Journal of Nonlinear and Convex Analysis
Volume 6, Number 3, 2005, 441–451
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

PALEY'S INEQUALITY AND HARDY'S INEQUALITY FOR THE FOURIER-BESSEL EXPANSIONS

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Abstract. Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the unit disc satisfying $\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty$. Then, $(\sum_{k=1}^{\infty} |a_{2^k}|^2)^{1/2} < \infty$, which is familiar as Paley's inequality. Another well-known inequality is Hardy's inequality: $\sum_{n=0}^{\infty} |a_n|/(n+1) < \infty$. In this paper, analogues of these inequalities with respect to the Fourier-Bessel expansions are established.

1. Introduction and Results

The classical Paley inequality [7] says that there exists a constant C such that $(\sum_{k=1}^{\infty} |a_{n_k}|^2)^{1/2} \le C\|F\|_{H^1}$ for $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^1(\mathbb{D})$, where $\{n_k\}_{k=1}^{\infty}$ is an Hadamard sequence, that is, a sequence of positive integers such that $n_{k+1}/n_k \ge \rho$ with a constant $\rho > 1$, and $H^1(\mathbb{D})$ is the Hardy space on the unit disc $\mathbb{D}$ which consists of the analytic functions $F(z)$ on $\mathbb{D}$ satisfying $\|F\|_{H^1} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})|\, d\theta < \infty$.

Another well-known inequality for the Hardy space is Hardy's inequality [1]: if $F(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to $H^1(\mathbb{D})$, then $\sum_{n=0}^{\infty} |a_n|/(n+1) \le C\|F\|_{H^1}$, where C is a constant independent of F.

For our purpose, we restate these inequalities in terms of the real Hardy space. Let $\Re H^1$ be the real Hardy space, that is, the space consisting of the boundary functions $f(\theta) = \lim_{r\to 1} \Re F(re^{i\theta})$ of $F \in H^1(\mathbb{D})$ and $\|f\|_{\Re H^1} = \|F\|_{H^1}$ with real $F(0)$. Then, Paley's inequality and Hardy's inequality turn to

$$\{\sum_{k=1}^{\infty} (|c_{n_k}|^2 + |c_{-n_k}|^2)\}^{1/2} \le C\|f\|_{\Re H^1},$$

and

$$\sum_{n=-\infty}^{\infty} \frac{|c_n|}{|n|+1} \le C\|f\|_{\Re H^1},$$

respectively, where $f(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$ in $\Re H^1$ and C is independent of f.

Kanjin and the author, [3] and [4], have established analogues of these inequalities with respect to the Jacobi expansions. The purpose of this paper is to obtain these types of inequalities with respect to the Fourier-Bessel expansions.

To formulate our inequalities, let us recall the Fourier-Bessel expansions, and give the definition of the nonperiodic real Hardy space on which we shall work.

2000 *Mathematics Subject Classification.* Primary 42C10; Secondary 42B30, 33C45.
Key words and phrases. Paley's inequality, Hardy's inequality, Fourier-Bessel series.

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Let $\nu > -1$ and $J_\nu(x)$ be the Bessel function of the first kind of order ν. We denote by $\lambda_n = \lambda_n^{(\nu)}, n = 1, 2, \ldots$ the successive positive zeros of $J_\nu(x)$:

$$0 < \lambda_1 < \lambda_2 < \ldots.$$

Let $\psi_n^{(\nu)}(x)$ be the functions defined by

$$\psi_n^{(\nu)}(x) = d_n^{(\nu)} \sqrt{\lambda_n x} J_\nu(\lambda_n x),$$

where

$$d_n = d_n^{(\nu)} = \frac{\sqrt{2}}{\sqrt{\lambda_n} |J_{\nu+1}(\lambda_n)|}.$$

Then, the system $\{\psi_n^{(\nu)}(x)\}_{n=1}^\infty$ is complete orthonormal in $L^2(0,1)$ with respect to the ordinary Lebesgue measure dx. When $\nu = -1/2$ and $\nu = 1/2$, the functions $\psi_n^{(\nu)}(x)$ are the cosine and the sine functions, respectively:

$$\psi_n^{(-1/2)}(x) = \sqrt{2}\cos(\pi(n-1/2)x), \qquad \psi_n^{(1/2)}(x) = \sqrt{2}\sin(\pi n x).$$

For a function $f(x)$ on $(0,1)$, we have the Fourier-Bessel expansion

$$f(x) \sim \sum_{n=1}^\infty c_n^{(\nu)}(f)\psi_n^{(\nu)}(x), \quad c_n^{(\nu)}(f) = \int_0^1 f(y)\psi_n^{(\nu)}(y)\, dy.$$

We turn to the definition of the nonperiodic real Hardy space according to [5, Ch.5, §3]. Let $\Delta = [0,1]$. A real-valued function $a(x)$ on Δ is a Δ-atom if there exists a subinterval $I \subset \Delta$ such that (1) supp $a(x) \subset I$; (2) $\int_I a(x)\, dx = 0$; (3) $\|a\|_\infty \le |I|^{-1}$, where $|I|$ is the length of the interval I. The function $a(x) = 1, x \in \Delta$ is a Δ-atom. Let $\mathcal{H}(\Delta)$ be the nonperiodic real Hardy space, that is, the space of functions representable in the form

$$f(x) = \sum_{k=0}^\infty \lambda_k a_k(x), \quad \sum_{k=0}^\infty |\lambda_k| < \infty, \tag{1}$$

where every $a_k(x)$ is a Δ-atom. We note that the above series converges in $L^1(\Delta)$ and also a.e. The norm $\|f\|_{\mathcal{H}(\Delta)}$ is defined by

$$\|f\|_{\mathcal{H}(\Delta)} = \inf \sum_{k=0}^\infty |\lambda_k|,$$

where the infimum is taken over all expression (1). Then, $\mathcal{H}(\Delta)$ is a Banach space, and $\|f\|_{L^1(\Delta)} \le \|f\|_{\mathcal{H}(\Delta)}$.

Our theorem is as follows:

Theorem. *Let $\nu \ge -1/2$. Then, the Fourier-Bessel coefficients $c_n^{(\nu)}(f)$ of a function $f \in \mathcal{H}(\Delta)$ satisfy*

$$\left(\sum_{k=1}^\infty |c_{n_k}^{(\nu)}(f)|^2 \right)^{1/2} \le C\|f\|_{\mathcal{H}(\Delta)}, \tag{2}$$

where $\{n_k\}_{k=1}^{\infty}$ *is an Hadamard sequence, and*

$$\sum_{n=1}^{\infty} \frac{|c_n^{(\nu)}(f)|}{n} \leq C\|f\|_{\mathcal{H}(\Delta)}, \tag{3}$$

where C *is independent of* f.

Remark. We can not replace the space $\mathcal{H}(\Delta)$ with the space $L^1(\Delta)$ in our theorem, that is, we can show the following: (1') There exists a function $f \in L^1(\Delta)$ such that the series $\sum_{k=1}^{\infty} |c_{n_k}^{(\nu)}(f)|^2$ diverges; (2') There exists a function $f \in L^1(\Delta)$ such that the series $\sum_{n=1}^{\infty} |c_n^{(\nu)}(f)|/n$ diverges.

We shall give a proof here only to (1'), and (2') can be proved in the same way. Suppose that $\sum_{k=1}^{\infty} |c_{n_k}^{(\nu)}(f)|^2 < \infty$ for all $f \in L^1(\Delta)$. Then, by the closed graph theorem, we have $\sum_{k=1}^{\infty} |c_{n_k}^{(\nu)}(f)|^2 \leq C\|f\|_{L^1(\Delta)}^2$. We consider the sequence $\{f_j\}_{j=1}^{\infty}$ of functions such that $f_j(x) = j\chi_j(x)/2$ where $\chi_j(x)$ is the characteristic function of the interval $(x_0 - 1/j, x_0 + 1/j) \cap \Delta$ with a fixed point $x_0 \in (0,1)$. Then, $\|f_j\|_{L^1(\Delta)} \leq 1$ and $c_n^{(\nu)}(f_j) \to \psi_n^{(\nu)}(x_0)$ as $j \to \infty$. Thus, by Fatou's lemma, we have

$$\sum_{k=1}^{\infty} |\psi_{n_k}^{(\nu)}(x_0)|^2 \leq \liminf_{j\to\infty} \sum_{k=1}^{\infty} |c_{n_k}^{(\nu)}(f_j)|^2 \leq C.$$

Given $x_0 \in (0,1)$, the asymptotic formulas (6), (7) and (9) lead to

$$|\psi_n^{(\nu)}(x_0)| \geq \sqrt{2}|\cos(\lambda_n x_0 - \pi D^{(\nu)})| - \frac{K}{n}$$

with a positive constant K independent of n which may depend on x_0 and ν. We have

$$\sum_{k=1}^{\infty} |\cos(\lambda_{n_k} x_0 - \pi D^{(\nu)})|^2 \leq K', \tag{4}$$

where K' are a constant depending only on x_0 and ν.

On the other hand, there exists a point $x_0 \in (0,1)$ such that the set of points $\{\langle \lambda_{n_k} x_0/\pi \rangle\}_{k=1}^{\infty}$ is dense in $(0,1)$ (cf. [2, Theorem 1.40]), where $\langle t \rangle$ denots the fractional part of t. The inequality (4) contradicts this fact, which completes the proof of (1').

In our proof of (2) of Theorem, the (H^1, BMO)-duality will play an essential role. Let us explain the duality. We put

$$\mathcal{N}_\Delta(f) = \sup_I \frac{1}{|I|} \int_I |f(x) - f_I|\, dx$$

for a function $f(x)$ on Δ, where $f_I = (1/|I|)\int_I f(x)\, dx$ and the supremum is taken all subintervals I of Δ. We here remark that

$$\frac{1}{|I|} \int_I |f(x) - f_I|\, dx \leq \frac{2}{|I|} \int_I |f(x) - c|\, dx$$

for every subinterval $I \subset \Delta$ and any constant c. The nonperiodic BMO space $BMO(\Delta)$ is defined by the space of functions $f \in L^1(\Delta)$ such that

$$\|f\|_{BMO(\Delta)} = \mathcal{N}_\Delta(f) + \left|\int_\Delta f(x)\,dx\right| < \infty.$$

The space $BMO(\Delta)$ is a Banach space with norm $\|f\|_{BMO(\Delta)}$. The (H^1, BMO)-duality in the nonperiodic case says that $(\mathcal{H}(\Delta))^* = BMO(\Delta)$ ([5, Ch.5, §3, Corollary 4]). 0In particular, for $b \in L^\infty(\Delta)(\subset BMO(\Delta))$ and $f \in \mathcal{H}(\Delta)$ we have

$$\left|\int_\Delta f(x)b(x)\,dx\right| \le C\|f\|_{\mathcal{H}(\Delta)}\|b\|_{BMO(\Delta)}, \tag{5}$$

where C is an absolute constant.

We, here at the end of this section, collect some asymptotic formulas which will be needed later.

$$\lambda_n = \pi(n + D^{(\nu)} + c_n^{(\nu)}), \quad D^{(\nu)} = (2\nu-1)/4, \qquad c_n^{(\nu)} = O(n^{-1}). \tag{6}$$

$$d_n = \sqrt{\pi}(1 + j_n^{(\nu)}), \qquad j_n^{(\nu)} = O(n^{-1}). \tag{7}$$

$$J_\nu(z) = O(z^\nu), \qquad z \to +0. \tag{8}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos(z - \pi D^{(\nu)}) + O(z^{-3/2}), \qquad z \to +\infty. \tag{9}$$

$$|\psi_n^{(\nu)}(x)| \le C\begin{cases}(nx)^{\nu+1/2}, & 0 < x \le 1/n,\\ 1, & 1/n < x \le 1.\end{cases} \tag{10}$$

For the Fourier-Bessel expansions and the above facts, we may consult [8] and [6].

2. Lemmas

We devote this section to the proofs of two lemmas which will be needed in the proof of our theorem.

Lemma 1. *Let $\nu \ge -1/2$. Then there exists a constant C such that*

$$|\psi_n^{(\nu)}(x_2) - \psi_n^{(\nu)}(x_1)| \le Cn^\delta|x_2 - x_1|^\delta \tag{11}$$

for $0 \le x_1 < x_2 \le 1$, where $\delta = 1$ for $\nu = -1/2$, and $\delta = \min\{1, \nu + 1/2\}$ for $\nu > -1/2$.

Proof. If $\nu = -1/2$, then $\psi_n^{(-1/2)}(x) = \sqrt{2}\cos(\pi(n-1/2)x)$, and thus we have the inequality (11).

Suppose that $\nu > -1/2$. We put $\phi_\nu(u) = \sqrt{u}J_\nu(u)$. By (6) and (7), we see that it is enough to show that

$$|\phi_\nu(u_2) - \phi_\nu(u_1)| \le C|u_2 - u_1|^\delta \tag{12}$$

for $0 \le u_1 < u_2$. It follows from (8) and (9) that $\sup_{0\le u}|\phi_\nu(u)| \le C$, which means that it suffices to show (12) for $0 \le u_1 < u_2$ and $u_2 - u_1 \le 1$. By the formula

$$\frac{d}{du}J_\nu(u) = \frac{1}{2}(J_{\nu-1}(u) - J_{\nu+1}(u)),$$

we have

$$\frac{d}{du}\phi_\nu(u) = \frac{1}{2}u^{-1/2}J_\nu(u) + \frac{1}{2}\{\sqrt{u}J_{\nu-1}(u) - \sqrt{u}J_{\nu+1}(u)\}.$$

By (9) we see that $\sup_{1\le u}|\phi'_\nu(u)| \le C$. Noting $0 < \delta \le 1$, we get (12) for $1 \le u_1 < u_2$ and $u_2 - u_1 \le 1$. The rest of the proof is to show (12) for $0 \le u_1 < u_2 \le 1$. For, if $0 \le u_1 < 1 \le u_2$, then we can divide the matter into two parts at the point 1. The series definition of the Bessel function leads to

$$\phi_\nu(u) = u^{\nu+1/2}h_\nu(u), \qquad h_\nu(u) = 2^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n(u/2)^{2n}}{n!\Gamma(\nu+n+1)}.$$

We note that h_ν is an entire function. We have

$$\begin{aligned}|\phi_\nu(u_2) - \phi_\nu(u_1)| &\le |u_2^{\nu+1/2}||h_\nu(u_2) - h_\nu(u_1)| + |u_2^{\nu+1/2} - u_1^{\nu+1/2}||h_\nu(u_1)| \\ &\le |u_2 - u_1| \sup_{0\le u\le 1}|h'_\nu(u)| + C|u_2 - u_1|^\delta \sup_{0\le u\le 1}|h_\nu(u)| \\ &\le C|u_2 - u_1|^\delta\end{aligned}$$

where C is independent of u_1 and u_2. □

Lemma 2. *Let $\nu \ge -1/2$. Then there exists a constant C such that*

$$(13)\qquad \left|\int_a^b \psi_m^{(\nu)}(x)\psi_n^{(\nu)}(x)\,dx\right| \le C\left\{(b-a)\left(\frac{m}{n}\right)^\delta + \frac{\log^+ n(b-a)}{n} + \frac{1}{n}\right\}$$

for $m \le n$ and $0 \le a < b \le 1$, where δ is the same as in Lemma 1. The notation '$\log^+ x$' means that $\log^+ x = \log x$ for $x \ge 1$ and $\log^+ x = 0$ for $x < 1$.

Proof. Let K be the greatest non-negative integer such that $2\pi K/\lambda_n \le b - a$. We put $x_k = a + (2\pi k/\lambda_n)$ for $k = 0, 1, 2, \dots, K$, and $x_{K+1} = b$. We write the integral in the following form.

$$\begin{aligned}\int_a^b \psi_m^{(\nu)}(x)\psi_n^{(\nu)}(x)\,dx &= \sum_{k=0}^{K}\left\{\int_{x_k}^{x_{k+1}}\left(\psi_m^{(\nu)}(x) - \psi_m^{(\nu)}(x_k)\right)\psi_n^{(\nu)}(x)\,dx \right.\\ &\qquad\left. + \psi_m^{(\nu)}(x_k)\int_{x_k}^{x_{k+1}}\psi_n^{(\nu)}(x)\,dx\right\} \\ &= \sum_{k=0}^{K}\{A_k^{(1)} + A_k^{(2)}\}, \qquad \text{say.}\end{aligned}$$

Noting $|\psi_n^{(\nu)}(x)| \le C$ for $\nu \ge -1/2$, we apply Lemma 1 to the terms $A_k^{(1)}$, and get

$$\begin{aligned}|A_k^{(1)}| &\le Cm^\delta\int_{x_k}^{x_{k+1}}|x - x_k|^\delta\,dx \le Cm^\delta\left(\frac{2\pi}{\lambda_n}\right)^\delta(x_{k+1} - x_k) \\ &\le C\left(\frac{m}{n}\right)^\delta(x_{k+1} - x_k).\end{aligned}$$

Here, we used (6) for the last inequality. We have

$$\sum_{k=0}^{K}|A_k^{(1)}| \le C\left(\frac{m}{n}\right)^{\delta}(b-a). \tag{14}$$

Let us estimate $A_k^{(2)}, k = 0, 1, \ldots, K$. We first deal with $A_0^{(2)}$ and $A_K^{(2)}$. Since $|\psi_n^{(\nu)}(x)| \le C$ for $\nu \ge -1/2$, it follows that

$$|A_j^{(2)}| \le C\int_{x_j}^{x_{j+1}} dx \le C\frac{2\pi}{\lambda_n} \le C\frac{1}{n}, \qquad j = 0, K. \tag{15}$$

For $A_1^{(2)}, \ldots, A_{K-1}^{(2)}$, we use the asymptotic formulas (6), (7) and (9). For $x \ge x_k \ge x_1 \ge (2\pi/\lambda_n)$, we have

$$\psi_n^{(\nu)}(x) = \sqrt{2}\cos(\lambda_n x - \pi D^{(\nu)}) + O(1/(nx)),$$

where 'O' depens only on ν. Thus we have for $k = 1, 2, \ldots, K-1$,

$$|A_k^{(2)}| \le C\left|\int_{x_k}^{x_{k+1}}\Big(\cos(\lambda_n x - \pi D^{(\nu)}) + O(1/(nx))\Big)\,dx\right|.$$

Since $\int_{x_k}^{x_{k+1}}\cos(\lambda_n x - \pi D^{(\nu)})\,dx = 0$ for $k = 1, 2, \ldots, K-1$, it follows that

$$|A_k^{(2)}| \le \frac{C}{n}\int_{x_k}^{x_{k+1}}\frac{dx}{x} = \frac{C}{n}(\log x_{k+1} - \log x_k),$$

which leads to

$$\sum_{k=1}^{K-1}|A_k^{(2)}| \le \frac{C}{n}(\log x_K - \log x_1) \le \frac{C}{n}\log K$$

$$\le \frac{C}{n}\log^{+}\frac{\lambda_n}{2\pi}(b-a) \le \frac{C}{n}(1 + \log^{+} n(b-a)). \tag{16}$$

Thus, by (14), (15) and (16), we have the desired inequality (13). □

3. Proof of the theorem

We come to the proof of the theorem. Let us prove the Paley type inequality (2) first. Let $\{r_k\}_{k=1}^{\infty}$ be a sequence such that $\sum_{k=1}^{\infty}|r_k|^2 < \infty$, and put $h_N(x) = \sum_{k=1}^{N} r_k\psi_{n_k}^{(\nu)}(x)$ for $N = 1, 2, \ldots$. By (5), we have

$$\left|\int_{\Delta} f(x)h_N(x)\,dx\right| \le C\|f\|_{\mathcal{H}(\Delta)}\|h_N\|_{BMO(\Delta)}$$

for $f \in \mathcal{H}(\Delta)$. Since

$$\int_{\Delta} f(x)h_N(x)\,dx = \sum_{k=1}^{N} c_{n_k}^{(\nu)}(f)r_k,$$

if we prove

$$\|h_N\|_{BMO(\Delta)} \le C\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2} \tag{17}$$

with a constant C independent of N and a sequence $\{r_k\}_{k=1}^\infty$, then

$$\left|\sum_{k=1}^N c_{n_k}^{(\nu)}(f) r_k\right| \le C\left(\sum_{k=1}^\infty |r_k|^2\right)^{1/2} \|f\|_{\mathcal{H}(\Delta)},$$

which leads to the inequality

$$\left(\sum_{k=1}^N |c_{n_k}^{(\nu)}(f)|^2\right)^{1/2} \le C\|f\|_{\mathcal{H}(\Delta)}.$$

Letting $N \to \infty$, we obtain the Paley type inequality (2).

We now prove (17). Since

$$\left|\int_\Delta h_N(x)\,dx\right| \le \|h_N\|_{L^2(\Delta)} = \left(\sum_{k=1}^N |r_k|^2\right)^{1/2},$$

it is enough to show $\mathcal{N}_\Delta(h_N) \le C(\sum_{k=1}^\infty |r_k|^2)^{1/2}$, that is, for every subinterval $I \subset \Delta$, there exists a constant c_I such that

$$\frac{1}{|I|}\int_I |h_N(x) - c_I|\,dx \le C\left(\sum_{k=1}^\infty |r_k|^2\right)^{1/2}, \tag{18}$$

where C is a constant independent of I, N and a sequence $\{r_k\}_{k=1}^\infty$. We set $I = [x_1, x_2] \subset \Delta$. It is enough to deal with the case where there is a positive integer M such that $1/n_{M+1} < |I| \le 1/n_M$. For, if $|I| > 1/n_1$, then

$$\begin{aligned}\frac{1}{|I|}\int_I |h_N(x)|\,dx &\le \left(\frac{1}{|I|}\int_I |h_N(x)|^2\,dx\right)^{1/2}\\ &\le n_1^{1/2}\left(\int_\Delta |h_N(x)|^2\,dx\right)^{1/2} = n_1^{1/2}\left(\sum_{k=1}^\infty |r_k|^2\right)^{1/2}.\end{aligned}$$

We show (18) with $c_I = h_M(x_1)$. Let

$$h_N(x) = h_M(x) + \sum_{k=M+1}^N r_k \psi_{n_k}^{(\nu)}(x) = h_M(x) + E_{M,N}(x), \quad \text{say.}$$

We have

$$\frac{1}{|I|}\int_I |h_N(x) - h_M(x_1)|\,dx \le \frac{1}{|I|}\int_I |h_M(x) - h_M(x_1)|\,dx + \frac{1}{|I|}\int_I |E_{M,N}(x)|\,dx. \tag{19}$$

Applying Schwarz's inequality and Lemma 1 to the integrand of the first integral on the right-hand side, we have

$$\begin{aligned}|h_M(x) - h_M(x_1)|^2 &\le \sum_{k=1}^M |r_k|^2 \sum_{k=1}^M |\psi_{n_k}^{(\nu)}(x) - \psi_{n_k}^{(\nu)}(x_1)|^2\\ &\le C\sum_{k=1}^M |r_k|^2 \sum_{k=1}^M n_k^{2\delta}|x - x_1|^{2\delta}\end{aligned}$$

$$\leq C|I|^{2\delta} \sum_{k=1}^{M} |r_k|^2 \sum_{k=1}^{M} n_k^{2\delta} \leq C|I|^{2\delta} n_M^{2\delta} \sum_{k=1}^{M} |r_k|^2,$$

where δ is as in Lemma 1. Here, we used the fact that $\sum_{k=1}^{M} n_k^{2\delta}$ is bounded by $Cn_M^{2\delta}$ since $\{n_k\}$ is an Hadamard sequence, that is, $n_{k+1}/n_k \geq \rho$ with some constant $\rho > 1$. Our choise of M leads to $|I|n_M \leq 1$, and thus $|h_M(x) - h_M(x_1)|^2 \leq C \sum_{k=1}^{M} |r_k|^2$. The first integral on the right-hand side of (19) is estimated as follows:

$$\begin{aligned} \frac{1}{|I|} \int_I |h_M(x) - h_M(x_1)|\, dx &\leq \left(\frac{1}{|I|} \int_I |h_M(x) - h_M(x_1)|^2\, dx \right)^{1/2} \\ &\leq C \left(\sum_{k=1}^{M} |r_k|^2 \right)^{1/2}. \end{aligned} \tag{20}$$

We come to estimating the second integral on the right-hand side of (19). We have

$$\begin{aligned} \left(\frac{1}{|I|} \int_I |E_{M,N}(x)|\, dx \right)^2 &\leq \frac{1}{|I|} \int_I |E_{M,N}(x)|^2\, dx \\ &\leq \sum_{j,k=M+1}^{N} \frac{|r_j||r_k|}{|I|} \left| \int_I \psi_{n_j}^{(\nu)}(x) \psi_{n_k}^{(\nu)}(x)\, dx \right|. \end{aligned}$$

Under the assumption $n_j \leq n_k$, by Lemma 2, we have

$$\frac{1}{|I|} \left| \int_I \psi_{n_j}^{(\nu)}(x) \psi_{n_k}^{(\nu)}(x)\, dx \right| \leq C \left\{ \left(\frac{n_j}{n_k} \right)^{\delta} + \frac{\log^+ n_k|I|}{|I|n_k} + \frac{1}{|I|n_k} \right\}.$$

The first term on the right-hand side of the above inequality is bounded by $(1/\rho^{\delta})^{k-j}$. We evaluate the second term as follows: We fix a positive number ν with $0 < \nu < 1$. There exists a constant C_ν such that

$$\frac{\log^+ n_k|I|}{|I|n_k} \leq C_\nu \left(\frac{1}{|I|n_k} \right)^{\nu} = C_\nu \left(\frac{1}{|I|n_j} \frac{n_j}{n_k} \right)^{\nu} \leq C_\nu \left(\frac{1}{\rho^{\nu}} \right)^{k-j}$$

for $j \geq M+1$. For the last inequality, we used the fact $|I|n_j > 1$ for $j \geq M+1$. In a similar way, we have $1/(|I|n_k) \leq (1/\rho)^{k-j}$. Therefore, we see that there exist a constant C and γ with $0 < \gamma < 1$ such that

$$\frac{1}{|I|} \left| \int_I \psi_{n_j}^{(\nu)}(x) \psi_{n_k}^{(\nu)}(x)\, dx \right| \leq C\gamma^{|k-j|}$$

for $j, k \geq M+1$. This leads to

$$\frac{1}{|I|} \int_I |E_{M,N}(x)|\, dx \leq C \left(\sum_{j,k=1}^{\infty} \gamma^{|k-j|} |r_j||r_k| \right)^{1/2}.$$

The last sum is evaluated by Schwarz's inequality as follows:

$$\sum_{j,k=1}^{\infty} \gamma^{|k-j|}|r_j||r_k| = \sum_{k=1}^{\infty}|r_k|^2 + 2\gamma\sum_{k=1}^{\infty}|r_{k+1}||r_k| + \cdots + 2\gamma^p\sum_{k=1}^{\infty}|r_{k+p}||r_k| + \dots$$
$$\leq (1 + 2\gamma + \cdots + 2\gamma^p + \dots)\sum_{k=1}^{\infty}|r_k|^2 \leq C\sum_{k=1}^{\infty}|r_k|^2.$$

Thus, we have

$$\frac{1}{|I|}\int_I |E_{M,N}(x)|\,dx \leq C\left(\sum_{k=1}^{\infty}|r_k|^2\right)^{1/2}.$$

Combining this and (20), we obtain (18) with $c_I = h_M(x_1)$, which completes the proof of (2) of Theorem.

We now turn to Hardy's inequality (3) of Theorem. Let $f \in \mathcal{H}(\Delta)$. There exist a sequence $\{a_k\}_{k=0}^{\infty}$ of Δ-atoms and a sequence of $\{\lambda_k\}_{k=0}^{\infty}$ of numbers such that

$$f(x) = \sum_{k=0}^{\infty}\lambda_k a_k(x) \quad \text{a.e. } x,$$

$$\sum_{k=0}^{\infty}|\lambda_k| \leq C\|f\|_{\mathcal{H}(\Delta)}, \tag{21}$$

where C is independent of f. Since $|\psi_n^{(\nu)}(x)| \leq C$ by (10), we have

$$c_n^{(\nu)}(f) = \sum_{k=0}^{\infty}\lambda_k c_n^{(\nu)}(a_k),$$

and thus

$$\sum_{n=1}^{\infty}\frac{|c_n^{(\nu)}(f)|}{n} \leq C\sum_{k=0}^{\infty}|\lambda_k|\sum_{n=1}^{\infty}\frac{|c_n^{(\nu)}(a_k)|}{n}.$$

It follows from (21) that to show Hardy's inequality it is enough to show

$$\sum_{n=1}^{\infty}\frac{|c_n^{(\nu)}(a)|}{n} \leq C \tag{22}$$

for every Δ-atom a, where C is independent of a Δ-atom a. Let us evaluate the coefficients $c_n^{(\nu)}(a)$. First, if $a = 1$, then by Schwarz's inequality and Parseval's identity we have

$$\sum_{n=1}^{\infty}\frac{|c_n^{(\nu)}(a)|}{n} \leq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\int_0^1 dx\right)^{1/2} \leq C$$

with some absolute constant C. Let us treat the othe cases of a. Let $I = [b, b+h]$ be the support interval of a. We have

$$|c_n^{(\nu)}(a)| = \left| \int_b^{b+h} a(y) \left(\psi_n^{(\nu)}(y) - \psi_n^{(\nu)}(b) \right) dy \right|$$

by the fact $\int a(y)\, dy = 0$. Our lemma leads to

$$|c_n^{(\nu)}(a)| \le C \int_b^{b+h} |a(y)| n^\delta (y-b)^\delta \, dy,$$

where δ means the one in the lemma. By Schwarz's inequality we see that the right-hand side of the inequality is bounded by $Cn^\delta \|a\|_2 h^{\delta+1/2}$, where $\|a\|_2 = (\int |a(y)|^2 \, dy)^{1/2}$. Since atoms satisfy the fact $h \le \|a\|_2^{-2}$, it follows that

$$|c_n^{(\nu)}(a)| \le Cn^\delta \|a\|_2^{-2\delta}. \tag{23}$$

To estimate the sum on the left-hand side of (22), we choose γ as $\gamma = \|a\|_2^2$ and write

$$\sum_{n=1}^{\infty} \frac{|c_n^{(\nu)}(a)|}{n} = \left(\sum_{n \le \gamma} + \sum_{n > \gamma} \right) \frac{|c_n^{(\nu)}(a)|}{n}. \tag{24}$$

We apply (23) to estimating the sum $\sum_{n \le \gamma}$. It follows that

$$\sum_{n \le \gamma} \frac{|c_n^{(\nu)}(a)|}{n} \le C \|a\|_2^{-2\delta} \sum_{n \le \gamma} n^{\delta - 1} \le C \|a\|_2^{-2\delta} \gamma^\delta \le C. \tag{25}$$

For the sum $\sum_{n > \gamma}$, we use Parseval's identity and Schwarz's inequality and get

$$\sum_{n > \gamma} \frac{|c_n^{(\nu)}(a)|}{n} \le \|a\|_2 \left(\sum_{n > \gamma} \frac{1}{n^2} \right)^{1/2} \le C \|a\|_2 \gamma^{-1/2} \le C. \tag{26}$$

Combining (26) and (25), we get (22), which completes the proof.

Finally, the author would like to express his sincere thanks to Professor Yuichi Kanjin for his many helpful suggestions in this paper.

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Manuscript received May 25, 2005
revised July 12, 2005

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