)$) and (t^n) is a sequence in $[0,1]$ converging to $t\in[0,1]$, then we have*

$$\limsup_{n\to\infty}\frac{1}{2r^n}\int_{t^n-r^n}^{t^n+r^n}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds \le \langle f_t,\mu^\infty_t\otimes\nu^\infty_t\rangle$$

$$(\textit{resp.}\ \liminf_{n\to\infty}\frac{1}{2r^n}\int_{t^n-r^n}^{t^n+r^n}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds \ge \langle f_t,\mu^\infty_t\otimes\nu^\infty_t\rangle)$$

for almost all $t\in[0,1]$.

Proof. It is enough to prove the case when f is upper semicontinuous. By an easy change of variable we have

$$\int_{t^n-r^n}^{t^n+r^n}\langle f_s,\mu^n_t\otimes\nu^n_t\rangle ds = \int_{t-r^n}^{t+r^n}\langle f_{s+(t^n-t)},\mu^n_t\otimes\nu^n_t\rangle ds.$$

Then the arguments of Lemma 3.2 can be applied to $\delta_{t^n-t}\otimes\mu_t^n\otimes\nu_t^n$ (which narrowly converges to $\delta_0\otimes\mu_t^\infty\otimes\nu_t^\infty$). Hence we have

$$\limsup_{n\to\infty}\frac{1}{2r^n}\int_{t-r^n}^{t+r^n}\langle f_{s+(t^n-t)},\mu_t^n\otimes\nu_t^n\rangle ds\le\langle f_t,\mu_t^\infty\otimes\nu_t^\infty\rangle$$

for almost all $t\in[0,1]$. □

4. Some limit results for Hamiltonians using the Lebesgue derivation theorem

Now we are able to formulate some results regarding the viscosity solutions for control problems when the dynamics are not globally continuous. For shortness, we introduce the following convergence. Let X be a compact metric space. Let (ψ^n,ψ^∞) be a sequence of mappings from X into a metric space W, (ψ^n) *$\mathcal{C}$-converges* to ψ^∞, if for any sequence (x^n) in X and for any $x\in ls\{x^n\}$ and for any subsequence (x^{n_k}) converging to x, we have $\lim_k\psi^{n_k}(x^{n_k})=\psi^\infty(x)$.

We begin with a sub-viscosity property of the value function V_J associated with upper semicontinuous dynamic f and continuous cost function J. Compare with Theorem 2.3.

Proposition 4.1. *Assume that $J:[0,1]\times\mathbf{R}\times Z\to\mathbf{R}$ is bounded and globally continuous, $f:\mathbf{R}\times\mathbf{R}\times Z\to\mathbf{R}$ is bounded, globally upper semicontinous and such that, for every $t\in[0,1]$, $f(t,.,.)$ is continuous on $\mathbf{R}\times Z$, and satisfying a Lipschitz condition: $|f(t,x_1,z)-f(t,x_2,z)|\le k|x_1-x_2|$ for all (t,x_1,z) and (t,x_2,z) in $[0,1]\times\mathbf{R}\times Z$. Let us consider the value function*

$$V_J^n(\tau,x):=\max_{\nu\in\mathcal{K}}\{\int_\tau^1[\int_Z J(t,u_{x,\nu}(t),z)\,\nu_t(dz)]\,dt\}$$

associated with the dynamic $(\mathcal{D}_{r^n})$

$$\begin{cases}\dot u_{x,\nu}(t)=\frac{1}{2r^n}\int_{t-r^n}^{t+r^n}[\int_Z f(s,u_{x,\nu}(t),z)\,\nu_t(dz)]ds, \text{ a.e. } t\in[\tau,1],\\ u_{x,\nu}(\tau)=x.\end{cases}$$

Let $\varphi^n,\varphi^\infty\in\mathcal{C}^1([0,1]\times\mathbf{R})$ for which $V_J^n-\varphi^n$ reaches a local maximum at (t^n,x^n). Assume further that (x^n) is bounded, $\nabla\varphi^n$ is ≥ 0 for all $n\in\mathbf{N}$, and $(\nabla\varphi^n,\frac{\partial\varphi^n}{\partial t})$ $\mathcal{C}$-converges to $(\nabla\varphi^\infty,\frac{\partial\varphi^\infty}{\partial t})$. Then there is a Lebesgue-negligible set N such that for $(t,x)\in ls\{t^n\}\setminus N\times ls\{x^n\}$, and for every subsequence (t^{n_k},x^{n_k}) of (t^n,x^n) converging to (t,x), we have

$$\begin{aligned}0&\le\limsup_k[\frac{\partial\varphi^{n_k}}{\partial t}(t^{n_k},x^{n_k})+H^+_{r_{n_k}}(t^{n_k},x^{n_k},\nabla\varphi^{n_k}(t^{n_k},x^{n_k}))]\\&\le\frac{\partial\varphi^\infty}{\partial t}(t,x)+H^+(t,x,\nabla\varphi^\infty(t,x)).\end{aligned}$$

Proof. Let us set $\hat f(t,x,\nu)=\int_Z f(t,x,z)\nu(dz)$ for all $(t,x,\nu)\in[0,1]\times\mathbf{R}\times\mathcal{M}^1_+(Z)$, and for each $n\in\mathbf{N}$,

$$\hat f_{r^n}(t,x,\nu):=\frac{1}{2r^n}\int_{t-r^n}^{t+r^n}\hat f(s,x,\nu)ds$$

for all $(t, x, \nu) \in [0, 1] \times \mathbf{R} \times \mathcal{M}_+^1(Z)$.Then it is not difficult to see that the function $\hat{f}_{r_n}$ is globally continuous on $[0, 1] \times E \times \mathcal{M}_+^1(Z)$ and uniformly Lipschitzean on $\mathbf{R}$. Consequently, by virtue of ([9], Theorem 3.2.3, [7], Theorem 4.3), the function V_J^n associated with the dynamic $(\mathcal{D}_{r^n})$ is a viscosity subsolution of the HJB equation

$$U_t(t, x) + H_{r_n}^+(t, x, \nabla U(t, x)) = 0,$$

that implies

$$\frac{\partial \varphi^n}{\partial t}(t^n, x^n) + H_{r^n}^+(t^n, x^n, \nabla\varphi^n(t^n, x^n)) \geq 0.$$

To simplify the notations, we may assume that $(t^n, x^n) \to (t, x)$. By the continuity of the function

$$\nu \mapsto \nabla\varphi^n(t^n, x^n).\hat{f}_{r^n}(t^n, x^n, \nu) + \hat{J}(t^n, x^n, \nu)$$

and by the compactness of $\mathcal{M}_+^1(Z)$ there is $\nu^n \in \mathcal{M}_+^1(Z)$ such that

$$\begin{aligned} H_{r_n}^+(t^n, x^n, \nabla\varphi^n(t^n, x^n)) &= \max_{\nu\in\mathcal{M}_+^1(Z)} \{\nabla\varphi^n(t^n, x^n).\hat{f}_{r_n}(t^n, x^n, \nu) + \hat{J}(t^n, x^n, \nu)\} \\ &= \nabla\varphi^n(t^n, x^n).\hat{f}_{r_n}(t^n, x^n, \nu^n) + \hat{J}(t^n, x^n, \nu^n). \end{aligned}$$

We may assume that (ν^n) narrowly converges to $\nu \in \mathcal{M}_+^1(Z)$. By Corollary 3.1, there is a Lebesgue-negligible set N such that

$$\begin{aligned} \limsup_n \hat{f}_{r^n}(t^n, x^n, \nu^n) &= \limsup_n \frac{1}{2r^n} \int_{t^n-r^n}^{t^n+r^n} \langle f_s, \delta_{x^n} \otimes \nu^n\rangle ds \\ &= \limsup_n \frac{1}{2r^n} \int_{t-r^n}^{t+r^n} \langle f_{s+(t^n-t)}, \delta_{x^n} \otimes \nu^n\rangle ds \\ &\leq \langle f_t, \delta_x \otimes \nu\rangle \end{aligned}$$

for each $t \in ls\{t^n\} \setminus N$. Combining this with the continuity of $\hat{J}$, the nonnegativity of $(\nabla\varphi^n(t^n, x^n))_{n\in\mathbf{N}}$ and the $\mathcal{C}$-convergence of $(\nabla\varphi^n, \frac{\partial\varphi^n}{\partial t})$ gives

$$\begin{aligned} 0 &\leq \limsup_n [\frac{\partial \varphi^n}{\partial t}(t^n, x^n) + H_{r^n}^+(t^n, x^n, \nabla\varphi^n(t^n, x^n))] \\ &= \limsup_n [\frac{\partial \varphi^n}{\partial t}(t^n, x^n) + \nabla\varphi^n(t^n, x^n).\hat{f}_{r_n}(t^n, x^n, \nu^n) + \hat{J}(t^n, x^n, \nu^n)] \\ &\leq \frac{\partial \varphi^\infty}{\partial t}(t, x) + \nabla\varphi^\infty(t, x).\hat{f}(t, x, \nu) + \hat{J}(t, x, \nu) \\ &\leq \frac{\partial \varphi^\infty}{\partial t}(t, x) + \max_{\nu'\in\mathcal{M}_+^1(Z)} \{\nabla\varphi^\infty(t, x).\hat{f}(t, x, \nu') + \hat{J}(t, x, \nu')\} \\ &= \frac{\partial \varphi^\infty}{\partial t}(t, x) + H^+(t, x, \nabla\varphi^\infty(t, x)) \end{aligned}$$

for $t \in ls\{t^n\} \setminus N$. □

Now is a super-viscosity property of V_J associated with lower semicontinuous dynamic f and continuous cost function J. Compare with Theorem 2.2.

Proposition 4.2. *Let $J : [0,1]\times \mathbf{R}\times Z \to \mathbf{R}$ be a bounded and globally continuous function, and let $f : \mathbf{R}\times\mathbf{R}\times Z \to \mathbf{R}$ be a bounded lower semicontinuous function such that, for every $t \in [0,1]$, $f(t,.,.)$ is continuous on $\mathbf{R}\times Z$, and satisfying a Lipschitz condition: $|f(t,x_1,z)-f(t,x_2,z)| \le k|x_1-x_2|$ for all (t,x_1,z) and (t,x_2,z) in $[0,1]\times\mathbf{R}\times Z$. Assume that the value function*

$$V_J^n(\tau,x) := \max_{\nu\in\mathcal{K}}\{\int_\tau^1 [\int_Z J(t,u_{x,\nu}(t),z)\,\nu_t(dz)]\,dt\}$$

associated with the dynamic $(\mathcal{D}_{r^n})$,

$$\begin{cases} \dot u_{x,\nu}(t) = \hat f_{r_n}(t,u_{x,\nu}(t),\nu_t) = \frac{1}{2r^n}\int_{t-r^n}^{t+r^n}[\int_Z f(s,u_{x,\nu}(t),z)\,\nu_t(dz)]ds, \ \ a.e.\ t\in[\tau,1],\\ u_{x,\nu}(\tau) = x\end{cases}$$

is a viscosity supersolution of the HJB equation

$$U_t(t,x) + H^+_{r_n}(t,x,\nabla U(t,x)) = 0.$$

Let $\varphi^n,\varphi^\infty \in \mathcal{C}^1([0,1]\times\mathbf{R})$ for which $V_J^n - \varphi^n$ reaches a local minimum at (t^n,x^n). Assume further that (x^n) is bounded, $\nabla\varphi^n$ is nonnegative for all $n\in\mathbf{N}$, and $(\nabla\varphi^n,\frac{\partial\varphi^n}{\partial t})$ $\mathcal{C}$-converges to $(\nabla\varphi^\infty,\frac{\partial\varphi^\infty}{\partial t})$. Then there is a Lebesgue-negligible set N such that for $(t,x)\in ls\{t^n\}\setminus N\times ls\{x^n\}$, and for every subsequence (t^{n_k},x^{n_k}) of (t^n,x^n) converging to (t,x), we have

$$\begin{aligned}&\frac{\partial\varphi^\infty}{\partial t}(t,x) + H^+(t,x,\nabla\varphi^\infty(t,x))\\ &\qquad\le \liminf_k[\frac{\partial\varphi^{n_k}}{\partial t}(t^{n_k},x^{n_k}) + H^+_{r_{n_k}}(t^{n_k},x^{n_k},\nabla\varphi^{n_k}(t^{n_k},x^{n_k}))] \le 0.\end{aligned}$$

Proof. As V_J^n is a viscosity supersolution of the HJB equation

$$U_t(t,x) + H^+_{r_n}(t,x,\nabla U(t,x)) = 0,$$

we have

$$\frac{\partial\varphi^n}{\partial t}(t^n,x^n) + H^+_{r^n}(t^n,x^n,\nabla\varphi^n(t^n,x^n)) \le 0.$$

It follows that

$$\begin{aligned}&\liminf_n[\frac{\partial\varphi^n}{\partial t}(t^n,x^n) + \nabla\varphi^n(t^n,x^n).\hat f_{r^n}(t^n,x^n,\nu) + \hat J(t^n,x^n,\nu)]\\ &\qquad\le \liminf_n[\frac{\partial\varphi^n}{\partial t}(t^n,x^n) + H^+_{r^n}(t^n,x^n,\nabla\varphi^n(t^n,x^n))] \le 0\end{aligned}$$

for each $\nu\in\mathcal{M}^1_+(Z)$. To simplify the notations, we may assume that (t^n,x^n) converges to (t,x). Recall that

$$\hat f_{r^n}(t^n,x^n,\nu) := 1/2r^n\int_{t^n-r^n}^{t^n+r^n}\langle f_s,\delta_{x^n}\otimes\nu\rangle ds$$

for each $\nu\in\mathcal{M}^1_+(Z)$. Let $\mathcal{Z}$ be a countable dense subset of $\mathcal{M}^1_+(Z)$. As f is bounded, globally lower semicontinuous, in view of Corollary 3.1, there is a

Lebesgue-negligible set N which does not depend on $\nu \in \mathcal{Z}$ such that

$$\liminf_n 1/2r^n \int_{t^n-r^n}^{t^n+r^n} \langle f_s, \delta_{x^n} \otimes \nu \rangle ds$$
$$= \liminf_n 1/2r^n \int_{t-r^n}^{t+r^n} \langle f_{s+(t^n-t)}, \delta_{x^n} \otimes \nu \rangle ds \geq \langle f_t, \delta_x \otimes \nu \rangle$$

for each $\nu \in \mathcal{Z}$ and for each $t \in ls\{t^n\} \setminus N$. As $(\nabla\varphi^n, \frac{\partial \varphi^n}{\partial t})$ $\mathcal{C}$-converges to $(\nabla\varphi^\infty, \frac{\partial \varphi^\infty}{\partial t})$ by hypothesis, and $\nabla\varphi^n(t^n, x^n)$ is ≥ 0 for all $n \in \mathbf{N}$, we get

$$\begin{aligned}
&\frac{\partial \varphi^\infty}{\partial t}(t,x) + \nabla\varphi^\infty(t,x).\langle f_t, \delta_x \otimes \nu\rangle + \hat{J}(t,x,\nu) \\
&\leq \liminf_n [\frac{\partial \varphi^n}{\partial t}(t^n,x^n) + \nabla\varphi^n(t^n,x^n).\hat{f}_{r^n}(t^n,x^n,\nu) + \hat{J}(t^n,x^n,\nu)] \\
&\leq \liminf_n [\frac{\partial \varphi^n}{\partial t}(t^n,x^n) + H^+_{r_n}(t^n,x^n,\nabla\varphi^n(t^n,x^n))] \leq 0
\end{aligned}$$

for each $\nu \in \mathcal{Z}$. Hence we deduce that

$$\begin{aligned}
&\frac{\partial \varphi^\infty}{\partial t}(t,x) + \sup_{\nu \in \mathcal{Z}}\{\nabla\varphi^\infty(t,x).\langle f_t, \delta_x \otimes \nu\rangle + \hat{J}(t,x,\nu)\} \\
&= \frac{\partial \varphi^\infty}{\partial t}(t,x) + \sup_{\nu' \in \mathcal{M}^1_+(Z)}\{\nabla\varphi^\infty(t,x).\langle f_t, \delta_x \otimes \nu'\rangle + \hat{J}(t,x,\nu')\} \\
&= \frac{\partial \varphi^\infty}{\partial t}(t,x) + H^+(t,x,\nabla\varphi^\infty(t,x)) \\
&\leq \liminf_n [\frac{\partial \varphi^n}{\partial t}(t^n,x^n) + H^+_{r_n}(t^n,x^n,\nabla\varphi^n(t^n,x^n))] \leq 0
\end{aligned}$$

for $t \in ls\{t^n\} \setminus N$. □

Remarks. If the family $(\hat{f}_{r_n}(.,.,\nu))_{\nu \in \mathcal{M}^1_+(Z)}$ is equi lower semicontinuous on $[0,1]\times\mathbf{R}$, then

$$V^n_J(\tau,x) := \max_{\nu \in \mathcal{K}}\{\int_\tau^1 [\int_n ZJ(t,u_{x,\nu}(t),z)\,\nu_t(dz)]\,dt\}$$

associated with the dynamic $(\mathcal{D}_{r^n})$,

$$\begin{cases} \dot{u}_{x,\nu}(t) = \hat{f}_{r_n}(t,u_{x,\nu}(t),\nu_t) = \frac{1}{2r^n}\int_{t-r^n}^{t+r^n}[\int_Z f(s,u_{x,\nu}(t),z)\,\nu_t(dz)]ds, \ \ a.e.\ t \in [\tau,1], \\ u_{x,\nu}(\tau) = x \end{cases}$$

is a viscosity supersolution of the HJB equation

$$U_t(t,x) + H^+_{r_n}(t,x,\nabla U(t,x)) = 0.$$

See ([9], Theorem 3.2.3).

Acknowledgements. We thank Lionel Thibault and Sylvie Marcellin for helpful discussions and comments.

References

[1] E.J. Balder, New fundamentals of Young measure convergence, Calculus of variations and optimal control (Haifa, 1998) (Boca Raton, FL) Chapman and Hall, 2000, 24–48.

[2] E. N. Barron and R. Jensen, Optimal control and semicontinuous viscosity solutions, Proc. A. M. S, Vol 113 (2), (1991), 397–402.

[3] E. N. Barron and R. Jensen, Generalized Viscosity Solutions for Hamilton-Jacobi Equations with Time-measurable Hamiltonians, Journal of Differential equations, Vol 68, (1987), 10–21.

[4] H. Benabdellah, Existence of solutions to the nonconvex sweeping process, J. Differential Eq. Vol 164, (2000), 286-295.

[5] C. Castaing and A. G. Ibrahim, Functional evolution equations governed by m-accretive operators, Adv. Math. Econ., Vol 5, (2003), 23–54.

[6] C. Castaing and A. Jofre, Optimal control problems and Variational problems, Preprint 03/03 Université de Montpellier II, January 2003.

[7] C. Castaing, A. Jofre and A. Salvadori, Control problems governed by functional evolution inclusions with Young measures, Journal of Nonlinear and Convex Analysis, Volume 5, Number 1, 2004, 131-152.

[8] C. Castaing, M. Moussaoui and A. Syam, Multivalued differential equations on closed convex sets in Banach spaces, Set-Valued Analysis, Vol 1, (1994), 329-353.

[9] C. Castaing and Paul Raynaud de Fitte, On the fiber product o Young measures with applications to a Control problem with measures, Adv. Math. Econ. Vol 6, (2004), 1-38.

[10] Castaing, C., Raynaud de Fitte, P., and Valadier, M. (2004). *Young measures on Topological Spaces. With Applications in Control Theory and Probability Theory*. Kluwer Academic Publishers, Dordrecht.

[11] C. Castaing, A. Salvadori and L. Thibault, Functional evolution equations governed by nonconvex sweeping process, Journal of Nonlinear and Convex Analysis, Vol 2, No 2, (2001), 217–241.

[12] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Vol 580, Springer Verlag, Berlin, (1977).

[13] G. Colombo and V. V. Goncharov, The sweeping process without convexity, Set-Valued Analysis, Vol 7, (1999), 357–374.

[14] R. M. Dudley, Convergence of Baire Measures, Studia Math. Vol 27, (1966), 7-17.

[15] R. J. Elliot, Viscosity solutions and optimal control, Pitman, London (1977)

[16] L. C. Evans and P. E. Souganides, Differential games and representation formulas for solutions of Hamilton-Jacobi-Issacs equations, Indiana Univ. Math. J, Vol 33, (1984), 773-797.

[17] H. Ishii, Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets, Bull. Fac. Engreg. Chuo. Univ., Vol 28 (1985), 33-77.

[18] S. Kempisty, Sur les fonctions semicontinues par rapport à chacune de deux variables. *Fund. Math.* 14, (1929), 237–241.

[19] M. Kunze and M. D. P. Monteiro Marques, BV Solutions to Evolution Problems with Time-dependent domains, Set-Valued Analysis, Vol 5 (1997), 57–72.

[20] P. L. Lions and B. Perthame, Remarks on Hamilton-Jacobi equations with measurable time dependent Hamiltonians, Nonlinear Analysis, Theory, Methods and Applications, Vol 11 , No 5, (1987), 613–621.

[21] M. D. P Monteiro Marquès, Differential inclusions in Nonsmooth Mechanical Problems, Shocks and Dry Friction, Birkhäuser (1993).

[22] J. J. Moreau, Evolution problem associated with a moving set in Hilbert space, J. Differential. Eq., Vol 26, (1977), 347–374.

[23] W. Sierpinski, Sur un problème concernant les ensembles mesurables superficiellement. *Fund. Math.* 1, (1920), 112–115.

[24] R. A. Poliquin, R. T. Rockafellar and L. Thibault, Local differentiability of distance functions, Trans. Amer. Math. Soc., Vol 352, No 11 (2000), 5231–5249.

[25] L. Thibault, Sweeping process with regular and nonregular sets, J. Differential Equations, Vol 193, (2003), 1–26.

[26] M. Valadier, Young measures, Methods of Convex Analysis, Lectures Notes in Mathematics, Vol 1445, (1990), 152-188.
[27] M. Valadier, A course of Young measures, Rendiconti dell'istituto di matematica dell' University di Trieste **26, suppl.** (1994), 349-394, Workshop di Teoria della Misura et Analisi Reale Grado, 1993 (Italia).

Manuscript received April 1, 2005

C. Castaing
Département de Mathématiques, Université Montpellier II, 34095 Montpellier Cedex 5, France.
E-mail address: `castaing@math.univ-montp2.fr`

A. Jofre
Departamento de Ingenieria Matematica, Universidade de Chile, Casilla, 170/3, Correo 3, Santiago, Chile.
E-mail address: `ajofre@dim.uchile`

A. Syam
Département de Mathématiques, Université Cadi Ayyad, Marrakech, Maroc.
E-mail address: `Syam@fstg-marrakech.ac.ma`