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UNIFORM NORMAL STRUCTURE AND STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Let K be a nonempty closed convex and bounded subset of a real Banach space E and $T : K \to K$ be uniformly L-Lipschitzian, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$, and asymptotically pseudocontractive with sequence $\{k_n\}$ where $\{k_n\}$ and $\{\varepsilon_n\}$ satisfy certain mild conditions. Let a sequence $\{x_n\}$ be generated from $x_1 \in K$ by $x_{n+1} := (1-\lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1)$ for all integers $n \geq 1$ where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences satisfying appropriate conditions, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Moreover if E has uniform normal structure with coefficient $N(E)$, $L < N(E)^{1/2}$ as well as has a uniformly Gâteaux differentiable norm and T satisfies an additional mild condition, then $\{x_n\}$ also converges strongly to a fixed point of T. The results presented in this paper are improvements, extension and complement of some earlier and recent ones in the literature.

1. Introduction

Let E be a real normed linear space with dual E^* and let $J : E \to 2^{E^*}$ denote the normalized duality mapping defined by

$$J(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. It is well known that if E is smooth then J is single-valued. In the sequel, we will denote the single-valued normalized duality map by J.

Let E be a normed linear space; $\emptyset \neq K \subset E$. A mapping $T : K \to K$ is said to be nonexpansive if for all $x, y \in K$ we have $\|Tx - Ty\| \leq \|x - y\|$. It is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all integers $n \geq 0$ and all $x, y \in K$. Clearly every nonexpansive map is asymptotically nonexpansive with sequence $k_n = 1$ $\forall n \geq 0$. There exist however asymptotically nonexpansive mappings which are not nonexpansive (see, e.g., [5]).

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] in 1972. Goebel and Kirk [4] proved that if K is a nonempty closed

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convex and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping of K has a fixed point. It is also remarkable that many authors have extensively studied the iterative approximation problems of fixed points of nonexpansive mappings and asymptotically nonexpansive mappings (see, e.g., [6, 8, 9, 13-18, 20, 22, 23]).

An important class of nonlinear mappings generalizing the class of asymptotically nonexpansive mappings was introduced by Schu [17] in 1991. Let K be a nonempty subset of a real Banach space E and $T : K \to E$ be any map. T is said to be asymptotically pseudocontractive if there exist $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x - y) \in J(x - y)$ such that the inequality

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2 \tag{1.1}$$

holds for all $x, y \in K$ and for all integers $n \geq 1$. It is trivial to see from inequality (1.1) that every asymptotically nonexpansive mapping is asymptotically pseudocontractive. But the converse is not valid in general; see, e.g., Chang [13, Example 1.1]. The mapping T is called uniformly asymptotically regular if for each $\varepsilon > 0$ there exists an integer $n_0 \geq 1$ such that $\|T^{n+1}x - T^n x\| \leq \varepsilon$ for all $n \geq n_0$ and all $x \in K$ and it is said to be uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ if $\|T^{n+1}x - T^n x\| \leq \varepsilon_n$ $\forall n \geq 0$ and $\forall x \in K$ where $\varepsilon_n \to 0$ as $n \to \infty$. T is called uniformly L-Lipschitzian if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ $\forall x, y \in K$ and for each integer $n \geq 1$. Also recall that a sequence $\{x_n\} \subset K$ is called an approximate fixed point sequence for T if $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

In 1991 Schu [16] constructed approximate fixed point sequences for the class of asymptotically pseudocontractive maps in Hilbert spaces. Furthermore Schu [17] also contructed an iterative algorithm which converges strongly to a fixed point of an asympotically pseudocontractive mapping in a smooth Banach space possessing a duality mapping $J : E \to E^*$ that is weakly sequentially continuous at 0. Unfortunately, L_p space, $1 < p < \infty, p \neq 2$, do not possess weakly sequentially continuous duality maps.

Recently Chidume and Zegeye [21] put forth the following two questions on the iterative approximation problems of fixed points of asymptotically pseudocontractive mappings.

Question 1. Is it possible to construct an approximate fixed point sequence for the class of asymptotically pseudocontractive maps in spaces more general than Hilbert spaces?

Question 2. Can an iterative algorithm be constructed which converges to a fixed point of an asymptotically pseudocontractive mapping in Banach spaces which include L_p spaces, $1 < p < \infty$?

Chidume and Zegeye [21] have given the following affirmative answers to Questions 1 and 2.

Theorem 1.1 ([21]). *Let K be a nonempty closed convex and bounded subsets of a real Banach space E. Let $T : K \to K$ be uniformly L-Lipschitzian, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with* *sequence* $\{k_n\}$ *such that for* $\lambda_n, \theta_n \in (0,1)$ $\forall n \geq 0$*, the following conditions are satisfied:* (i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty; \lambda_n(1+\theta_n) \leq 1$; (ii) $\frac{\lambda_n}{\theta_n} \to 0, \theta_n \to 0, (\frac{\theta_{n-1}}{\theta_n} - 1)/\lambda_n\theta_n \to 0, \frac{\varepsilon_{n-1}}{\lambda_n\theta_n^2} \to 0$; (iii) $k_{n-1} - k_n = o(\lambda_n\theta_n)^2$; (iv) $k_n - 1 = o(\theta_n)$. *Let a sequence* $\{x_n\}$ *be iteratively generated from* $x_1 \in K$ *by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1), \ \forall n \geq 1.$$

Then $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$*.*

Theorem 1.2 ([21])**.** *Suppose* E *is a real reflexive Banach space with uniform normal structure and suppose* E *has a uniformly Gâteaux differentiable norm. Let* K *be a nonempty closed convex and bounded subset of* E*. Let* $T : K \to K$ *be uniformly* L*-Lipschizian with* $L < N(E)^{1/2}$*, uniformly asymptotically regular with sequence* $\{\varepsilon_n\}$ *and asymptotically pseudocontractive with sequence* $\{k_n\}$*. Let* $\lambda_n, \theta_n \in (0,1)$ $\forall n \geq 1$ *satisfy conditions* (i)–(iv) *of Theorem 1.1 and let* $\lim_{n\to\infty} \frac{k_n-1}{k_n-t_n} = 0$ *where* $t_n = 1/(1+\theta_n)$*. Suppose that*

$$\|y_n - T^m y\|^2 \leq \langle y_n - T^m y, J(y_n - y)\rangle, \ \forall m, n \geq 1, \ \forall y \in C$$

where $C = \{y \in K : \phi(y) = \min_{z\in K}\phi(z)\}$*. Then the sequence* $\{x_n\}$ *generated from* $x_1 \in K$ *by*

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1), \ \forall n \geq 1$$

converges strongly to a fixed point of T*.*

Remark 1.1. Lim and Xu [20, p. 1346] reminded us of the following fact: A Banach space with uniform normal structure is reflexive and all uniformly convex or uniformly smooth Banach spaces have uniform normal structure. Therefore the reflexivity assumption on E in Theorem 1.2 can be removed.

The purpose of this paper is to continue the study of Questions 1 and 2 and to give also affirmative answers to them. Let K be a nonempty closed convex and bounded subset of a real Banach space E and $T : K \to K$ be uniformly L-Lipsdchitzian, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with sequence $\{k_n\}$ where $\{k_n\}$ and $\{\varepsilon_n\}$ satisfy certain mild conditions. Let a sequence $\{x_n\}$ be generated from $x_1 \in K$ by

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1), \ \forall n \geq 1$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences satisfying approximate conditions, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. This also provides an affirmative answer to Question 1. Moreover if E has uniform normal structure with coefficient $N(E)$ and $L < N(E)^{1/2}$ and has a uniformly Gâteaux differentiable norm and T satisfies an additional mild condition, then $\{x_n\}$ also converges strongly to a fixed point of T. This also provides an affirmative answer to Question 2. Compared with those restrictions (i)-(iv) in Chidume and Zegeye's Theorem 1.1, our restrictions on sequences $\{\lambda_n\}, \{\theta_n\}, \{k_n\}$ and $\{\varepsilon_n\}$ are quite concise and very convenient to test in applications. Also due to Lim and Xu [20, p. 1346], our theorems remove the reflexivity assumption on E.

2. Preliminaries

Let K be a nonempty bounded closed convex subset of a real Banach space E and let $d(K) := \sup\{\|x-y\| : x, y \in K\}$ be the diameter of K. For any $x \in K$ let $r(x, K) := \sup\{\|x-y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ be the Chebyshev radius of K relative to itself. The normal structure coefficient of E is defined (e.g., [2]) as the number:

$$N(E) := \inf\{d(K)/r(K) : K \text{ is a bounded closed convex subset of } E \text{ with } d(K) > 0\}.$$

A space E such that $N(E) > 1$ is said to have uniform normal structure. It is known that a space with uniform normal structure is reflexive and that all uniformly convex Banach spaces and all uniformly smooth Banach spaces have uniform normal structure (e.g., [1]; see also [20]).

Recall (e.g., see [19]) that a Banach limit LIM is a bounded linear functional on l^∞ such that

$$\|\mathrm{LIM}\| = 1, \ \liminf_{n\to\infty} t_n \le \mathrm{LIM}_n t_n \le \limsup_{n\to\infty} t_n, \tag{2.1}$$

and $\mathrm{LIM}_n t_n = \mathrm{LIM}_n t_{n+1}$ for all $\{t_n\} \in l^\infty$.

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $T : K \to K$ is called pseudoconractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 \tag{2.2}$$

for all $x, y \in K$. As a result of Kato [7], it follows that inequality (2.2) is equivalent to

$$\|x-y\| \le \|x-y+t((I-T)x-(I-T)y)\|$$

for each $x, y \in K$ and for all $t > 0$ where I is the identity operator. In order to establish the main results of this paper, we need the following lemmas.

Lemma 2.1 (see e.g., [3, 13]). *Let E be a real normed linear space. Then for any $x, y \in E$ and $j(x+y) \in J(x+y)$, we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Lemma 2.2 (see Lemma 2.5 in [24]). *Let $\{\lambda_n\}$ be a sequence of real numbers in $[0,1]$, and $\{\gamma_n\}$ and $\{\mu_n\}$ be sequences of nonnegative real numbers. Assume that $\sum_{n=0}^\infty \lambda_n = \infty$ and $\sum_{n=0}^\infty \mu_n < \infty$. Then there hold the following statements:*

(i) *If for any given $\varepsilon > 0$, there exists a positive integer n_0 such that*

$$\gamma_{n+1} \le (1-\lambda_n)\gamma_n + \varepsilon \cdot \lambda_n + \mu_n, \ \forall n \ge n_0, \tag{2.3}$$

then $\limsup_{n\to\infty} \gamma_n = 0$;

(ii) *If there exists a positive integer n_1 such that*

$$\gamma_{n+1} \le (1-\lambda_n)\gamma_n + \lambda_n \cdot \sigma_n + \mu_n, \ \forall n \ge n_1, \tag{2.4}$$

where $\{\sigma_n\}$ is a sequence of nonnegative real numbers satisfying $\lim_{n\to\infty} \sigma_n = 0$, *then*

$$\lim_{n\to\infty} \gamma_n = 0.$$

Lemma LX ([20, Theorem 1]). *Suppose E is a Banach space with uniform normal structure, K is a nonempty bounded subset of E and $T : K \to K$ is a uniformly L-Lipschitizan mapping with $L < N(E)^{1/2}$. Suppose also there exists a nonempty closed convex subset C of K with the following property (P): $x \in C \Rightarrow \omega_w(x) \subset C$ where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e., the set $\{y \in E : y = \text{weak} - \lim_j T^{n_j} x$ for some $n_j \to \infty\}$. Then T has a fixed point in K.*

Lemma SR ([12]). *Let E be a Banach space with a uniformly Gateaux differentiable norm, K be a nonempty closed convex subset of E and $\{x_n\}$ be a bounded sequence in E. Let LIM be a Banach limit and $y \in K$. Then*

$\text{LIM}_n\|x_n - y\|^2 = \min_{z\in K}\text{LIM}_n\|x_n - z\|^2 \Longleftrightarrow \text{LIM}_n\langle x - y, J(x_n - y)\rangle \le 0,\ \forall x \in K.$

Lemma CZ ([21, Lemma 3.1]). *Let E be a real Banach space. Suppose K is a nonempty closed convex and bounded subset of E and $T : K \to K$ is a uniformly asymptotically regular, uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with sequence $\{k_n\}$. Then for $u \in K$ and $\{t_n\} \subset (0, 1)$ such that $t_n \to 1$ as $n \to \infty$, there exists a sequence $\{y_n\} \subset K$ satisfying the following condition:*

$$y_n = \frac{t_n}{k_n}T^n y_n + \left(1 - \frac{t_n}{k_n}\right)u. \tag{2.5}$$

Furthermore, $\|y_n - Ty_n\| \to 0$ as $n \to \infty$.

3. Main Results

Now we state and prove the main results of this paper.

Theorem 3.1. *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $T : K \to K$ be a uniformly L-Lipschitzian, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with sequence $\{k_n\}$ such that for $\lambda_n, \theta_n \in (0, 1)$ $\forall n \ge 1$, the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty,\ \lambda_n(1 + \theta_n) \le 1$;

(ii) $\lim_{n\to\infty} \frac{\lambda_n}{\theta_n} = 0,\ \lim_{n\to\infty}\theta_n = 0,\ \lim_{n\to\infty}(\frac{\theta_{n-1}}{\theta_n} - 1)/\lambda_n\theta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \frac{\varepsilon_{n-1}}{\theta_n} < \infty,\ \sum_{n=1}^{\infty} \frac{|k_{n-1}-k_n|}{\theta_n} < \infty,\ \sum_{n=1}^{\infty} \lambda_n(k_n - 1) < \infty.$

Let a sequence $\{x_n\}$ be iteratively generated from $x_1 \in K$ by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1),\ \forall n \ge 1. \tag{3.1}$$

Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Proof. Note that (3.1) can be rewritten as

$$x_{n+1} := (1 - \lambda_n - \lambda_n\theta_n)x_n + \lambda_n T^n x_n + \lambda_n\theta_n x_1,\ \forall n \ge 1.$$

Since $\lambda_n(1 + \theta_n) \le 1$, it is easy to see that the sequence $\{x_n\}$ is well defined.

Let $\{y_n\}$ denote the sequence defined as in (2.5) with $t_n = \frac{1}{1+\theta_n}$. Then following the same estimate technique as in the proof of [21, Theorem 3.2] and using (3.1) and Lemma 2.1, we get

(3.2)

$$\|x_{n+1} - y_n\|^2 = \|x_n - y_n - \lambda_n((x_n - T^n x_n) + \theta_n(x_n - x_1))\|^2$$

$$
\begin{aligned}
&\le \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 \\
&\quad + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) - (x_n - T^n x_n) + \theta_n(x_1 - y_n), J(x_{n+1} - y_n)\rangle \\
&\le \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 \\
&\quad + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) + [\theta_n(x_1 - y_n) - (y_n - \frac{1}{k_n}T^n y_n)] \\
&\quad - [(x_{n+1} - \frac{1}{k_n}T^n x_{n+1}) - (y_n - \frac{1}{k_n}T^n y_n)] \\
&\quad + [(x_{n+1} - \frac{1}{k_n}T^n x_{n+1}) - (x_n - T^n x_n)], J(x_{n+1} - y_n)\rangle.
\end{aligned}
$$

Observe that from the properties of y_n and T we have

$$
\theta_n(x_1 - y_n) - (y_n - \frac{1}{k_n}T^n y_n) + (1 - \frac{1}{k_n})x_1 = 0, \tag{3.3}
$$

$$
\langle (x_{n+1} - \frac{1}{k_n}T^n x_{n+1}) - (y_n - \frac{1}{k_n}T^n y_n), J(x_{n+1} - y_n)\rangle \ge 0. \tag{3.4}
$$

Thus from (3.2) it follows that

$$
\begin{aligned}
\|x_{n+1} - y_n\|^2 &\le \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 \\
&\quad + 2(2+L)\lambda_n^2\|x_n - T^n x_n + \theta_n(x_n - x_1)\| \cdot \|x_{n+1} - y_n\| \\
&\quad + 2\lambda_n\frac{(k_n - 1)}{k_n}(\|T^n x_n\| + \|x_1\|)\|x_{n+1} - y_n\|.
\end{aligned} \tag{3.5}
$$

But since K is bounded, $\{x_n\}, \{y_n\}$ and $\{T^n x_n\}$ are also bounded. Thus there exists $M_1 > 0$ such that $\max\{\|T^n x_n\| + \|x_1\|, \|x_n - T^n x_n + \theta_n(x_n - x_1)\|\} \le M_1$. Then from (3.5) we get

$$
\begin{aligned}
\|x_{n+1} - y_n\|^2 &\le \|x_n - y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 \\
&\quad + 2(2+L)\lambda_n^2 M_1\|x_{n+1} - y_n\| \\
&\quad + 2\lambda_n(k_n - 1)M_1\|x_{n+1} - y_n\|.
\end{aligned} \tag{3.6}
$$

Moreover observe that $T_n := \frac{1}{k_n}T^n$ is pseudocontractive. Thus by using (3.3), we obtain

$$
\begin{aligned}
&\|y_{n-1} - y_n\| \\
&\le \|y_{n-1} - y_n + \frac{1}{\theta_n}((I - T_n)y_{n-1} - (I - T_n)y_n)\| \\
&\le |\frac{\theta_{n-1}}{\theta_n} - 1|(\|y_{n-1}\| + \|x_1\|) + \frac{1}{\theta_n}\|\frac{1}{k_{n-1}}T^{n-1}y_{n-1} - \frac{1}{k_n}T^n y_{n-1}\| \\
&\le |\frac{\theta_{n-1}}{\theta_n} - 1|(\|y_{n-1}\| + \|x_1\|) + \frac{1}{\theta_n k_{n-1}}\varepsilon_{n-1} \\
&\quad + \frac{1}{\theta_n}\frac{|k_{n-1} - k_n|}{k_n k_{n-1}}(\|T^n y_{n-1}\| + \|x_1\|).
\end{aligned} \tag{3.7}
$$

Thus from (3.6),(3.7) and Lemma 2.1, we derive for some $M \geq M_1$,

$$\begin{aligned}
&\|x_{n+1} - y_n\|^2 \\
&\leq \|x_n - y_{n-1}\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 + 2(2+L)\lambda_n^2 M \\
&\quad + 2\lambda_n(k_n - 1)M + M|\frac{\theta_{n-1}}{\theta_n} - 1| + M\frac{\varepsilon_{n-1}}{\theta_n k_{n-1}} + \frac{1}{\theta_n}\frac{|k_{n-1} - k_n|}{k_{n-1}k_n}M \\
&\leq \|x_n - y_{n-1}\|^2 - 2\lambda_n\theta_n\|x_{n+1} - y_n\|^2 + 2M(2+L)\lambda_n^2 \\
&\quad + 2M\lambda_n(k_n - 1) + M|\frac{\theta_{n-1}}{\theta_n} - 1| + M\frac{\varepsilon_{n-1}}{\theta_n} + M\frac{|k_{n-1} - k_n|}{\theta_n},
\end{aligned}$$

which hence implies that

$$\begin{aligned}
&\|x_{n+1} - y_n\|^2 \\
&\leq (1 - \frac{2\lambda_n\theta_n}{1 + 2\lambda_n\theta_n})\|x_n - y_{n-1}\|^2 + \frac{2M(2+L)}{1 + 2\lambda_n\theta_n}\cdot\lambda_n^2 \\
&\quad + \frac{2M}{1 + 2\lambda_n\theta_n}\cdot|\frac{\theta_{n-1}}{\theta_n} - 1| + \frac{2M}{1 + 2\lambda_n\theta_n}(\frac{\varepsilon_{n-1}}{\theta_n} + \frac{|k_{n-1} - k_n|}{\theta_n} + \lambda_n(k_n - 1)).
\end{aligned}$$

Since $\lim_{n\to\infty} 2/(1 + 2\lambda_n\theta_n) = 2$, there is a positive integer n_1 so that $1 < 2/(1 + 2\lambda_n\theta_n) < 3$ $\forall n \geq n_1$. Hence it is easy to see that there exists some $M_0 \geq M$ such that for all $n \geq M_0$,

(3.8)

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + M_0(\lambda_n^2 + |\frac{\theta_{n-1}}{\theta_n} - 1|) \\
&\quad + M_0(\frac{\varepsilon_{n-1}}{\theta_n} + \frac{|k_{n-1} - k_n|}{\theta_n} + \lambda_n(k_n - 1)) \\
&\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + \lambda_n\theta_n\cdot M_0(\frac{\lambda_n}{\theta_n} + |\frac{\theta_{n-1}}{\theta_n} - 1|/\lambda_n\theta_n) \\
&\quad + M_0(\frac{\varepsilon_{n-1}}{\theta_n} + \frac{|k_{n-1} - k_n|}{\theta_n} + \lambda_n(k_n - 1)).
\end{aligned}$$

Now for all $n \geq 1$ we define $\sigma_n = M_0(\frac{\lambda_n}{\theta_n} + |\frac{\theta_{n-1}}{\theta_n} - 1|/\lambda_n\theta_n)$ and

$$\mu_n = M_0(\frac{\varepsilon_{n-1}}{\theta_n} + \frac{|k_{n-1} - k_n|}{\theta_n} + \lambda_n(k_n - 1)).$$

Then (3.8) reduces to

$$\|x_{n+1} - y_n\|^2 \leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + \lambda_n\theta_n\cdot\sigma_n + \mu_n \ \forall n \geq n_1.$$

According to conditions (i)–(iii), we have

$$\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty, \ \lim_{n\to\infty}\sigma_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\mu_n < \infty.$$

Thus by Lemma 2.2 (ii) we infer that $x_{n+1} - y_n \to 0$. Consequently $x_n - y_n \to 0$. Next we prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Indeed by Lemma CZ, we know that

$\|y_n - Ty_n\| \to 0$ as $n \to \infty$. Hence from the uniformly L-Lipschitzian continuity of T we obtain

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Ty_n\| + \|Tx_n - Ty_n\| \to 0 \text{ as } n \to \infty.$$

The proof is complete. □

In what follows, for the bounded sequence $\{y_n\}$ defined by Eq. (2.5) with $u = x_1$ and a Banach limit LIM, let the function $\phi : E \to [0, \infty)$ be defined by $\phi(z) = \text{LIM}_n \|y_n - z\|^2$ for each $z \in K \subset E$. Clearly ϕ is continuous, convex and satisfies $\phi(z) \to \infty$ as $\|z\| \to \infty$. For our next theorem, we need the following propositions.

Proposition 3.1 ([21, Proposition 3.5]). *Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex and bounded subset of E and $T : K \to K$ be a uniformly L-Lipschitzian mapping such that $L < N(E)^{1/2}$ and asymptotically pseudocontactive mapping with sequence $\{k_n\}$ such that $(k_n - 1)/(k_n - t_n) \to 0$ as $n \to \infty$ where t_n is as in Lemma CZ and uniformly asymptotically regular. Suppose that*

$$\|y_n - T^m y\|^2 \leq \langle y_n - T^m y, J(y_n - y)\rangle, \ \forall m, n \geq 1, \ \forall y \in C$$

where $C = \{y \in K : \phi(y) = \min_{z \in K} \phi(z)\}$. Then $\{y_n\}$ converges strongly to a fixed point of T.

Proposition 3.2. *Suppose E is a real uniform convex Banach space which has uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex and bounded subset of E and $T : K \to K$ be a uniformly L-Lipschitzian mapping such that $L < N(E)^{1/2}$ and asymptotically pseudocontractive mapping with sequence $\{k_n\}$ such that $(k_n - 1)/(k_n - t_n) \to 0$ as $n \to \infty$ where t_n is as in Lemma CZ and uniformly asymptotically regular. Then*

(a) *$C = \{y \in K : \phi(y) = \min_{z \in K} \phi(z)\}$ is a singleton, say $\{z_0\}$;*

(b) *the following statements are equalivalent:*

  (i) $z_0 \in F(T)$,

  (ii) $\|y_n - T^m z_0\|^2 \leq \langle y_n - T^m z_0, J(y_n - z_0)\rangle \ \forall m, n \geq 1$,

  (iii) *$T^n z_0 \to z_0$ weakly as $n \to \infty$,*

  (iv) *$y_n \to z_0$ strongly as $n \to \infty$.*

Proof. (1) Since E is reflexive and ϕ is continuous, convex and $\phi(z) \to \infty$ as $\|z\| \to \infty$, ϕ attains its infimun over K (see e.g., [19, 20]). Hence $C := \{y \in K : \phi(y) = \min_{z \in K} \phi(z)\}$ is nonempty, closed and convex. By Lemma SR, we infer that $u \in C$ if and only if

$$\text{LIM}_n \langle z - u, J(y_n - u)\rangle \leq 0, \ \forall z \in K. \tag{3.9}$$

Now we claim that C consists of one point. Indeed let $u, v \in C$ and $u \neq v$. Then by [10, Theorem 1], there exists $\delta > 0$ such that

$$\begin{aligned}&\langle v - u, J(y_n - u) - J(y_n - v)\rangle \\ &\qquad = \langle y_n - u - (y_n - v), J(y_n - u) - J(y_n - v)\rangle \geq \delta > 0\end{aligned}$$

for each $n \geq 1$ which implies that

$$\mathrm{LIM}_n\langle v-u, J(y_n-u)-J(y_n-v)\rangle \geq \delta > 0.$$

But it follows from (3.9) that for $u, v \in C$,

$$\mathrm{LIM}_n\langle v-u, J(y_n-u)\rangle \leq 0$$

and

$$\mathrm{LIM}_n\langle u-v, J(y_n-v)\rangle \leq 0.$$

Thus we have

$$\mathrm{LIM}_n\langle v-u, J(y_n-u)-J(y_n-v)\rangle \leq 0.$$

This arrives at a contradiction. Hence $u = v$. Therefore C is a singleton, say $\{z_0\}$.

(2) At first, we prove that (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i). Indeed if z_0 is a fixed point of T in K, then it is easy to see that for all $m, n \geq 1$

$$\begin{aligned}\|y_n - T^m z_0\|^2 &= \|y_n - z_0\|^2 \\ &= \langle y_n - z_0, J(y_n - z_0)\rangle = \langle y_n - T^m z_0, J(y_n - z_0)\rangle.\end{aligned}$$

Suppose that for all $m, n \geq 1$,

$$\|y_n - T^m z_0\|^2 \leq \langle y_n - T^m z_0, J(y_n - z_0)\rangle. \tag{3.10}$$

Then we claim that $T^n z_0 \to z_0$ weakly as $n \to \infty$. Indeed, let $y = w - \lim_j T^{m_j} z_0$ be any element of the weak ω-limit set $\omega_w(z_0)$ of T at z_0. Note that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ by Lemma CZ. From the weak lower semicontinuity of ϕ and (3.10) we obtain

$$\begin{aligned}\phi(y) &\leq \liminf_{j\to\infty} \phi(T^{m_j} z_0) \leq \limsup_{m\to\infty} \phi(T^m z_0) \\ &= \limsup_{m\to\infty} (\mathrm{LIM}_n \|y_n - T^m z_0\|^2) \\ &\leq \limsup_{m\to\infty} (\mathrm{LIM}_n \langle y_n - T^m z_0, J(y_n - z_0)\rangle) \\ &= \limsup_{m\to\infty} (\mathrm{LIM}_n \langle y_n - Ty_n + (Ty_n - T^2 y_n) + \cdots \\ &\quad + (T^m y_n - T^m z_0), J(y_n - z_0)\rangle) \\ &\leq \limsup_{m\to\infty} (\mathrm{LIM}_n [\|y_n - Ty_n\| + L\|y_n - Ty_n\| + \cdots \\ &\quad + L\|y_n - Ty_n\|] d + \mathrm{LIM}_n k_m \|y_n - z_0\|^2) \\ &= \phi(z_0) = \min_{z\in K} \phi(z),\end{aligned}$$

where $d = d(K)$ the diameter of K. Thus by the definition of C, we have $y \in C = \{z_0\}$ which implies that $y = z_0$. This shows that $\omega_w(z_0) = \{z_0\}$. Thus $T^n z_0 \to z_0$ weakly as $n \to \infty$.

Since $T^n z_0 \to z_0$ weakly as $n \to \infty$, $C = \{z_0\}$ satisfies the property (P). It follows from Lemma LX that z_0 is a fixed point of T in K.

Secondly, we prove that (i)$\Longleftrightarrow$(iv). Indeed if $\{y_n\}$ converges strongly to z_0, then according to $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$, the point z_0 is a fixed point of T in K. Conversely, suppose that z_0 is a fixed point of T in K. Then according to (i)$\Longrightarrow$(ii), we have

$$\|y_n - T^m z_0\|^2 \leq \langle y_n - T^m z_0, J(y_n - z_0)\rangle, \ \forall m, n \geq 1.$$

Hence by Proposition 3.1 we conclude that $\{y_n\}$ converges strongly to some $z_* \in F(T)$. Since

$$\phi(z_*) = \text{LIM}_n\|y_n - z_*\|^2 = 0 \le \text{LIM}_n\|y_n - z_0\|^2 = \phi(z_0) = \min_{z\in K}\phi(z),$$

it follows from $C = \{z_0\}$ that $z_* = z_0$. Thus $\{y_n\}$ converges strongly to z_0. □

Theorem 3.2. *Suppose E is a real Banach space with uniform normal structure and suppose E has a uniformly Gâteaux differentiable norm. Let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be uniformly L-Lipschitzian with $L < N(E)^{1/2}$, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with sequence $\{k_n\}$. Let $\lambda_n, \theta_n \in (0, 1)$ $\forall n \ge 1$ satisfy conditions (i)-(iii) of Theorem 3.1 and let $\lim_{n\to\infty}\frac{k_n-1}{k_n-t_n} = 0$ where $t_n = 1/(1+\theta_n)$. Suppose that*

$$\|y_n - T^m y\|^2 \le \langle y_n - T^m y, J(y_n - y)\rangle, \ \forall m, n \ge 1, \ \forall y \in C$$

where $C = \{y \in K : \phi(y) = \min_{z\in K}\phi(z)\}$. Then the sequence $\{x_n\}$ generated from $x_1 \in K$ by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1), \ \forall n \ge 1$$

converges strongly to a fixed point of T.

Proof. From the proof of Theorem 3.1, we can see that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Moreover according to Proposition 3.1, we known that $\{y_n\}$ converges strongly to a fixed point of T. Consequently, $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.3. *Suppose E is a real uniformly convex Banach space which has a uniformly Gateâux differentiable norm. Let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be uniformly L-Lipschitizian with $L < N(E)^{1/2}$, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with sequence $\{k_n\}$. Let $\lambda_n, \theta_n \in (0, 1)$ $\forall n \ge 1$ satisfy conditions* (i)–(iii) *of Theorem 3.1 and let $\lim_{n\to\infty}\frac{k_n-1}{k_n-t_n} = 0$ where $t_n = 1/(1+\theta_n)$. Suppose that*

$$\|y_n - T^m z_0\|^2 \le \langle y_n - T^m z_0, J(y_n - z_0)\rangle, \ \forall m, n \ge 1$$

where $\{y \in K : \phi(y) = \min_{z\in K}\phi(z)\} = \{z_0\}$. Then the sequence $\{x_n\}$ generated from $x_1 \in K$ by

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T^n x_n - \lambda_n\theta_n(x_n - x_1), \ \forall n \ge 1$$

converges strongly to z_0 and $z_0 \in F(T)$.

Proof. From the proof of Theorem 3.1, we can see that $\|x_n - y_n\| \to 0$ as $n \to \infty$. Moreover according to Proposition 3.2 (b), we known that $\{y_n\}$ converges strongly to z_0 and $z_0 \in F(T)$. Therefore $\{x_n\}$ converges strongly to the fixed point z_0 of T. □

References

[1] A. G. Aksoy and M. A. Khamsi, Nonstandard Methods in Fixed Point Theory, Springer, New York, 1990.

[2] W. L. Bynum, Normal structure coefficients for Banach spaces, Pacific. J. Math. 86(1980), 427-436.

[3] S. S. Chang, Some problems and results in the study of nonlinear analysis, Nonlinear Anal. 30(1997,) 4197-4208.

[4] K. Goebel and W. A. Kirk, A fixed point theory for asymptotically nonexpansive mapping, Proc. Amer. Math. Soc. 35(1972), 171-174.
[5] K. Goebel and W. A. Kirk, Topics in Matric Fixed Point Theory, in: Cambridge Studies in Advanced Mathmatics, Vol. 28, Cambridge University Press, Cambridge, 1990.
[6] L. C. Zeng, A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 266(1998), 245-250.
[7] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan 19 (1967), 508-520.
[8] L. C. Zeng, Ishikawa iteration process for approximating fixed points of nonexpansive mappings, J. Math. Res. & Exposition 23(1)(2003), 33-39.
[9] L. C. Zeng, Approximating fixed points of asymptotically nonexpansive mappings in Banach spaces, Acta Math. Scientia 23A(1)(2003), 31-37.
[10] B. Prus, A characterization of uniform convexity and applications to accretive operators, Hiroshima J. Math. 11(1981), 229-234.
[11] S. Reich, Iterative methods for accretive sets, in: Nonlinear Equations in Abstract Spaces, Academic Press, New York, 1978, pp. 317-326.
[12] S. Reich, Product formulas, nonlinear semigroups and accretive operators, J. Funct. Anal. 36(1980), 147-168.
[13] S. S. Chang, Sone results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 129(3)(2000), 845-853.
[14] B. E. Rhoades, Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56(1976), 741-750.
[15] B. E. Rhoades, Fixed point iteration methods for certain nonlinear mappings, J. Math. Anal. Appl. 183(1994), 118-120.
[16] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158(1991), 407-413.
[17] J. Schu, Approximation of fixed points of asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 112(1)(1991), 143-151.
[18] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive maps, Bull. Austral. Math. Soc. 43(1991), 153-159.
[19] W. Takahashi, Nonlinear Functional Analysis, Kindikagaku, Tokyo, 1998(in Japanese).
[20] T. C. Lim and H. K. Xu, Fixed point theorems for asymptotically nonexpansive mappings, Nonlinear Anal. 22(1994), 1345-1355.
[21] C. E. Chidume and H. Zegeye, Approximate fixed point sequences and convergence theorems for asymptotically pseudocontractive mappings, J. Math. Anal. Appl. 278(2003), 354-366.
[22] K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178(1993), 301-308.
[23] K. K. Tan and H. K. Xu, Fixed point iteration processes for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 122(1994), 733-739.
[24] XU, H. K., Iterative algorithms for nonlinear operators, Journal of London Mathematical Society, Vol. 66, No. 2, pp. 240-256, 2002.

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