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A NOVEL CLASS OF OPERATORS AND SOME RELATED CONSTANTS

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Abstract. In this paper we will introduce a new class of operators related to the nuclear operators. Study some of its properties and estimate some constants concerning an example of these operators. Define a partially ordering amoung prenuclear operators and show that this relation is obtained by aplication of infinite matrices on the classical Banach space of all convergent to zero sequences c_0.

1. Introduction

The close connection between the sumability properties of the eigenvalues of the nuclear operator on the space X and the upper bounds for the norms of suitable projections from a Banach space X onto an arbitrary finite dimensional subspace Y of it and the close connection between the projection constant and the nuclear norm during the trace duality theorem [3] are challenging tasks to introduce a new class of operators related to the nuclear operators, namely prenuclear, left, right nuclear operators define some related constants, prove parallel results to that introduced in [7] which are for projection operators. We also define a partially ordering amonge prenuclear operators and show that this ordering is an application of infinite matrices on t he classical Banach space c_0.

In fact, the space X is isomertic isomorphic to a Hilbert space if and only if any nuclear operator on X has absolutely summable eigenvalues. On the other hand the space X is isomorphic to a Hilbert space if and only if every subspace of which is complemented.

2. Notations and basic definitions

X, Y, Z, W and E denote Banach spaces, X^* denotes the conjugate space of X, $B(X, Y)$ denotes the class of all linear bounded operators from a Banach space X into a Banach space Y while A and B, C, D and T denote elemnts in $B(X, Y)$ and $\widehat{[Y]}$ denotes the closed linear span of the set Y.

An operator $A \in B(X, Y)$ is said to be nuclear if and only if it has the following representation

$$A(x) = \sum_{n\in\mathcal{N}} f_n(x)y_n,\ x \in X,\ \text{simply } A = \sum_{n\in\mathcal{N}} f_n \otimes y_n, \tag{2.1}$$

where $f_n \in X^*$, $y_n \in Y$ and $\{\|f_n\|\|y_n\|\}_{n=1}^{\infty} \in l_1$.
The nuclear norm $\nu(A)$ of the nuclear operator A is defined as:

$$\nu(A) := \inf\{\|\{\|f_n\|\|y_n\|\}_{n=1}^{\infty}\|_{l_1}\},$$

where the infimum is taken over all represntations of A given in (2.1). The operator ideal of all nuclear operators from X into Y will be denoted by $N(X, Y)$, see [5].

The projective tensor product $X \otimes^{\wedge} Y$ between the normed spaces X and Y is defined as the completion of the largest cross norm on the space $X \otimes Y$ and the norm on the space $X \otimes Y$ is defined by

$$\|\sum_{i=1}^{n} x_i \otimes y_i\|_{X\otimes^{\wedge}Y} = \inf\{\sum_{j=1}^{m} \|u_j\|\|v_j\|\}, \tag{2.2}$$

where the infimum is taken over all equivalent representations $\sum_{j=1}^{m} u_j \otimes v_j \in X \otimes Y$ of $\sum_{i=1}^{n} x_i \otimes y_i$; see [2] and [9].

A Banach space X is said to be separably injective if and only if for every separable Banach spaces Y, every subspace Z , $Z \subset Y$ and every linear bounded operator T from Z into X there is a linear bounded operator $\widehat{T}$ extending T from Y into X.

We have the following theorem:

Theorem 1. [10], [11], [12] *The Banach space X is separably injective if and only if it is linearly isomorphic to the space c_0 of all convergent to zero sequences. Moreover, for every separable Banach spaces Y, every subspace Z, $Z \subset Y$ and every linear bounded operator T from Z into X there a linear bounded operator $\widehat{T}$ extending T from Y into X such that $\|\widehat{T}\| \le 2\|T\|$.*

The following lemma give a complete characterization of the linear bounded operators on the Banach space c_0:

Lemma 1. [1] *An infinite matrix $T = \{\tau_{ij}\}_{i,j\in\mathcal{N}}$ defines a bounded linear operator $T \in B(c_0)$ if and only if it satisfies the following conditions:*

(1) *$\{\tau_{nk}\}_{n=1}^{\infty} \in c_0$ for all $k \in \mathcal{N}$, i.e. each column is a convergent to zero sequence.*
(2) *$\{\{\tau_{nk}\}_{k=1}^{\infty}\}_{n=1}^{\infty} \in l_\infty(l_1)$, i.e. $\sup_{n\in\mathcal{N}} \sum_{k\in\mathcal{N}} |\tau_{nk}| < \infty$.*

In this case, we have

(1) *The operator T is given by the foll owing formula:*
$$T(\{y_n\}_{n=1}^{\infty}) = \{\sum_{k\in\mathcal{N}} \tau_{nk} y_k\}_{n=1}^{\infty},$$
(2)
$$\|T\| = \sup_{n\in\mathcal{N}} \sum_{k\in\mathcal{N}} |\tau_{nk}|.$$

We introduce the following new definition:

Definition 1. (1) An operator $A \in B(X, Y)$ is said to be prenuclear if and only the representation $A(x) = \sum_{n\in\mathcal{N}} f_n(x) y_n$ converges for every $x \in X$, where $f_n \in X^*$, $y_n \in Y$. Simply we write $A = \{f_n \otimes y_n\}_{n=1}^{\infty}$.

(2) On the space of all prenuclear operators from a Banach space X into a Banach space Y a partial ordering $\le$ can be defined as follows:
We write $A = \{g_n \otimes y_n\}_{n=1}^{\infty} \le B = \{f_n \otimes x_n\}_{n=1}^{\infty}$ if and only if there exists a non negative real number M such that
$$\sup_{n\in\mathcal{N}} |g_n(x) f(y_n)| \le M \sup_{n\in\mathcal{N}} |f_n(x) f(x_n)| \ \forall x \in X, \forall f \in Y^*.$$

Equivalently, we can write $A \leq B$ if and only if there exists a non negative real number M such that

$$\|\{g_n(x)f(y_n)\}_{n=1}^{\infty}\|_{c_0} \leq M\,\|\{f_n(x)f(x_n)\}_{n=1}^{\infty}\|_{c_0}\,\forall(x,\,f) \in X \times Y^*. \tag{2.3}$$

(3) The smallest such constants M denoted by $M(A, B)$ is called the ordered constant between the two prenuclear operators A and B.

We have the following:

Lemma 2. *Let $A = \{g_n \otimes y_n\}_{n=1}^{\infty}$ be prenuclear operator. Then the sequence of rank one operators $\{g_n \otimes y_n\}_{n=1}^{\infty}$ from X into Y is b ounded.*

Proof. Since the sequence $\{g_n(x)y_n\}_{n=1}^{\infty}$ is convergent to zero sequence for every $x \in X$, the sequence $\{g_n(x)y_n\}_{n=1}^{\infty}$ is bounded at each point $x \in X$, using the uniform boundedness principle, the sequence $\{\|g_n \otimes y_n\|\}_{n=1}^{\infty}$ is bounded. □

Lemma 3. *Let $A = \{g_n \otimes y_n\}_{n=1}^{\infty}$ be prenuclear operator. Then the operator C defined by*

$$C(\sum_{i=1}^{m} x_i \otimes f_i) = \{\sum_{i=1}^{m} g_n(x_i)f_i(y_n)\}_{n=1}^{\infty} = \{g_n \otimes y_n(\sum_{i=1}^{m} x_i \otimes f_i)\}_{n=1}^{\infty},$$

is linear bounded operator from the projective tensor product $X \otimes^{\wedge} Y^*$ into c_0.

Proof. Let $\sum_{i=1}^{m} x_i \otimes f_i \in X \otimes^{\wedge} Y^*$ and $\sum_{j=1}^{l} u_j \otimes h_j$ be any of it s equivalent representations. Using lemma (2), we see that

$$\begin{aligned}\|C(\sum_{i=1}^{m} x_i \otimes f_i)\|_{c_0} &= \|\{g_n \otimes y_n(\sum_{i=1}^{m} x_i \otimes f_i)\}_{n=1}^{\infty}\|_{c_0}\\ &= \|\{g_n \otimes y_n(\sum_{j=1}^{l} u_j \otimes h_j)\}_{n=1}^{\infty}\|_{c_0}\\ &= \sup_{n\in\mathcal{N}} |\sum_{j=1}^{l} g_n(u_j)h_j(y_n)|\\ &\leq \sup_{n\in\mathcal{N}} \|g_n \otimes y_n\| \sum_{j=1}^{l} \|u_j)\|\|h_j\|.\end{aligned}$$

Taking the infimum over all such equivalent representations, we see that

$$\|C\| \leq \sup_{n\in\mathcal{N}} \|g_n \otimes y_n\|. \qquad \square$$

Lemma 4. *For any $x \in X$ the set $Y_x := \left\{\{g_n(x)f(y_n)\}_{n=1}^{\infty} : f \in Y^*\right\} = \left\{\{g_n \otimes y_n(x \otimes f)\}_{n=1}^{\infty} : f \in Y^*\right\}$ is closed linear subspace of the space c_0.*

Proof. Let $\left\{\{g_n(x)f^i(y_n)\}_{n=1}^{\infty}\right\}_{i=1}^{\infty}$ be a convergent sequence in the space Y_x, since the convergent in c_0 is the uniform coordinatewise convergent, $g_n(x)\lim_{i\to\infty} f^i(y_n) = \lim_{i\to\infty} f^i(g_n(x)y_n) < \infty$ uniformly for each $n \in \mathcal{N}$. Once $f^i \in Y^*$ for each $i \in \mathcal{N}$,

$\lim_{i\to\infty} f^i(x)$ exists for every $x \in \widehat{[y_n]}$. Define the linear bounded functional f on $\widehat{[y_n]}$ by the following formula:

$$f(x) := \lim_{i\to\infty} f^i(x).$$

Using Hahn-Banach theorem, f can be extended to a linear bounded functional $g \in Y^*$, the sequence $\{g_n(x)g(y_n)\}_{n=1}^{\infty}$ is the unique uniform coordinatewise limit of the given sequence. □

Remark. (1) For every $g \otimes y \in X^* \otimes Y$, the functional again denoted by $g \otimes y$ and defined by

$$g \otimes y(x \times f) := g(x)f(y)$$

is a bounded bilinear form on the product $X \times Y^*$.

(2) Let $T = \{\tau_{ij}\}_{i,j\in\mathcal{N}} \in B(C_0)$ and $B = \{g_n \otimes y_n\}_{n=1}^{\infty}$ be prenuclear. Then the operator $A(x) = \sum_{n\in\mathcal{N}}\sum_{k\in\mathcal{N}} \tau_{nk} g_k(x) y_k$ is such that $\left\|\{\sum_{k\in\mathcal{N}} \tau_{nk} f_k(x) f(x_k)\}_{n=1}^{\infty}\right\|_{c_0} \leq \|T\| \left\|\{g_n(x)f(y_n)\}_{n=1}^{\infty}\right\|_{c_0}$ for every $(x, f) \in X \times Y^*$.

We also give the follwing interesting result:

Theorem 2. *Let $A = \{f_n \otimes x_n\}_{n=1}^{\infty}$ and $B = \{g_n \otimes y_n\}_{n=1}^{\infty}$ be two prenuclear operators and $A \leq B$. Then for every $x \in X$ there esists a n infinite matrix $T = \{\tau_{ij}\}_{i,j\in\mathcal{N}} \in B(c_0)$ such that*

$$\{f_n(x)x_n\}_{n=1}^{\infty} = \{\sum_{k\in\mathcal{N}} \tau_{nk} g_k(x) y_k\}_{n=1}^{\infty}.$$

Proof. On the space Y_x, define the linear bounded operator

$$\acute{T}_x : Y_x \to c_0$$

by

$$\acute{T}_x(\{g_n(x)f(y_n)\}_{n=1}^{\infty}) := \{f_n(x)f(x_n)\}_{n=1}^{\infty}.$$

To show that $\acute{T}_x$ is bounded, we have

$$\begin{aligned}\left\|\acute{T}_x(\{g_n(x)f(y_n)\}_{n=1}^{\infty})\right\|_{c_0} &= \left\|\{f_n(x)f(x_n)\}_{n=1}^{\infty}\right\|_{c_0} \\ &\leq M(A, B)\left\|\{g_n(x)f(y_n)\}_{n=1}^{\infty}\right\|_{c_0}.\end{aligned}$$

Using the fact that c_0 is separable Banach space and theorem (1), $\acute{T}_x$ can be extended to a linear bounded operator $T_x \in B(c_0)$, using (1), the matrix representation of the operator T_x has the required properties. □

3. Right and Left-Nuclear-Decomposable operators:

In this section we will introduce a novel class of operators having parallel characterizations as that proved for the left complemented operators [7] and we also collect some of its properties. Our definitions for the Right and Left-Nuclear-Decomposable operators are some way opposite to the factorization of operators through Banach spaces that given in [4] and [6].

Definition 2. (1) An operator $A \in B(X, Y)$ (respectively $A \in B(X)$) is said to be Right-Nuclear-Decomposable (respectively l_p-Right-Nuclear-Decomposable) with respect to a Banach space Z if and only if there exists a non-nuclear linear bounded operator T from Y into Z (respectively from X into Z) such that the compositi on TA is nuclear (respectively TA is nuclear and the sequence of nuclear norms $\{\nu(TA^n)\}_{n=1}^{\infty}$ is an element of the space l_p. i.e.,

$$\big\{\|\{\nu(TA^n)\}_{n=1}^{\infty}\|_{l_p}\big\}^p = \sum_{n\in\mathcal{N}} [\nu(TA^n)]^p < \infty.)$$

(2) The space Z in this case is said to be Right-Nuclearlizer or l_p-Right-Nuclearlizer of the operator A respectively.

$RNDO(X,\ Y;\ Z)$ denotes the class of all Right-Nuclear-Decomposable operators from a Banach space X into a Banach space Y with respect to the same Banach space Z, $l_p - RNDO(X;\ Z)$ denotes the class of all l_p-Right-Nuclear-Decomposable operators on a Banach space X with respect to the same Banach space Z. If $A \in RNDO(X,\ Y;\ Z)$, then the class of all nonnuclear operators $T : Y \to Z$ such that TA is nuclear will be denoted by $RNO(A;\ Z)$ and for $A \in l_p - RNDO(X;\ Z)$, the class of all nonnuclear operators $T : X \to Z$ such that TA is nuclear and $\{\nu(TA^n)\}_{n=1}^{\infty} \in l_p$ will be denoted by $l_p - RNO(A;\ Z)$. The class of all Right-nuclearlizer and l_p-Right-nuclearlizer of A will be denoted respectively by $RN(A)$ and $l_p - RN(A)$.

Definition 3. (1) If $A \in RNDO(X,\ Y;\ Z)$, then the right-relative nuclear constant of the operator A denoted by $R(A,\ Z)$ is defined by:

$$R(A,\ Z) := \inf\{\nu(TA) :\ T \in RNO(A;\ Z)\}.$$

(2) If $A \in l_p - RNDO(X;\ Z)$, then the p-right-relative nuclear constant of the operator A denoted by $R^p(A,\ Z)$ is defined by:

$$R^p(A,\ Z) := \inf\{\|\{\nu(TA^n)\}_{n=1}^{\infty}\|_{l_p} : T \in l_p - RNO(A;\ Z)\}.$$

(3) The absolute Right-Nuclear and p-Right-Nuclear constant s of A are defined respectively by:

$$R(A) = \sup\{R(A,\ Z) :\ Z \in RN(A)\}.$$

and

$$R^p(A) = \sup\{R^p(A,\ Z) :\ Z \in l_p - RN(A)\}.$$

A parallel arguments can be defined from the left.

Definition 4. (1) An operator $A \in B(X, Y)$ (respectively $A \in B(X)$)is said to be Left-Nuclear-Decomposable (respectively l_p-Left-Nuclear-Decomposable) with respect to a Banach space Z if and only if there exists a non-nuclear linear bounded operator T from Z into X (respectively T from Z into X) such that the composition AT is nuclear (respectively AT is nuclear and the sequence of nuclear norms $\{\nu(A^nT)\}_{n=1}^{\infty}$ is an element of the space l_p. i.e.,

$$\big\{\|\{\nu(A^nT)\}_{n=1}^{\infty}\|_{l_p}\big\}^p = \sum_{n\in\mathcal{N}} [\nu(A^nT)]^p < \infty.)$$

(2) The space Z in this case is said to be Left-Nuclearlizer or l_p-Left-Nuclearlizer-of the operator A respectively.

$LNDO(X, Y; Z)$ denotes the class of all Left-Nuclear-Decomposable operators from X into Y with respect to the same Banach space Z. $l_p - LNDO(X; Z)$ denote the class of all l_p-Left-Nuclear-Decomposable operators on a Banach space X with respect to the same Banach space Z. If $A \in LNDO(X, Y; Z)$, then the class of all nonnuclear operators $T : Z \to X$ such that AT is nuclear will be denoted by $LNO(A; Z)$. If $A \in l_p - LNDO(X; Z)$, then the class of all nonnuclear operators $T : Z \to X$ such that AT is nuclear and $\{\nu(A^nT)\}_{n=1}^{\infty} \in l_p$ will be denoted by $l_p - LNO(A; Z)$. The class of all Left-nuclearlizer spaces and l_p-Left-nuclearlizer spaces of A will be denoted respectively by $LN(A)$ and $l_p - LN(A)$.

Definition 5. (1) If $A \in LNDO(X, Y; Z)$, then the left-relative nuclear constant of the operator A denoted by $L(A, Z)$ is defined by:

$$L(A, Z) := \inf\{\nu(TA) : T \in LNO(A; Z)\}.$$

(2) If $A \in l_p - LNDO(X; Z)$, then the p-left-relative nuclear constant of the operator A denoted by $L^p(A, Z)$ is defined by:

$$L^p(A, Z) := \inf\{\|\{\nu(A^nT)\}_{n=1}^{\infty}\|_{l_p} : T \in l_p - LNO(A; Z)\}.$$

(3) The absolute Left-Nuclear and p-Left-Nuclear constants are defined respectively by:

$$L(A) = \sup\{L(A, Z) : Z \in LN(A)\}.$$

and

$$L^p(A) = \sup\{L^p(A, Z) : Z \in l_p - LN(A)\}.$$

4. Some preliminary lemmas

We have the following main properties:

Lemma 5. (1) *The classes $RNO(A; Z)$ and $l_p - RNO(A; Z)$ are linear subspaces of the spaces $B(Y, Z)$ and $B(X, Z)$ respectively.*

(2) *The classes $LNO(A; Z)$ and $l_p - LNO(A; Z)$ are also linear subspaces.*

(3) *The class $L(X, Y; Z)$ of the intersection of all classes $RNO(A; Z)$ over all nonnuclear operators $A \in RNDO(X, Y; Z)$*

$$L(X, Y; Z) := \bigcap_{A \in RNDO(X, Y; Z)/N(X, Y)} RNO(A; Z) \subset B(Y, Z),$$

is linear subspace of $B(Y, Z)$.

(4) *The class $R(X, Y; Z)$ of the intersection of all classes $LNO(A; Z)$ over all nonnuclear operators $A \in LNDO(X, Y; Z)$*

$$R(X, Y; Z) := \bigcap_{A \in LNDO(X, Y; Z)/N(X, Y)} LNO(A; Z),$$

is a linear subspace of $B(Z, X)$.

(5) *If $A \in RNDO(X, Y; Z)$, $T \in RNO(A; Z)$ and W is any Banach space with some $T_0 : Z \to W$ such that T_0T is not nuclear, then $A \in RNDO(X, Y; W)$.*

(6) *If* $A \in N(X,\ Y)$, *then* $A \in RNDO(X,\ Y;\ Z)$ *and* $A \in LNDO(X,\ Y;\ Z)$ *for any Banach space* Z.

(7) *If* $A \in B(X)$ *and* $\{\|A^n\|\}_{n=1}^{\infty} \in l_p$, *then*

$$A \in \begin{cases} l_p - RNDO(X;\ Z), & if\, A \in RNDO(X;\ Z); \\ l_p - LNDO(X;\ Z), & if\, A \in LNDO(X;\ Z). \end{cases}$$

In fact, if $T \in RNO(A;\ Z)$ *(respectively* $T \in LNO(A;\ Z)$*), we have* $\nu(TA^n) \leq \nu(TA)\|A^{n-1}\|$ *(respectively* $\nu(A^nT) \leq \nu(AT)\|A^{n-1}\|$*).*

(8) *If* $A \in l_p - RNDO(X;\ Z)$ *(respectively* $A \in l_p - LNDO(X;\ Z)$*),* $B \in B(X)$, $AB = BA$ *and* $\{\|B^n\|\}_{n=1}^{\infty} \in l_q$, *then* $AB \in l_r - RNDO(X;\ Z)$ *(respectively* $A \in l_r - LNDO(X;\ Z)$*), where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
In fact, if $T \in RNO(A;\ Z)$ *(respectively* $T \in LNO(A;\ Z)$*), we have* $\nu(T[AB]^n) = \nu(TA^nB^n) \leq \nu(TA^n)\|B^n\|$ *(respectively* $\nu([BA]^nT) = \nu(B^nA^nT) \leq \nu(A^nT)\|B^n\|$*), using the Minkowisky inequality, we get the proof.*

(9) *If* $A \in LNDO(X,\ Y;\ Z)$, *then* $A \in LNDO(X,\ Y;\ W)$ *where* W *is a subspace of* l_∞.

(10) *If* $A \in RNDO(X,\ Y;\ Z)$, *then* $A \in RNDO(X,\ Y;\ l_\infty)$.

(11) *Let* A *be a projection from* X *onto a closed subspace* Y *and* $A_\alpha = \alpha A$. *Then for* $A \in RNDO(X,\ Y;\ Z)$, *we have*

$$A_\alpha \begin{cases} \in l_p - RNDO(X,\ Y;\ Z), & if\ |\alpha^p| < 1; \\ \notin l_p - RNDO(X,\ Y;\ Z), & if\ |\alpha^p| \geq 1. \end{cases}$$

Moreover in the first, case we have

$$R_p(A,\ Z) \leq R(A,\ Z)^p \big[\frac{|\alpha^p|}{1 - |\alpha^p|}\big].$$

We also have the same from the left, in particular the projection operators are not Power-Nuclear-Decomposable operators.

(12) *Let* $A \in B(X,\ Y)$, $B \in B(Y,\ E)$, *and* $C \in B(W,\ X)$. *Then*
- *If* $A \in LNDO(X,\ Y;\ Z)$, *and* B *homeomorphism, then* $BA \in LNDO(X,\ E;\ Z)$ *and* $AC \in LNDO(W,\ Y;\ Z)$.
- *If* $A \in RNDO(X,\ Y;\ Z)$ *and* B *homeomorphism, then* $BA \in RNDO(X,\ E;\ Z)$ *and* $AC \in RNDO(W,\ Y;\ Z)$.

We have the following theorem:

Theorem 3. *Let* $A \in LNDO(X,\ Y;\ l_\infty)$. *Then* $A \in LNDO(X,\ Y;\ Z)$ *for every Banach space* Z. *Moreover the absolute Left-Nuclear-Decomposable constant* $L(A)$ *of the operator* A *attains its supremum at* l_∞. *i.e.,*

$$L(A) = L(A,\ l_\infty).$$

Proof. Let l_∞ be a left nuclearlizer of the operator A. Then there exists a linear bounded non-nucle ar operator $T : l_\infty \to X$ such that $AT : l_\infty \to Y$ is nuclear. If Z is any Banach space, $J : Z \to l_\infty$ is the isometric embedding of Z into l_∞, then TJ

is not nuclear with $A(TJ) = (AT)J$ nuclear and $\nu(A(TJ)) \leq \|J\| \, \nu(AT) = \nu(AT)$, from that we conclude for the infimum,

$$L(A,\ Z) \leq L(A,\ l_\infty),\ \textit{for every Banach space } Z.$$

Taking the supremum over all Banach spaces Z we get the proof. □

5. EXAMPLES

we have the following examples:

(1) The identity operator I_X on an infinite dimensional Banach space X is not Left and not Right-Nuclear-Decomposable with respect to any Banach space Z, if X is a Banach space having Schauder basis, then I_X is prenuclear.

(2) Consider the operator $A \in B(l_1)$ defined by

$$A(\{x_i\}_{i=1}^\infty) = \{\frac{\alpha x_i}{i}\}_{i=1}^\infty,$$

where α is a real number with $|\alpha| \leq 1$.

- A is prenuclear non-nuclear. In fact A has the representation

$$A(\{x_i\}_{i=1}^\infty) = \sum_{n \in \mathcal{N}} f_n(\{x_i\}_{i=1}^\infty) e_n,$$

where $f_n = \{0,\ 0,\ \ldots,\ \frac{\alpha}{n},\ 0,\ 0,\ \ldots\} \in l_\infty$, the nonzero coordinate is the n-coordinate. i.e., $f_n(\{x_i\}_{i=1}^\infty) = \frac{\alpha x_n}{n}$ and $e_n = \{0,\ 0,\ \ldots,\ 1,\ 0,\ 0,\ \ldots\}$, $n \in \mathcal{N}$ are the canonical basis elements of the space l_1.
On the other hand,

- A is Right-Nuclear-Decomposable with respect to l_1. In fact, consider the linear bounded non-nuclear operator T on l_1 defined by

$$T(\{x_i\}_{i=1}^\infty) = \{\frac{x_i}{i}\}_{i=1}^\infty,$$

we have

$$TA(\{x_i\}_{i=1}^\infty) = \{\frac{\alpha x_i}{i^2}\}_{i=1}^\infty,$$

which is linear bounded nuclear operator. This showed also that T is Left-Nuclear-Decomposable with respect to l_1.

- A is l_p-Power-Nuclear-Decomposable with respect to l_1. In fact, we have

$$(TA^n)(\{x_i\}_{i=1}^\infty) = \{\frac{\alpha x_i}{i^{n+1}}\}_{i=1}^\infty,$$

and

$$\nu(TA^n) \leq \sum_{i \in \mathcal{N}} \frac{|\alpha|^n}{i^{n+1}},$$

$$\|\{\nu(TA^n)\}_{n=1}^\infty\|_{l_p} \leq \frac{|\alpha|}{(1 - |\alpha|^p)^{\frac{1}{p}}}.$$

- Finally, we have

$$R^p(A) \leq \frac{|\alpha|}{(1 - |\alpha|^p)^{\frac{1}{p}}}.$$

(3) Consider the last operator as an operator from l_q into l_1 and T as an operator from l_1 into l_∞. For this two operators A is Right-Nuclear-Decomposable with respect to l_∞ and T is Left-Nuclear-Decomposable with respect to l_q.

6. Problems

We have the following problems:

Problem 1. Under what conditions are the spaces $L(X, Y; Z)$ and $R(X, Y; Z)$ not empty?

Problem 2. Under what conditions are there infinite matries $T = \{\tau_{ij}\}_{i,j\in\mathcal{N}} \in B(c_0)$ satiesfying theorem (2) valied for every $x \in X$?

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