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EXISTENCE OF NONNEGATIVE SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS VIA PSEUDOMONOTONE OPERATORS AND NONSMOOTH CRITICAL POINT THEORY

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Abstract. We study the existence of nonnegative solutions for nonlinear Dirichlet boundary value problems driven by the ordinary scalar p-Laplacian and with a nonsmooth potential. Our approach involves using the method of upper and lower solutions with nonsmooth critical point theory for locally Lipschitz functions. Our analysis covers the so-called "sublinear" and "superlinear" cases.

1. Introduction

In this paper, we study the existence of nonnegative solutions for the following nonlinear second order boundary value problem

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) \text{ a.e on } T=[0,b] \\ x(0)=x(b)=0, \quad 1<p<\infty. \end{array}\right\} \tag{1.1}$$

Here $j(t,x)$ is a function measurable in $t \in T$ and locally Lipschitz-in general nonsmooth-in $x \in \mathbb{R}$. By $\partial j(t,x)$ we denote the generalized (Clarke) subdifferential of $j(t,\cdot)$ (see Section 2). In the past the question of existence of nonnegative solutions was examined primarily in the context of semilinear (i.e. $p=2$) equations with a smooth potential (i.e. $j(t,\cdot) \in C^1(\mathbb{R})$ and so $\partial j(t,\cdot) = f(t,\cdot) \in C(\mathbb{R})$). In this direction we mention the works of Erbe-Hu-Wang [8], Erbe-Wang [9], Liu-Li [14] and Y. Li [13]. In all of these works the right hand side nonlinearity is jointly continuous on $T \times \mathbb{R}$ and in the first three papers it is also nonnegative. Only the recent work of Y. Li [13] allows the nonlinearity to have a varying sign. The works of Erbe-Hu-Wang [8] and Erbe-Wang [9] cover the strictly sublinear and superlinear cases, while Liu-Li [14] and Y. Li [13] include in their considerations the asymptotically linear case. In all these works the approach is similar and it is based on compression-expansion type fixed point theorems (see for example Guo-Lakshmikantham [11]). For problems driven by the ordinary scalar p-Laplacian, we have the works of De Coster [4], Wang [16] and Agarwal-Lü-O'Regan [1]. De Coster [4] studies the so-called "sub-super linear case" which loosely speaking means that the ratio $\dfrac{f(t,x)}{|x|^{p-2}x}$ (f being the right hand side nonlinearity) is greater than the first eigenvalue $\lambda_1 > 0$ near 0^+ and near $+\infty$. No interaction with $\lambda_1 > 0$ is allowed (uniform nonresonance). The nonlinearity $f(t,x)$ is a Carathéodory function

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and the approach of De Coster is degree theoretic with parallel use of the time map. In Wang [16] and Agarwal-Lü-O'Regan [1] the right hand side nonlinearity is nonnegative. Wang [16] deals with the sublinear and superlinear problems, while Agarwal-Lü-O'Regan [1] study eigenvalue problems and for a variety of asymptotic conditions at 0^+ and $+\infty$, determine the eigenvalue interval for which the problem has a nonnegative solution. Both works base their method of proof on fixed point theorems of compression-expansion type.

Our work here extends the aforementioned semilinear works and complements the work of De Coster [4] since we cover also the sublinear and superlinear cases. Moreover, in certain circumstances we allow partial interaction with $\lambda_1 > 0$ (nonuniform nonresonance). In contrast to Wang [16] and Agarwal-Lü-O'Regan [1] here the potential function changes sign. Moreover, we do not have a smooth potential function. Finally, our approach is different and either it uses the method of upper and lower solutions or it is variational based on nonsmooth critical point theory (see Chang [2] and Kourogenis-Papageorgiou [12]). For the convenience of the reader, in the next section we recall some basic definitions and facts from this theory, which will be used in the sequel.

2. Mathematical Background

The nonsmooth critical point theory that we use is based on the subdifferential theory of Clarke [3] for locally Lipschitz functions. Let $X = (X, \|\cdot\|)$ be a Banach space. By X^* we denote its topological dual and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ we can find a neighborhood U of x and $k > 0$ (depending on U) such that $|\varphi(y) - \varphi(z)| \le k\|y - z\|$ for all $y, z \in U$. We know that if $\psi : X \to \mathbb{R}$ is continuous convex, then it is locally Lipschitz. Also if $\psi \in C^1(X)$, then clearly ψ is locally Lipschitz.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the generalized directional derivative of φ at x in the direction $h \in X$ is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x; \cdot)$ is sublinear and continuous. Thus it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi^0(x; h) \ \text{ for all } \ h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of φ. If φ is also convex, then $\partial\varphi(x)$ coincides with the subdifferential in the sense of convex analysis given by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le \varphi(y) - \varphi(x) \ \text{ for all } \ y \in X\}.$$

Moreover, if $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$. For $\varphi, \psi :\to \mathbb{R}$ locally Lipschitz functions and $\lambda \in \mathbb{R}$, we have

$$\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi \ \text{ and } \ \partial(\lambda\varphi) = \lambda\partial\varphi.$$

A point $x \in X$ is said to be a critical point of φ, if $0 \in \partial\varphi(x)$. It is easy to check that if $x \in X$ is a local extremum of φ (i.e. a local maximum or a local minimum), then $0 \in \partial\varphi(x)$ (i.e. $x \in X$ is a critical point of φ). If $x \in X$ is a critical point of φ, then $c = \varphi(x)$ is a critical value of φ.

From the smooth theory (i.e. $\varphi \in C^1(X)$), we know that in variational methods a central role is played by a compactness type condition, known as the Palais-Smale condition (PS-condition for short). In the present nonsmooth setting this condition takes the following form:

> "A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the *nonsmooth PS-condition*, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $|\varphi(x_n)| \leq M$ for some $M > 0$ and all $n \geq 1$ and
> $$m(x_n) = \inf\,[\|x^*\| : x^* \in \partial\varphi(x_n)] \to 0 \quad \text{as } n \to \infty,$$
> has a strongly convergent subsequence".

Using this condition, we have a nonsmooth version of the well-known Mountain Pass Theorem (see Chang [2] and Kourogenis-Papageorgiou [12]).

Theorem 2.1. *If X is a reflexive Banach space, $\varphi : X \to \mathbb{R}$ is locally Lipschitz and satisfies the nonsmooth PS-condition, there exist $r > 0$ and $x_1 \in X$ such that $\|x_1\| \geq r$ and $\max\{\varphi(x_1), \varphi(0)\} < \inf\,[\varphi(x) : \|x\| = r]$ and $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = x_1\}$, then*
$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))$$
is a critical value and $c \geq \inf\,[\varphi(x) : \|x\| = r]$.

As we already indicated in the introduction, our hypotheses on $j(t,x)$ will involve the spectrum of the negative ordinary scalar p-Laplacian with Dirichlet boundary conditions, so let us briefly describe this spectrum. Consider the following nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = \lambda|x(t)|^{p-2}x(t) \text{ a.e on } T = [0,b] \\ x(0) = x(b) = 0,\ 1 < p < \infty,\ \lambda \in \mathbb{R}. \end{array}\right\} \tag{2.1}$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue, if problem (2.1) has a nontrivial solution. It is well-known (see Del Pino-Elgueta-Manasevich [5]) that problem (2.1) has an increasing sequence $\{\lambda_n\}_{n\geq 1}$ of eigenvalues given by $\lambda_n = (\frac{n\pi_p}{b})^p$, $n \geq 1$, where $\pi_p = \dfrac{2\pi(p-1)^{1/p}}{p\sin(\frac{\pi}{p})}$. Note that $\pi_2 = \pi$. It is worth pointing out that the same sequence forms the spectrum of the negative ordinary vector p-Laplacian with Dirichlet boundary conditions (this is no longer true if we have periodic boundary conditions). The first eigenvalue $\lambda_1 = (\frac{\pi_p}{b})^b$ is simple (i.e. the corresponding eigenspace is one dimensional) and has the following variational characterization

$$\lambda_1 = \inf\left[\frac{\|x'\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(0,b),\ x \neq 0\right]. \tag{2.2}$$

The infimum is attained at the eigenfunctions corresponding to $\lambda_1 > 0$. Let u_1 be the normalized (i.e. $\|u_1\|_p = 1$) eigenfunction corresponding to $\lambda_1 > 0$. Then $u_1(t) > 0$ for all $t \in (0,b)$.

Finally let us recall some definitions and facts from the theory of nonlinear operators of monotone type. For details we refer the reader to Denkowski-Migorski-Papageorgiou [7].

Let X be a reflexive Banach space. An operator $A : D \subseteq X \to 2^{X^*}$ is said to be monotone, if for all $x, y \in D$ and all $x^* \in A(x), y^* \in A(y)$, we have $\langle x^* - y^*, x - y\rangle \geq 0$. If $(x^* - y^*, x - y) = 0$ implies that $x = y$, then we say that A is strictly monotone. A monotone operator $A : D \subseteq X \to 2^{X^*}$ is said to be maximal monotone, if $\langle x^* - y^*, x - y\rangle \geq 0$ for all $x \in D$ and all $x^* \in A(x)$, imply $y \in D$ and $y^* \in A(y)$. The monotone operator A is maximal monotone if and only if its graph $GrA = \{(x, x^*) \in X \times X^* : x^* \in A(x)\}$ is maximal with respect to inclusion among the graphs of all monotone operators. An operator $A : X \to X^*$ which is single-valued and everywhere defined, is said to be demicontinuous, if $x_n \to x$ in X, implies that $A(x_n) \xrightarrow{w} A(x)$ in X^*. A monotone, demicontinuous operator is maximal monotone.

An operator $A : D \subseteq\to 2^{X^*}$ is said to be coercive, if D is bounded or D is unbounded and $\inf\,[\|x^*\| : x^* \in A(x)] \to +\infty$ as $\|x\| \to +\infty$.

An operator $A : X \to 2^{X^*}$ is said to be pseudomonotone if

(a) for every $x \in X$, $A(x)$ is nonempty, weakly compact and convex;
(b) A is upper semicontinuous for every finite dimensional subspace Z of X into X^* equipped with the weak topology (i.e. for all $U \subseteq X^*$ weakly open $A|_Z^+(U) = \{x \in Z : A(x) \subseteq U\}$ is open);
(c) If $x_n \xrightarrow{w} x$ in X, $x_n^* \in A(x_n)$ and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \leq 0$, then for every $y \in X$, there exists $x^*(y) \in A(x)$ such that $\langle x^*(y), x-y\rangle \leq \liminf_{n\to\infty}\langle x_n^*, x_n-y\rangle$.

If A is bounded (i.e. it maps bounded sets to bounded sets) and satisfies condition (c), then it satisfies condition (b) too. An operator $A : D \subseteq X \to 2^{X^*}$ is said to be generalized pseudomonotone, if for all $x_n^* \in A(x_n)$ $n \geq 1$ which satisfy $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \leq 0$, we have $x^* \in A(x)$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

The following proposition relates all these notions.

Proposition 2.2. *If X is a reflexive Banach space and $A : X \to 2^{X^*}$, then*

(a) *If A is maximal monotone, it is also generalized pseudomonotone;*
(b) *If A is pseudomonotone, it is also generalized pseudomonotone;*
(c) *If A is generalized pseudomonotone, bounded and for all $x \in X$, $A(x)$ is nonempty, w-compact and convex, then A is pseudomonotone;*
(d) *If A is pseudomonotone and coercive, then A is surjective.*

3. Sublinear Problems

In this section we deal with the so-called "sublinear case". Roughly speaking, in the context of semilinear (i.e. $p = 2$), smooth (i.e. $\partial j(t, \cdot) = f(t, \cdot) \in C(\mathbb{R})$) problems, this means that the ratio (slope) $\dfrac{f(t,x)}{x}$ is below the first eigenvalue $\lambda_1 > 0$ near 0^+ and near $+\infty$. Therefore $f(t, \cdot)$ exhibits a linear or sublinear behavior near 0^+ and near $+\infty$ and this justifies the name "sublinear case". In all the works mentioned in the introduction, asymptotically at 0^+ and at $+\infty$, there is no interaction with $\lambda_1 > 0$. In contrast here asymptotically at $+\infty$, we allow partial interaction with $\lambda_1 > 0$ (nonuniform nonresonance, see hypothesis $H(j)_1(iv)$ below).

Our hypotheses on the nonsmooth potential $j(t,x)$ are the following:

$\underline{H(j)_1}$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \le r$ and all $u \in \partial j(t,x)$, we have $|u| \le \alpha_r(t)$;

(iv) there exists $\theta \in L^\infty(T)_+$ such that $\theta(t) \le \lambda_1$ a.e. on T with strict inequality on a set of positive measure and $\limsup\limits_{x\to+\infty} \dfrac{u}{x^{p-1}} \le \theta(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$;

(v) $\liminf\limits_{x\to 0^+} \dfrac{u}{x^{p-1}} > \lambda_1$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$.

Remark 3.1. The following nonsmooth locally Lipschitz functions satisfy hypotheses $H(j)_1$. For simplicity we have dropped the t-dependence: $j_1(x) = \max\{|x|^r, \mu |x|^p\}$ with $1 \le r < p$ and $0 < \mu < \lambda_1$. Also let $h(x) = \frac{1}{2}(x^2 + x)$ for all $x \in [0,1]$ and extend h by periodicity to all of $\mathbb{R}$ (period 1). Denote the extension by $j_2(x)$. Evidently $\partial j_2(x) = \begin{cases} x - [x] + \frac{1}{2} & \text{if } x \notin \mathbb{N}_0 \\ [0,1] & \text{if } x \in \mathbb{N}_0 \end{cases}$, i.e. $\partial j_2(x)$ is a sawtooth type function (here $[x]$ denotes the largest integer less or equal to x and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We start with a simple auxiliary result which will help us to deal with the nonuniform nonresonance condition in hypothesis $H(j)_1(iv)$.

Lemma 3.2. *If $\theta \in L^\infty(T)_+$ is such that $\theta(t) \le \lambda_1$ a.e. on T with strict inequality on a set of positive measure, then there exists $\xi_1 > 0$ such that $\psi(x) = \|x'\|_p^p - \int_0^b \theta(t)|x(t)|^p dt \ge \xi_1 \|x'\|_p^p$ for all $x \in W_0^{1,p}(0,b)$.*

Proof. By virtue of the variational characterization of $\lambda_1 > 0$ (see (2.2)), we see that $\psi \ge 0$. Suppose that the lemma was not true. Exploiting the positive p-homogeneity of the function ψ, we can find $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ with $\|x_n'\|_p = 1$, $n \ge 1$, such that $\psi(x_n) \downarrow 0$ as $n \to \infty$.

From the Poincaré inequality, we have that $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W_0^{1,p}(0,b) \ \text{ and } \ x_n \to x \ \text{ in } \ C(T)$$

(recall that $W_0^{1,p}(0,b)$ is embedded compactly in $C(T)$). Also since $x_n' \xrightarrow{w} x'$ in $L^p(T)$, we have $\|x'\|_p \le \liminf\limits_{n\to\infty} \|x_n'\|_p^p$. Thus in the limit as $n \to \infty$, we obtain

$$\begin{aligned} &\|x'\|_p^p \le \int_0^b \theta(t)|x(t)|^p dt \le \lambda_1 \|x\|_p^p, \qquad (3.1)\\ \Rightarrow &\|x'\|_p^p = \lambda_1 \|x\|_p^p \ \text{ (see (2.2))},\\ \Rightarrow &x = 0 \ \text{ or } \ x = \pm u_1. \end{aligned}$$

If $x = 0$, then $\|x_n'\|_p \to 0$, a contradiction to the fact that $\|x_n'\|_p = 1$ for all $n \ge 1$. Thus $x = \pm u_1$, which means that $|x(t)| > 0$ for all $t \in (0,b)$. Therefore from (3.1) and hypothesis $H(j)_1(iv)$, we have

$$\|x'\|_p^p \leq \int_0^b \theta(t)|x(t)|^p dt < \lambda_1 \|x\|_p^p,$$

a contradiction to (2.2). $\square$

Because of hypotheses $H(j)_1(iii)$ and (iv), given $\varepsilon > 0$, we can find $\gamma_\varepsilon \in L^1(T)_+ \setminus \{0\}$ such that for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$, we have

$$u \leq (\theta(t)+\varepsilon)x^{p-1} + \gamma_\varepsilon(t). \tag{3.2}$$

We consider the following auxiliary Dirichlet problem:

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = (\theta(t)+\varepsilon)|x(t)|^{p-2}x(t) + \gamma_\varepsilon(t) \text{ a.e on } T \\ x(0) = x(b) = 0, \ \ \varepsilon > 0. \end{array} \right\} \tag{3.3}$$

In the next proposition we show that for $\varepsilon > 0$ small, problem 3.3 has a positive solution.

Proposition 3.3. *If $\theta \in L^\infty(T)_+$ and $\theta(t) \leq \lambda_1$ a.e. on T with strict inequality on a set of positive measure, then for $\varepsilon > 0$ small, problem (3.3) has a solution $\overline{x} \in C^1(T)$ such that $\overline{x}(t) > 0$ for all $t \in (0,b)$, $x'(b) < 0 < x'(0)$.*

Proof. Let $A : W_0^{1,p}(0,b) \to W^{-1,q}(0,b) = W_0^{1,p}(0,b)^*$ $(\frac{1}{p} + \frac{1}{q} = 1)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt \ \text{ for all } \ x,y \in W_0^{1,p}(0,b).$$

Hereafter by $\langle \cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{-1,q}(0,b), W_0^{1,p}(0,b))$. It easy to check that A is monotone demicontinuous, hence it is maximal monotone. Also let $J_\varepsilon : W_0^{1,p}(0,b) \to L^q(T) \subseteq W^{-1,q}(0,b)$ be defined by

$$J_\varepsilon(x)(\cdot) = (\theta(\cdot)+\varepsilon)|x(\cdot)|^{p-2}x(\cdot).$$

Because of the compact embedding of $L^q(T)$ into $W^{-1,q}(0,b)$, the map $x \to J_\varepsilon(x)$ is completely continuous. Let $K_\varepsilon : A - J_\varepsilon : W_0^{1,p}(0,b) \to W^{-1,q}(0,b)$. We claim that K_ε is pseudomonotone. Since K_ε is everywhere defined and bounded, it suffices to show that K_ε is generalized pseudomonotone. For this purpose suppose that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(0,b)$ and assume that $\limsup\limits_{n\to\infty}\langle K_\varepsilon(x_n), x_n - x\rangle \leq 0$. We need to show that

$$\langle K_\varepsilon(x_n), x_n\rangle \to \langle K_\varepsilon(x), x\rangle \ \text{ and } \ K_\varepsilon(x_n) \xrightarrow{w} K_\varepsilon(x) \ \text{ in } \ W^{-1,q}(0,b). \tag{3.4}$$

We have

$$\langle K_\varepsilon(x_n), x_n - x\rangle = \langle A(x_n), x_n - x\rangle - \langle J_\varepsilon(x_n), x_n - x\rangle.$$

Note that $\langle J_\varepsilon(x_n), x_n - x\rangle \to 0$ as $n \to \infty$. As a result it follows that

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0.$$

Now since A is maximal monotone, it is generalized pseudomonotone and so we have

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \ \text{ and } \ A(x_n) \xrightarrow{w} A(x) \ \text{ in } \ W^{-1,q}(0,b). \tag{3.5}$$

Also from the complete continuity of J_ε we have

$$J_\varepsilon(x_n) \to J_\varepsilon(x) \text{ in } W^{-1,q}(0,b). \tag{3.6}$$

From (3.5) and (3.6), we conclude that (3.4) holds and so K_ε is pseudomonotone.

For every $x \in W_0^{1,p}(0,b)$, we have

$$\begin{aligned}\langle K_\varepsilon(x), x\rangle &= \langle A(x), x\rangle - \langle J_\varepsilon(x), x\rangle \\ &= \|x'\|_p^p - \int_0^b \theta_\varepsilon(t)|x(t)|^p dt - \varepsilon\|x\|_p^p \\ &\geq \xi_1\|x'\|_p^p - \varepsilon\|x\|_p^p \text{ (see Lemma 3.2)} \\ &\geq \xi_1\|x'\|_p^p - \frac{\varepsilon}{\lambda_1}\|x'\|_p^p \text{ (see (2.2))} \\ &= (\xi - \frac{\varepsilon}{\lambda_1})\|x'\|_p^p. \end{aligned} \tag{3.7}$$

If we choose $\varepsilon < \xi_1\lambda_1$, from (3.7) and the Poincaré inequality we infer that K_ε is coercive. Since a pseudomonote, coercive operator is surjective, we can find $\overline{x} \in W_0^{1,p}(0,b)$ such that

$$K_\varepsilon(\overline{x}) = \gamma_\varepsilon$$

(recall that $L^1(T) \subseteq W^{-1,q}(0,b)$). From this equality it follows easily that $\overline{x} \in C^1(T)$ and it solves problem (3.3).

Next we show that $\overline{x}(t) > 0$ for all $t \in (0,b)$. To this end, if we use as a test function $-x^- \in W_0^{1,p}(0,b)$, we obtain

$$\begin{aligned}\|(\overline{x}^-)'\|_p^p &= \int_0^b (\theta(t)+\varepsilon)|\overline{x}^-(t)|^p dt + \int_0^b \gamma_\varepsilon(t)(-\overline{x}^-)(t)dt \\ &\leq \int_0^b (\theta(t)+\varepsilon)|\overline{x}^-(t)|^p dt \text{ (since } \gamma_\varepsilon \geq 0), \\ \Rightarrow \xi_1\|(\overline{x}^-)'\|_p^p &\leq \varepsilon\|\overline{x}^-\|_p^p \text{ (see Lemma 3.2)} \\ \Rightarrow \xi_1\|(\overline{x}^-)'\|_p^p &\leq \frac{\varepsilon}{\lambda_1}\|(\overline{x}^-)'\|_p^p \text{ (see (2.2))}. \end{aligned} \tag{3.8}$$

Recall that $\varepsilon < \lambda_1\xi_1$. Now from (3.8) we infer that $\|(\overline{x}^-)'\|_p = 0$ and this by Poincaré's inequality implies that $\overline{x}^- = 0$, hence $\overline{x}(t) \geq 0$ for all $t \in T$. Furthermore, from (3.3) and since $\gamma_\varepsilon \neq 0$, we see that $\overline{x} \neq 0$. Finally note that

$$(|x'(t)|^{p-2}x'(t))' \leq (\theta(t)+\varepsilon)\overline{x}(t)^{p-1} \text{ a.e. on } T \text{ (see (3.3))}.$$

Invoking Theorem 5 of Vazquez [15], we obtain that

$$\overline{x}(t) > 0 \text{ for all } t \in (0,b), \text{ with } x'(b) < 0 < x'(0). \qquad \square$$

Consider the functions

$$\underline{\alpha}(t,x) = \min[u : u \in \partial j(t,x)] \text{ and } \overline{\alpha}(t,x) = \max[u : u \in \partial j(t,x)].$$

Because of hypotheses $H(j)_1(i)$ and (ii) and by redefining if necessary, $j(\cdot,\cdot)$ on a set $N \times \mathbb{R}$ with N being a Lebesgue-null subset of T, we may assume that $j(\cdot,\cdot)$

is Borel measurable on $T \times \mathbb{R}$ and that for all $t \in T$ $j(t, \cdot)$ is locally Lipschitz. From the definition of the generalized directional derivative, we have

$$j^0(t, x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{j(t, x' + \lambda h) - j(t, x')}{\lambda} = \inf_{\substack{\varepsilon > 0 \\ \varepsilon \in Q}} \sup_{\substack{|x' - x| < \varepsilon \\ 0 < \lambda < \varepsilon \\ x', \lambda \in Q}} \frac{j(t, x' + \lambda h) - j(t, x')}{\lambda}. \tag{3.9}$$

Since the function $(t, x) \to j(t, x)$ is Borel measurable, from (3.9) it follows that $(t, x, h) \to j^0(t, x; h)$ is Borel measurable from $T \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. We also know that

$$\partial j(t, x) = \{u \in \mathbb{R} : uh \leq j^0(t, x; h) \text{ for all } h \in \mathbb{R}\}.$$

Let $\{h_n\}_{n \geq 1}$ be the enumeration of the rationals in $\mathbb{R}$. Exploiting the continuity of the map $h \to j^0(t, x; h)$, we have

$$\mathrm{Gr}\partial j = \bigcap_{n \geq 1} \{(t, x, u) \in T \times \mathbb{R} \times \mathbb{R} : uh_n \leq j^0(t, x; h_n)\} \in B(T) \times B(\mathbb{R}) \times B(\mathbb{R}),$$

where $B(T)$ (resp. $B(\mathbb{R})$) is the Borel σ-field of T (resp. of $\mathbb{R}$). For every $\eta \in \mathbb{R}$ we have

$$\{(t, x) \in T \times \mathbb{R} : \underline{\alpha}(t, x) < \eta\} = \mathrm{proj}_{T \times \mathbb{R}}[\mathrm{Gr}\partial j \cap (T \times \mathbb{R} \times (-\infty, \eta))].$$

The generalized subdifferential is compact-valued. Thus we can use Theorem 2.6.36, p.216, of Denkowski-Migorski-Papageorgiou [6] and obtain that

$$\{(t, x) \in T \times \mathbb{R} : \underline{\alpha}(t, x) < \eta\} \in B(T \times \mathbb{R}) = B(T) \times B(\mathbb{R}),$$
$$\Rightarrow \underline{\alpha} \text{ is Borel measurable.}$$

Since $\overline{\alpha}(t, x) = \max[u : u \in \partial j(t, x)] = -\min[v : v \in \partial(-j)(t, x)]$, the above argument (with j replaced by $-j$) reveals that $\overline{\alpha}$ is Borel measurable too.

Note that hypothesis $H(j)_1(iii)$ implies that for every $u \in W_0^{1,p}(0, b)$, $\underline{\alpha}(\cdot, u(\cdot))$, $\overline{\alpha}(\cdot, u(\cdot)) \in L^1(T)$. Using these two functions, we can now give the definitions of upper and lower solutions for problem (1.1).

Definition:

(a) A function $\underline{v} \in W^{1,p}(0, b)$ is a "lower solution" for problem (1.1), if $\underline{v}(0), \underline{v}(b) \leq 0$ and $\int_0^b |\underline{v}'(t)|^{p-2}\underline{v}'(t)y'(t)dt \leq \int_0^b \underline{\alpha}(t, \underline{v}(t))y(t)dt$ for all $y \in W_0^{1,p}(0, b)$, with $y(t) \geq 0$ for $t \in T$.

(b) A function $\overline{v} \in W^{1,p}(0, b)$ is an "upper solution" for problem (1.1), if $\overline{v}(0), \overline{v}(b) \geq 0$ and $\int_0^b |\overline{v'}(t)|^{p-2}\overline{v'}(t)y'(t)dt \leq \int_0^b \overline{\alpha}(t, \overline{v}(t))y(t)dt$ for all $y \in W_0^{1,p}(0, b)$, with $y(t) \geq 0$ for $t \in T$.

Let $\overline{x} \in C^1(T)$ be the solution of problem (3.3) obtained in Proposition 3.3. Then because of (3.2) and (3.3), we have

$$\left\{ \begin{array}{l} -(|\overline{x}'(t)|^{p-2}\overline{x}'(t))' = (\theta(t) + \varepsilon)|\overline{x}(t)|^{p-1} + \gamma_\varepsilon(t) \geq \overline{\alpha}(t, x(t)) \text{ a.e on } T \\ \overline{x}(0) = \overline{x}(b) = 0. \end{array} \right\}$$

Hence $\overline{x}$ is an upper solution for problem (1.1).

On the other hand, because of hypothesis $H(j)_1(v)$, given $\delta > 0$, we can find $\mu = \mu(\delta) > \lambda_1$ such that for almost all $t \in T$, all $x \in (0, \delta]$ and all $u \in \partial j(t, x)$, we have

$$u \geq \mu x^{p-1} > \lambda_1 x^{p-1}. \tag{3.10}$$

Let $u_1 \in C^1(T)$ be the normalized eigenfunction corresponding to $\lambda_1 > 0$. Then we can find $\eta = \eta(\delta) > 0$ small enough such that $\underline{x}(t) = \eta u_1(t) \leq \delta$ for all $t \in T$. On the other hand if $C_0^1(T) = \{x \in C^1(T) : x(0) = x(b) = 0\}$ and $C_0^1(T)_+ = \{x \in C_0^1(T) : x(t) \geq 0 \text{ for all } t \in T\}$, then

$$\mathrm{int} C_0^1(T)_+ = \{x \in C_0^1(T)_+ : x'(b) < 0 < x'(0)\}.$$

Thus Proposition 3.3 implies that $\overline{x} \in \mathrm{int} C_0^1(T)_+$. This means that we can always choose $\eta > 0$ such that $\underline{x}(t) < \overline{x}(t)$ for all $t \in (0, b)$. Moreover, we have

$$\left\{ \begin{array}{l} -(|\underline{x}'(t)|^{p-2}\underline{x}'(t))' = \lambda_1 |\underline{x}(t)|^{p-1} < \mu \underline{x}(t)^{p-1} \leq \underline{\alpha}(t, x(t)) \text{ a.e on } T \text{ (see (3.10))} \\ \underline{x}(0) = \underline{x}(b) = 0. \end{array} \right\}$$

Here $\underline{x} \in C_0^1(T)$ is a lower solution for problem (1.1).

Next employing truncation and penalization techniques, we produce a solution $x \in C_0^1(T)$ of (1.1) such that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \in T$. Evidently this is a positive solution of problem (1.1).

Theorem 3.4. *If hypotheses $H(j)_1$ hold, then problem (1.1) has a solution $x \in C_0^1(T)$ such that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \in T$.*

Proof. We introduce the truncation map $\tau : W_0^{1,p}(0, b) \to W_0^{1,p}(0, b)$ defined by

$$\tau(x)(t) = \begin{cases} \overline{x}(t) & \text{if } \overline{x}(t) \leq x(t) \\ x(t) & \text{if } \underline{x}(t) \leq x(t) \leq \overline{x}(t) \\ \underline{x}(t) & \text{if } x(t) \leq \underline{x}(t). \end{cases}$$

Clearly τ is continuous and bounded and the same can be said if we view τ as a map from $L^p(T)$ into itself. Moreover, for all $x \in W_0^{1,p}(0, b)$, we have $\|\tau(x)\|_p^p \leq \|x\|_p^p + c_1$ for some $c_1 > 0$.

In addition we introduce a penalty function $\sigma : T \times \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(t, x) = \begin{cases} (x - \overline{x}(t))^{p-1} & \text{if } \overline{x}(t) \leq x \\ 0 & \text{if } \underline{x}(t) \leq x \leq \overline{x}(t) \\ -(\underline{x}(t) - x)^{p-1} & \text{if } x \leq \underline{x}(t). \end{cases}$$

It is clear from this definition that $\sigma(t, x)$ is a Carathéodory function (i.e. measurable in $t \in T$ and continuous in $x \in \mathbb{R}$, hence jointly measurable), which is nondecreasing in the $x \in \mathbb{R}$ variable.

Let $A : W_0^{1,p}(0, b) \to W^{-1,q}(0, b)$ be the maximal monotone operator introduced in the proof of Proposition 3.3 and let $N_\sigma : W_0^{1,p}(0, b) \to L^q(T)$ be the Nemitsky operator corresponding to $\sigma(t, x)$, namely $N_\sigma(x)(\cdot) = \sigma(\cdot, x(\cdot))$ for all $x \in W_0^{1,p}(0, b)$. Evidently N_σ is bounded and continuous and

$$\langle N_\sigma(x), x \rangle \geq c_2 \|x\|_p^p - c_3 \text{ for all } x \in W_0^{1,p}(0, b) \text{ and some } c_2, c_3 > 0. \tag{3.11}$$

Also we introduce the multifunction $G : W_0^{1,p}(0,b) \to 2^{L^1(T)}$ defined by

$$G(x) = \{u \in L^1(T) : u(t) \in \partial j(t, \tau(x)(t)) \text{ a.e. on } T\}.$$

In our discussion earlier, we saw that the multifunction $(t,x) \to \partial j(t,x)$ is graph measurable, i.e. $\mathrm{Gr}\partial j \in B(T) \times B(\mathbb{R}) \times B(\mathbb{R})$. The map $\xi : T \times \mathbb{R} \to T \times \mathbb{R} \times \mathbb{R}$ defined by $\xi(t,u) = (t, \tau(x)(t), u)$ is clearly measurable. Therefore $\xi^{-1}(\mathrm{Gr}\partial j) = \mathrm{Gr}\partial j(\cdot, \tau(x)(\cdot)) \in \mathcal{L}_T \times B(\mathbb{R})$, with $\mathcal{L}_T$ being the Lebesgue σ-field of T. Invoking the Yankov-von Neumann-Aumann selection theorem (see Denkowski-Migorski-Papageorgiou [6], p.432) and using hypothesis $H(j)_1(iii)$, we infer that for all $x \in W_0^{1,p}(0,b)$ we have $G(x) \neq \emptyset$. Moreover, it is clear that $G(x)$ is closed convex and by the Dunford-Pettis theorem it is also w-compact in $L^1(T)$. For any $x \in W_0^{1,p}(0,b)$ and any $u \in G(x)$, we have

$$\begin{aligned} \langle u, x\rangle &= \int_0^b u(t)x(t)dt \le \|u\|_1\|x\|_\infty \\ &\le c_4\|x\|_\infty \text{ for some } c_4 > 0 \text{ (see hypothesis } H(j)_1(iii)) \\ &\le c_5(\varepsilon) + \varepsilon\|x'\|_p^p \text{ for any } \varepsilon > 0 \text{ and some } c_5(\varepsilon) > 0. \end{aligned} \tag{3.12}$$

In the last inequality, we have used Young's inequality with $\varepsilon > 0$ and the fact that $W_0^{1,p}(0,b)$ is embedded continuously (in fact compactly) into $C(T)$.

We consider the following auxiliary boundary value problem

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' + \sigma(t, x(t)) \in \partial j(t, \tau(x)(t)) \text{ a.e on } T \\ x(0) = x(b) = 0. \end{array} \right\} \tag{3.13}$$

This boundary value problem can be equivalently rewritten as the following operator inclusion:

$$U(x) = A(x) + N_\sigma(x) - G(x) \ni 0.$$

We remark that $U : W_0^{1,p}(0,b) \to 2^{W^{-1,q}(0,b)} \setminus \{\emptyset\}$ and has w-compact and convex values (recall that $L^1(T)$ is embedded continuously in $W^{-1,q}(0,b)$). Note that $\langle A(x), x\rangle = \|x'\|_p^p$. Using this together with (3.11) and (3.12), we obtain

$$\langle U(x), x\rangle \ge \|x'\|_p^p + c_2\|x\|_p^p - \varepsilon\|x'\|_p^p - c_6(\varepsilon) \text{ with } c_6(\varepsilon) = c_5(\varepsilon) + c_3 > 0.$$

Let $\varepsilon = \frac{1}{2}$. We have

$$\langle U(x), x\rangle \ge \frac{1}{2}\|x'\|_p^p - c_7 \text{ for some } c_7 > 0.$$

Thus U is coercive. We claim that it is also pseudomonotone. By virtue of Proposition 2.2(c) it suffices to show that U is generalized pseudomonotone. Suppose that $x_n^* \in U(x_n)$ $n \ge 1$, $x_n \xrightarrow{w} x$ in $W_0^{1,p}(0,b)$, $x_n^* \xrightarrow{w} x^*$ in $W^{-1,q}(0,b)$ and $\limsup_{n\to\infty}\langle x_n^*, x_n - x\rangle \le 0$. We need to show that $x^* \in U(x)$ and $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$. Since $W_0^{1,p}(0,b)$ is embedded compactly in $C(T)$, we have that $x_n \to x$ in $C(T)$. For all $n \ge 1$ we have $x_n^* = A(x_n) + N_\sigma(x_n) - u_n$ with $u_n \in G(x_n)$. Then

$$\langle N_\sigma(x_n), x_n\rangle = \int_0^b \sigma(t, x_n(t))(x_n - x)(t)dt \to 0$$

and $\langle u_n, x_n - x\rangle = \displaystyle\int_0^b u_n(t)(x_n - x)(t)dt \to 0$ (see hypothesis $H(j)_1(iii)$).

It follows that

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0. \tag{3.14}$$

Because A is maximal monotone it is generalized pseudomonotone and so from (3.14) we infer that

$$A(x_n) \xrightarrow{w} A(x) \text{ in } W^{-1,q}(0,b) \text{ and } \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle. \tag{3.15}$$

Hypothesis $H(j)_1(iii)$ implies that $\{u_n\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable and so by the Dunford-Pettis theorem, we may assume that $u_n \xrightarrow{w} u$ in $L^1(T)$. Also note that by passing to a subsequence if necessary, we can say that $\tau(x_n) \to \tau(x)$ in $L^p(T)$ and $\tau(x_n)(t) \to \tau(x)(t)$ for all $t \in T$.

The subdifferential multifunction $z \to \partial j(t,z)$ has closed graph and so using Proposition 4.7.44, p.484, of Denkowski-Migorski-Papageorgiou [6], we obtain

$$\begin{aligned} u(t) \in \overline{conv}\limsup_{n\to\infty}\partial j(x, \tau(x_n)(t)) \subseteq \partial j(t, \tau(x)(t)) \text{ a.e. on } T,\\ \Rightarrow u \in G(x). \end{aligned}$$

Therefore in the limit as $n \to \infty$, we have

$$x^* = A(x) + N_\sigma(x) - u \text{ with } u \in G(x), \text{ i.e. } x^* \in U(x).$$

Moreover, because of (3.15), we have

$$\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle \text{ as } n \to \infty.$$

This proves the pseudomonotonicity of U. Because U is also coercive, it is surjective and this implies that the auxiliary problem (3.13) has a solution $x \in W^{1,p}(0,b)$.

On (3.13) we act with the test function $(\underline{x} - x)^+ \in W_0^{1,p}(0,b)$. After integration by parts, we have

$$\begin{aligned} &-\int_0^b |x'(t)|^{p-2}x'(t)[(\underline{x}-x)^+]'(t)dt - \int_0^b \sigma(t,x(t))(\underline{x}-x)^+(t)dt\\ &= -\int_0^b u(t)(\underline{x}-x)^+(t)dt,\ u \in G(x). \end{aligned} \tag{3.16}$$

Since $\underline{x} \in W_0^{1,p}(0,b)$ is a lower solution for problem (1.1), we have

$$\int_0^b |\underline{x}'(t)|^{p-2}x'(t)[(\underline{x}-x)^+]'(t)dt \leq \int_0^b \underline{\alpha}(t,\underline{x}(t))(\underline{x}-x)^+(t)dt. \tag{3.17}$$

Adding (3.16) and (3.17), we obtain

$$\begin{aligned} &\int_0^b (|\underline{x}'(t)|^{p-2}\underline{x}'(t) - |x'(t)|^{p-2}x'(t))[(\underline{x}-x)^+]'(t)dt\\ &\qquad - \int_0^b \sigma(t,x(t))(\underline{x}-x)^+(t)dt\\ &\qquad\qquad \leq \int_0^b (\underline{h}(t,\underline{x}(t)) - u(t))(\underline{x}-x)^+(t)dt. \end{aligned} \tag{3.18}$$

Note that

$$\begin{aligned}(3.19)\quad &\int_0^b (|\underline{x}'(t)|^{p-2}\underline{x}'(t) - |x'(t)|^{p-2}x'(t))[(\underline{x}-x)^+]'(t)dt \\ &= \int_{\{\underline{x}\geq x\}} (|\underline{x}'(t)|^{p-2}\underline{x}'(t) - |x'(t)|^{p-2}x'(t))(\underline{x}-x)'(t)dt \geq 0.\end{aligned}$$

From the definition of the penalty function $\sigma(t,x)$, we have

$$\begin{aligned}(3.20)\quad -\int_0^b \sigma(t,x(t))(\underline{x}-x)^+(t)dt &= -\int_{\{\underline{x}\geq x\}} \sigma(t,x(t))(\underline{x}-x)(t)dt \\ &= \int_{\{\underline{x}\geq x\}} (\underline{x}-x)(t)^{p-1}(\underline{x}-x)(t)dt = \|(\underline{x}-x)^+\|_p^p.\end{aligned}$$

Finally from the definition of $\underline{\alpha}(t,x)$, we have

$$(3.21)\qquad \int_0^b (\underline{\alpha}(t,\underline{x}(t)) - u(t))(\underline{x}-x)^+(t)dt \leq 0.$$

Using (3.19), (3.20) and (3.21) in (3.18), we obtain

$$\begin{aligned}&\|\underline{x}-x)^+\|_p = 0, \\ \Rightarrow &\underline{x}-x = 0, \text{ i.e. } \underline{x} \leq x.\end{aligned}$$

In a similar fashion we show that $x \leq \overline{x}$. Therefore $\sigma(t,x(t)) = 0$ for all $t \in T$ and $\tau(x) = x$. Hence problem (3.13) becomes

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) \text{ a.e on } T \\ x(0) = x(b) = 0. \end{array}\right\}$$

Therefore $x \in C_0^1(T)$, it solves problem (1.1) and $x(t) > 0$ for all $t \in (0,b)$. $\square$

For the semilinear case, we can have a second existence theorem. Now the asymptotic condition at zero is in terms of the potential function $j(t,x)$ rather than in terms of $\partial j(t,x)$ (see hypothesis $H(j)_2(v)$ below). The new condition is in general less restrictive than the previous one. On the other hand asymptotically at $+\infty$, we impose a bound from below of the ratio $\dfrac{\partial j(t,x)}{x^{p-1}}$, which we did not need before. Moreover, now we produce a nontrivial nonnegative solution, but we can not say that it is positive on $(0,b)$. In order that it is positive, we need to impose an extra unilateral growth condition of $\partial j(t,x)$ (see $H(j)_3(vi)$ below).

More precisely the new hypotheses on the nonsmooth potential $j(t,x)$, are the following:

$\underline{H(j)_2}$**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot,0) \in L^1(T)$, $\int_0^b j(t,0)dt \leq 0$ and

- (i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;
- (ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;
- (iii) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \leq r$ and all $u \in \partial j(t,x)$, we have $|u| \leq \alpha_r(t)$;

(iv) there exists $\theta_1, \theta_2 \in L^\infty(T)$ such that $0 \leq \theta_2(t) \leq \lambda_1$ a.e. on T with the second inequality strict on a set of positive measure and

$$\theta_1(t) \leq \liminf_{x\to+\infty} \frac{u}{x^{p-1}} \leq \limsup_{x\to+\infty} \frac{u}{x^{p-1}} \leq \theta_2(t)$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$;

(v) $\liminf_{x\to 0^+} \frac{pj(t,x)}{x^p} > \lambda_1$ uniformly for almost all $t \in T$.

Remark 3.5. Consider the following nonsmooth locally Lipschitz potential function. Again for simplicity we drop the t-dependence.

$$j(x) = \begin{cases} x^2 \ln|x| + 1 & \text{if } x < 0 \\ \cos(2\pi x^p) & \text{if } x \in [0,1], \\ \frac{\mu}{p}x^p + \frac{p-\mu}{p} & \text{if } x > 1 \end{cases} \qquad \text{with } 0 < \mu < \lambda_1.$$

Note that in this case hypothesis $H(j)_2(v)$ is satisfied but not hypothesis $H(j)_1(v)$.

Now our approach will be variational. For this purpose we introduce the Lipschitz continuous truncation function $\tau_1 : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\tau_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0. \end{cases}$$

We set $j_1(t,x) = j(t, \tau_1(x))$. Evidently for almost all $t \in T$, $j_1(t,\cdot)$ is locally Lipschitz and we have

$$\partial j_1(t,x) \subseteq \begin{cases} 0 & \text{if } x < 0 \\ \operatorname{conv} \bigcup_{\theta\in[0,1]} \partial j(t,0) & \text{if } x = 0 \\ \partial j(t,x) & \text{if } x > 0 \end{cases} \tag{3.22}$$

(see Denkowski-Migorski-Papageorgiou [6], p.611). Let $\varphi_1 : W_0^{1,p}(0,b) \to \mathbb{R}$ be the energy functional defined by

$$\varphi_1(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j_1(t,x(t))dt, \quad x \in W_0^{1,p}(0,b).$$

We know that φ_1 is locally Lipschitz.

Proposition 3.6. *If hypotheses $H(j)_2$ hold, then φ_1 satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ be a sequence such that

$$|\varphi_1(x_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty.$$

Because $\partial\varphi(x_n) \subseteq W^{-1,q}$ is w-compact and the norm functional in a Banach space is weakly lower semicontinuous, by virtue of the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We have

$$x_n^* = A(x_n) - u_n \text{ with } u_n \in L^1(T), \ u_n(t) \in \partial j_1(t, x_n(t)) \text{ a.e. on } T, \ n \geq 1,$$

(see Denkowski-Migorski-Papageorgiou [6], p.617). We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ is bounded. Suppose that this is not true. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to \infty$ as $n \to \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(0,b) \text{ and } y_n \to y \text{ in } C(T).$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$, we have

$$\begin{aligned} &|\langle A(x_n), x_n^-\rangle - \int_0^b u_n(t)x_n^-(t)dt| \leq \varepsilon_n\|x_n^-\| \text{ with } \varepsilon_n \downarrow 0, \\ \Rightarrow &\|(x_n^-)'\|_p^p \leq \varepsilon_n\|(x_n^-)'\|_p \text{ (see (3.22))}, \\ \Rightarrow &\{x_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(0,b) \text{ is bounded.} \end{aligned}$$

Therefore $y_n^- = \frac{x_n^-}{\|x_n\|} \to 0$ in $W_0^{1,p}(0,b)$ and from this we infer that $y \geq 0$. Then we have $x_n(t) \to +\infty$ as $n \to \infty$ for all $t \in \{y > 0\}$. Moreover, by virtue of hypotheses $H(j)_2(iii), (iv)$, for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$, we have

$$|u| \leq \widehat{\alpha}(t) + \widehat{c}|x|^{p-1} \text{ with } \widehat{\alpha} \in L^1(T)_+,\ \widehat{c} > 0. \tag{3.23}$$

Using as a test function $y_n^+ \in W_0^{1,p}(0,b)$, we have

$$|\langle A(x_n), y_n^+\rangle - \int_0^b u_n(t)y_n^+(t)dt| \leq \varepsilon_n\|y_n^+\| \text{ with } \varepsilon_n \downarrow 0.$$

Dividing with $\|x_n\|^{p-1}$, we obtain

$$\left| \|(y_n^+)'\|_p^p - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} y_n^+(t)dt \right| \leq \varepsilon_n' \text{ with } \varepsilon_n' \downarrow 0. \tag{3.24}$$

Because of (3.23), we have

$$\frac{|u_n(t)|}{\|x_n\|^{p-1}} \leq \frac{\widehat{\alpha}(t)}{\|x_n\|^{p-1}} + \widehat{c}y_n^+(t)^{p-1} \text{ a.e. on } T, \tag{3.25}$$

$$\Rightarrow \left\{ \frac{u_n}{\|x_n\|^{p-1}} \right\}_{n\geq 1} \subseteq L^1(T) \text{ is uniformly integrable.}$$

So by the Dunford-Pettis theorem, there exists $h \in L^1(T)$ such that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^1(T) \text{ as } n \to \infty.$$

For $\varepsilon > 0$ and $n \geq 1$, we consider the set

$$C_{\varepsilon,n} = \{t \in T : x_n(t) > 0, \frac{u_n(t)}{x_n(t)^{p-1}} \leq \theta_2(t) + \varepsilon\}$$

and set $\chi_{\varepsilon,n}(t) = \chi_{C_{\varepsilon,n}}(t)$. Because of hypothesis $H(j)_2(iv)$, we have that

$$\chi_{\varepsilon,n}(t) \to 1 \text{ a.e. on } \{y > 0\}. \tag{3.26}$$

Note that

$$\int_{\{y>0\}} (1 - \chi_{\varepsilon,n}(t)) \frac{|u_n(t)|}{\|x_n\|^{p-1}} dt \to 0 \text{ (see (3.25) and (3.26))},$$

$$\Rightarrow \chi_{\varepsilon,n}\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \ \text{ in } \ L^1(\{y>0\}).$$

Then we have

$$\chi_{\varepsilon,n}(t)\frac{u_n(t)}{\|x_n\|^{p-1}} = \chi_{\varepsilon,n}(t)\frac{u_n(t)}{x_n(t)^{p-1}}y_n(t)^{p-1} \le \chi_{\varepsilon,n}(t)(\theta_2(t)+\varepsilon)y_n(t)^{p-1}.$$

Taking weak limits in $L^1(\{y>0\})$, we obtain

$$h(t) \le (\theta_2(t)+\varepsilon)y(t)^{p-1} \ \text{ a.e. on } \ \{y>0\}.$$

Because $\varepsilon > 0$ was arbitrary, it follows that

$$h(t) \le \theta_2(t)y(t)^{p-1} \ \text{ a.e. on } \ \{y>0\}.$$

Moreover, from (3.25) it is clear that

$$h(t) = 0 \ \text{ a.e. on } \ \{y=0\}.$$

Since $T = \{y>0\} \cup \{y=0\}$, we have

$$h(t) \le \theta_2(t)y(t)^{p-1} \ \text{ a.e. on } T.$$

If we pass to the limit as $n \to \infty$ in (3.24) and since $y \ge 0$, we obtain

$$\|y'\|_p^p \le \int_0^b h(t)y(t)dt \le \int_0^b \theta_2(t)y(t)^p dt \le \lambda_1\|y\|_p^p, \tag{3.27}$$

$$\Rightarrow y = 0 \ \text{ or } \ y = u_1 \ \text{ (see (2.2))}.$$

If $y = 0$, then $y_n \to 0$ in $W_0^{1,p}(0,b)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$.

Therefore $y = u_1$ and so $y(t) > 0$ for all $t \in (0,b)$. From (3.27) we have

$$\|y'\|_p^p < \lambda_1\|y\|_p^p,$$

a contradiction to (2.2). This proves that $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ is bounded and so we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W_0^{1,b}(0,b) \text{ and } \ x_n \to x \ \text{ in } \ C(T).$$

From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$, we have

$$|\langle A(x_n), x_n - x\rangle - \int_0^b u_n(t)(x_n-x)(t)dt| \le \varepsilon_n\|x_n\|.$$

Note that $\int_0^b u_n(t)(x_n-x)(t)dt \to 0$, so it follows that

$$\lim\langle A(x_n), x_n - x\rangle = 0.$$

But A being maximal monotone, it is generalized pseudomonotone (see Proposition 2.2(a)) and so

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle,$$

$$\Rightarrow \|x_n'\|_p \to \|x'\|_p.$$

Because $x_n' \xrightarrow{w} x'$ in $L^p(T)$ and $L^p(T)$ is uniformly convex, from the Kadec-Klee property we have $x_n' \to x'$ in $L^p(T)$ and so $x_n \to x$ in $W_0^{1,p}(0,b)$. Therefore φ satisfies the nonsmooth PS-condition. $\square$

Now using the nonsmooth least action principle, we can show that problem (1.1) has a nontrivial nonnegative solution.

Theorem 3.7. *If hypotheses $H(j)_2$ hold, then problem (1.1) has a solution $x \in C_0^1(T)$ such that $x \neq 0$ and $x(t) \geq 0$ for all $t \in T$.*

Proof. By virtue of hypothesis $H(j)_2(iv)$, given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that for almost all $t \in T$, all $x \geq M_2$ and all $u \in \partial j(t,x)$, we have

$$u \leq (\theta_2(t) + \varepsilon)x^{p-1}.$$

On the other hand hypothesis $H(j)_2(iii)$ implies that for almost all $t \in T$ and all $0 \leq x \leq M_2$, we have

$$|u| \leq \alpha_\varepsilon(t) \ \text{ with } \ \alpha_\varepsilon \in L^1(T)_+.$$

So we can say that for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$

$$u \leq (\theta_2(t) + \varepsilon)x^{p-1} + \alpha_\varepsilon(t). \tag{3.28}$$

Since for almost all $t \in T$, $j(t,\cdot)$ is locally Lipschitz, it is differentiable at all $x \in \mathbb{R}\backslash D(t)$ with $|D(t)|_1 = 0$ (by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$) and $\frac{d}{dr}j(t,r) \in \partial j(t,r)$. Hence for almost all $t \in T$ and all $x \geq 0$, we have

$$\begin{aligned} j(t,x) &= j(t,0) + \int_0^x \frac{d}{dr}j(t,x)dr \\ &\leq j(t,0) + \int_0^b (\theta_2(t) + \varepsilon)r^{p-1}dr + \alpha_\varepsilon(t)x \ \ \text{(see (3.28))} \\ &= j(t,0) + \frac{1}{p}\theta_2(t)x^p + \frac{\varepsilon}{p}x^p + \alpha_\varepsilon(t)x. \end{aligned} \tag{3.29}$$

Using (3.29) for all $x \in W_0^{1,p}(0,b)$ we have

$$\begin{aligned} \varphi_1(x) \geq &\frac{1}{p}\|x'\|_p^p - \frac{1}{p}\int_0^b \theta_2(t)|x(t)|^p dt - \frac{\varepsilon}{p}\|x\|_p^p - c_7\|x'\|_p \ \text{ for some } \ c_7 > 0 \\ &\left(\text{recall } \int_0^b j(t,0)dt \leq 0\right) \\ \geq &\frac{\xi_1}{p}\|x'\|_p^p - \frac{\varepsilon}{p\lambda_1}\|x'\|_p^p - c_7\|x'\|_p \ \text{ (see (2.2) and Lemma 3.2).} \end{aligned}$$

So by choosing $\varepsilon < \xi_1\lambda_1$, we infer that φ_1 is coercive, thus it is bounded below. This combined with Proposition 3.6 implies that there exists $x \in W_0^{1,p}(0,b)$ such that

$$\begin{aligned} &\varphi_1(x) = \inf_{W_0^{1,p}(0,b)} \varphi_1, \\ \Rightarrow &0 \in \partial\varphi_1(x), \\ \Rightarrow &A(x) = u \ \text{ with } \ u \in L^1(T), \ u(t) \in \partial j_1(t,x(t)) \text{ a.e. on } T. \end{aligned}$$

From this equality it follows that $x \in C_0^1(T)$ and it solves problem (1.1). Using as a test function $-x^- \in W_0^{1,p}(0,b)$, we obtain

$$\begin{aligned}
&\|(x^-)'\|_p^p = -\int_0^b u(t)x^-(t)dt = 0 \;\; \text{(see (3.22))} \\
\Rightarrow &x^- = 0, \;\; \text{i.e. } \; x \geq 0.
\end{aligned}$$

Because of hypothesis $H(j)_2(v)$, we can find $\delta > 0$ and $\mu = \mu(\delta) > \lambda_1$ such that for almost all $t \in T$ and all $x \in (0, \delta]$, we have

$$j(t,x) \geq \frac{\mu}{p} x^p. \tag{3.30}$$

Since $u_1 \in C_0^1(T)$, we can find $\eta > 0$ small enough so that $\eta u_1(t) \in (0, \delta]$ for all $t \in (0, b)$. Then

$$\begin{aligned}
\varphi_1(nu_1) &= \frac{\eta^p}{p}\|u_1'\|_p^p - \int_0^b j(t, \eta u_1(t))dt \\
&\leq \frac{\eta^p}{p}\|u_1'\|_p^p - \frac{\mu\eta^p}{p}\|u_1\|_p^p \\
&= \frac{\eta^p}{p}(1 - \frac{\mu}{\lambda_1})\|u_1'\|_p^p \;\; \text{(see (2.2))}, \\
\Rightarrow \varphi_1(\eta u_1) &< 0 \;\; \text{since} \;\; \mu > \lambda_1, \\
\Rightarrow \inf_{W_0^{1,p}(0,b)} \varphi_1 &= \varphi_1(x) < 0 \leq \varphi(0) \;\; \text{(recall that } \textstyle\int_0^b j(t,0)dt \leq 0), \\
\Rightarrow x &\neq 0, \; x \geq 0.
\end{aligned}$$

$\square$

To guarantee that the solution is positive, we need to impose an extra unilateral condition on $\partial j(t, \cdot)$. More precisely we assume:

<u>$H(j)_3$</u>**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$, $\int_0^b j(t,0)dt \leq 0$ and $(i), (ii), (iii), (iv)$, (v) are the same as hypotheses $H(j)_2(i), (ii), (iii), (iv), (v)$ respectively and

(vi) for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$,
$$-\widehat{c}x^{p-1} \leq u \;\; \text{with} \;\; \widehat{c} > 0.$$

Remark 3.8. The nonsmooth locally Lipschitz $j(x)$ given after hypotheses $H(j)_2$ satisfies also hypotheses $H(j)_3$.

Theorem 3.9. *If hypotheses $H(j)_3$ hold, then problem (1.1) has a solution $x \in C_0^1(T)$ such that $x(t) > 0$ for all $t \in (0, b)$ and $x'(b) < 0 < x'(0)$.*

Proof. Let $x \in C_0^1(T)$ be the nontrivial nonnegative solution of problem (1.1) obtained in Theorem 3.7. We have

$$-(|x'(t)|^{p-2}x'(t))' = u(t) \;\; \text{a.e. on } T \text{ with } \; u \in L^1(T), \; u(t) \in \partial j_1(t, x(t)) \;\; \text{a.e. on } T.$$

Since $x'(t) = 0$ a.e. on $\{x = 0\}$ by Stampacchia's Theorem, (see Denkowski-Migorski-Papageorgiou [6], p.349), using (3.22) and hypothesis $H(j)_3(vi)$, we have

$$(|x'(t)|^{p-2}x'(t))' \leq \widehat{c}x(t)^{p-1} \;\; \text{a.e. on } T.$$

The strict maximum principle of Vazquez [15] implies that

$$x(t) > 0 \text{ for all } t \in (0,b),\ x'(b) < 0 < x'(0). \qquad \square$$

Remark 3.10. Theorem 3.9 implies that $x \in \mathrm{int} C_0^1(T)_+$.

4. Superlinear Problems

In this section we deal with superlinear problems. Namely now the ratio $\frac{\partial j(t,x)}{x^{p-1}}$ stays above $\lambda_1 > 0$ near $+\infty$ and below $\lambda_1 > 0$ near 0^+.

The first result actually concerns the "linear" case, i.e. $\partial j(t,\cdot)$ has at most $(p-1)$-polynomial growth (see hypothesis $H(j)_4(iii)$), hence if $p = 2$, it exhibits linear growth, which justifies the nomenclature "linear". However, note that as x moves from 0 to $+\infty$, the ratio $\frac{u}{x^{p-1}}$ can cross the first eigenvalue $\lambda_1 > 0$. As before at $+\infty$ we allow nonuniform nonresonance.

The hypotheses on the nonsmooth potential $j(t,x)$ are the following:

$\underline{H(j)_4}$**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$|u| \le \alpha(t) + c|x|^{p-1} \text{ with } \alpha \in L^\infty(T)_+,\ c > 0;$$

(iv) there exists $\theta \in L^\infty(T)_+$ such that $\theta(t) \ge \lambda_1$ a.e. on T with strict inequality on a set of positive measure and

$$\liminf_{x\to+\infty} \frac{u}{x^{p-1}} \ge \theta(t)$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$;

(v) $\limsup\limits_{x\to 0^+} \dfrac{pj(t,x)}{x^p} < \lambda_1$ uniformly for almost all $t \in T$.

Remark 4.1. The following nonsmooth locally Lipschitz function satisfies hypotheses $H(j)_4$.

$$j(x) = \begin{cases} xe^x & \text{if } x < 0 \\ \frac{\mu}{p}x^p e^{-x^p} & \text{if } x \in [0,1] \quad \text{with } \mu < \lambda_1 < \eta - 1. \\ \frac{\eta}{p}x^p + \frac{1}{p}\sin(\frac{\pi}{2}x^p) + \frac{\mu-(\eta+1)}{p} & \text{if } x > 1 \end{cases}$$

As before $j_1(t,x) = j(t,\tau_1(x))$ and we consider the locally Lipschitz energy functional $\varphi_1 : W_0^{1,p}(0,b) \to \mathbb{R}$ defined by

$$\varphi_1(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j_1(t,x(t))dt,\ x \in W_0^{1,p}(0,b).$$

Proposition 4.2. *If hypotheses $H(j)_4$ hold, then φ_1 satisfies the PS-condition.*

Proof. Let $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ be a sequence such that

$$|\varphi_1(x_n)| \le M_3 \text{ for some } M_3 > 0, \text{ all } n \ge 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty.$$

As before we can find $x_n^* \in \partial\varphi(x_n)$ such that $\|x_n^*\| = m(x_n),\ n \geq 1$. We have

$$x_n^* = A(x_n) - u_n \ \text{ with } \ u_n \in L^1(T),\ u_n(t) \in \partial j_1(t, x_n(t)) \text{ a.e. on } T, n \geq 1.$$

We use as a test function $x_n^- \in W_0^{1,p}(0,b)$ and obtain

$$\begin{aligned} &|\langle A(x_n), x_n^-\rangle - \int_0^b u_n(t)x_n^-(t)dt| \leq \varepsilon_n \|x_n^-\| \ \text{ with } \ \varepsilon_n \downarrow 0, \\ \Rightarrow &\|(x_n^-)'\|_p^p \leq \varepsilon_n \|(x_n^-)'\|_p \ \text{ (see (3.22))}, \\ \Rightarrow &\{x_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(0,b) \ \text{ is bounded.} \end{aligned}$$

Suppose that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ is not bounded. We may assume that $\|x_n\| \to \infty$. We set $y_n = \dfrac{x_n}{\|x_n\|}$, $n \geq 1$. We can say that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(0,b)$ and $y_n \to y$ in $C(T)$. Since $\{x_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$ is bounded, we have that $y \geq 0$. Then for all $t \in \{y > 0\}$ we have $x_n(t) \to +\infty$ as $n \to \infty$. Because of hypothesis $H(j)_4(iii)$ we have that $\left\{\dfrac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^1(T)$ is uniformly integrable and so we can say that $\dfrac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h$ in $L^1(T)$. As in the proof of Proposition 3.6, we can show that

$$\begin{aligned} h(t) &\geq \theta(t)y(t)^{p-1} \ \text{ a.e. on } \{y > 0\} \\ \text{and } \ h(t) &= 0 \text{ a.e on } \ \{y = 0\}. \end{aligned}$$

Therefore we have

$$h(t) \geq \theta(t)y(t)^{p-1} \ \text{ a.e. on } T \tag{4.1}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$, we have that

$$\begin{aligned} &\left|\langle A(y_n), y_n - y\rangle - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}}(y_n - y)(t)dt\right| \leq \varepsilon_n \frac{\|y_n - y\|}{\|x_n\|^{p-1}}, \\ \Rightarrow &\langle A(y_n), y_n - y\rangle \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

From this convergence, as before, via the Kadec-Klee property, we conclude that

$$y_n \to y \text{ in } W_0^{1,p}(0,b), \ \text{ i.e. } \ y \neq 0 \ \text{ (since } \|y_n\| = 1 \text{ for all } n \geq 1).$$

Then $A(y_n) \xrightarrow{w} A(y)$ in $W^{-1,q}(0,b)$ and for all $v \in W_0^{1,p}(0,b)$, we have

$$\begin{aligned} &|\langle A(y_n), v\rangle - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} v(t)dt| \leq \ \varepsilon_n \|v\|, \\ \Rightarrow &\langle A(y), v\rangle = \int_0^b h(t)v(t)dt \ \text{ for all } \ v \in W_0^{1,p}(0,b), \end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{l} -(|y'(t)|^{p-2}y'(t))' = h(t) \text{ a.e on } T = [0,b] \\ y(0) = y(b) = 0. \end{array} \right\} \tag{4.2}$$

From (4.1), (4.2) and Proposition 4.1 of Godoy-Gossez-Paczca [10], we have that $y(\cdot)$ changes sign, a contradiction. This proves the boundedness of $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b)$, from which we infer that φ_1 satisfies the nonsmooth PS-condition (see the proof of Proposition 3.6). $\square$

Proposition 4.3. *If hypotheses $H(j)_4$ hold, then $\varphi_1(\eta u_1) \to -\infty$ as $n \to +\infty$.*

Proof. From hypotheses $H(j)_4(iii)$ and (iv), we know that given $\varepsilon > 0$ we can find $\alpha_\varepsilon \in L^1(T)$ such that for almost all $t \in T$ all $x \geq 0$ and all $u \in \partial j(t,x)$, we have

$$u \geq (\theta(t) - \varepsilon)x^{p-1} - \alpha_\varepsilon(t). \tag{4.3}$$

Recall that for almost all $t \in T$, $j(t,\cdot)$ is differentiable at all $x \in \mathbb{R}\backslash D(t)$, $|D(t)|_1 = 0$ and we have $\frac{d}{dr}j(t,r) \in \partial j(t,r)$. So using (4.3), we see that for almost all $t \in T$ and all $x \geq 0$, we have

$$j(t,x) = j(t,0) + \int_0^x \frac{d}{dr}j(t,r)dr \geq \frac{\theta(t)}{p}x^p - \frac{\varepsilon}{p}x^p - \alpha_\varepsilon(t)x. \tag{4.4}$$

Thus if $\eta > 0$, we have

$$\begin{aligned} \varphi_1(\eta u_1) &= \frac{\eta^p}{p}\|u_1'\|_p^p - \int_0^b j(t,\eta u_1(t))dt \ \ (\text{since } \eta u_1(t) \geq 0 \text{ for all } t \in T) \\ &\leq \frac{\eta^p}{p}\|u_1'\|_p^p - \frac{\eta^p\lambda_1}{p}\|u_1\|_p^p - \frac{\eta^p}{p}\int_0^b (\theta(t)-\lambda_1)u_1(t)^p dt + \frac{\varepsilon\eta^p}{p}\|u_1\|_p^p \\ &\quad + \eta c_\varepsilon\|u_1\|_p \ \text{ with } c_\varepsilon > 0 \text{ (see (4.4))} \\ &= -\frac{\eta^p}{p}\int_0^b (\theta(t)-\lambda_1)u_1(t)^p dt + \frac{\varepsilon\eta^p}{p}\|u_1\|_p^p + \eta c_\varepsilon\|u_1\|_p. \end{aligned} \tag{4.5}$$

Let $\zeta = \frac{1}{p}\int_0^b(\theta(t) - \lambda_1)u_1(t)^p dt$. Because of hypothesis $H(j)_4(iv)$, we have that $\zeta > 0$ and so if we choose $\varepsilon \in (0,\zeta)$ (recall $\|u_1\|_p = 1$), from (4.5) we conclude that

$$\varphi(\eta u_1) \to -\infty \ \text{ as } \ n \to +\infty. \qquad \square$$

The next proposition will allow us to satisfy the Mountain Pass geometry and so eventually apply Theorem 2.1.

Proposition 4.4. *If hypotheses $H(j)_4$ hold, then $\varphi_1(x) \geq \beta_1\|x\|^p - \beta_2\|x\|^\sigma$ for some $\beta_1,\ \beta_2 > 0$, all $x \in W_0^{1,p}(0,b)$ and with $\sigma > p$.*

Proof. By virtue of hypothesis $H(j)_4(v)$, we can find $\mu < \lambda_1$ and $\delta = \delta(\mu) > 0$ such that for almost all $t \in T$ and all $x \in (0,\delta]$, we have

$$j(t,x) \leq \frac{\mu}{p}x^p.$$

On the other hand by virtue of hypothesis $H(j)_4(iii)$ and the mean value theorem for locally Lipschitz functions (see Denkowski-Migorski-Papageorgiou [6], p.609), for almost all $t \in T$ and all $x > \delta$, we have

$$j(t,x) \leq \bar{c}x^\sigma \ \text{ for some } \ \bar{c} > 0 \ \text{ and with } \ \sigma > p$$

(remark that in hypothesis $H(j)_4(iii)$ $\alpha \in L^\infty(T)_+$). So finally we can say that for almost all $t \in T$ and all $x \geq 0$, we have

$$j(t,x) \leq \frac{\mu}{p}x^p + \bar{c}x^\sigma. \tag{4.6}$$

Using (4.6), for all $x \in W_0^{1,p}(0,b)$ we have

$$\begin{aligned}\varphi_1(x) &= \frac{1}{p}\|x'\|_p^p - \int_0^b j_1(t,x(t))dt \\ &\geq \frac{1}{p}\|x'\|_p^p - \frac{\mu}{p\lambda_1}\|x'\|_p^p - \eta\|x\|_\sigma^\sigma \text{ for some } \eta > 0 \text{ (see (4.6) and (2.2))} \\ &\geq \beta_1\|x\|^p - \beta_2\|x\|^\sigma \text{ for some } \beta_1, \beta_2 > 0 \text{ (recall that } \mu < \lambda_1). \qquad \square\end{aligned}$$

Now we have all the necessary geometry to apply Theorem 2.1 (the nonsmooth Mountain Pass Theorem).

Theorem 4.5. *If hypotheses $H(j)_4$ hold, then problem (1.1) has a solution $x \in C_0^1(T)$ such that $x \neq 0$ and $x(t) \geq 0$ for all $t \in T$.*

Proof. By virtue of Proposition 4.4, we can find $r > 0$ small such that if $\|x\| = r$, then $\varphi_1(x) > 0$. On the other hand for $\eta > 0$ large we have $\varphi_1(\eta u_1) < 0$ (see Proposition 4.3). Since $\varphi_1(0) = 0$ and φ_1 satisfies the nonsmooth PS-condition (see Proposition 4.2), we see that we can apply Theorem 2.1 and obtain $x \in W_0^{1,p}(0,b)$ such that

$$0 \in \partial\varphi_1(x) \text{ and } \varphi_1(0) = 0 < \inf[\varphi_1(y) : \|y\| = r] \leq \varphi_1(x), \text{ i.e. } x \neq 0.$$

From the inclusion $0 \in \partial\varphi_1(x)$, we deduce that $x \in C_0^1(T)$ and it solves (1.1) with j replaced by j_1. Finally as in the proof of Theorem 3.7, we verify that $x(t) \geq 0$ for all $t \in T$. $\square$

As before by imposing an extra unilateral growth condition on $\partial j(t,\cdot)$, we can verify that the solution x is positive. So we assume:

$\underline{H(j)_5}$**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $(i), (ii), (iii), (iv), (v)$ are the same as hypotheses $H(j)_4(i), (ii), (iii), (iv), (v)$ respectively and

(vi) for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$, we have

$$-\widehat{c}x^{p-1} \leq u \text{ with } \widehat{c} > 0.$$

Theorem 4.6. *If hypotheses $H(j)_5$ hold, then (1.1) has a solution $x \in C_0^1(T)$ such that $x(t) > 0$ for all $t \in (0,b)$.*

In the next existence theorem we will take care of the strictly "superlinear" case, which is not covered by the previous analysis. So now our hypotheses on the nonsmooth potential $j(t,x)$ are the following:

$\underline{H(j)_6}$**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T and

(i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$|u| \leq \alpha(t) + c|x|^{r-1} \text{ with } \alpha \in L^\infty(T)_+,\ c > 0,\ 1 \leq r < \infty;$$

(iv) there exists $M > 0$ such that for almost all $t \in T$ and all $x \geq M > 0$ we have

$$0 < \gamma \leq \eta j(t,x) \leq -j^0(t,x;-x) \text{ with } \eta > p;$$

(v) $\limsup_{x\to 0^+} \frac{pj(t,x)}{x^p} < \lambda_1$ uniformly for almost all $t \in T$.

Remark 4.7. Hypothesis $H(j)_6(iv)$ is the well-known Ambrosetti-Rabinowitz condition adapted to the present nonsmooth setting. As in the smooth case we will see that it implies that $j(t,\cdot)$ has an η-polynomial growth at $+\infty$, i.e. if $p = 2$, then $j(t,\cdot)$ is strictly superquadratic and so $\partial j(t,\cdot)$ is strictly sublinear. The following nonsmooth, locally Lipschitz function satisfies hypotheses $H(j)_6$

$$j(x) = \begin{cases} |x| & \text{if } x < 0 \\ \theta \sin x^p & \text{if } x \in [0,1] \\ \frac{1}{\eta}x^\eta + \theta\sin(1) - \frac{1}{\eta} & \text{if } x > 1 \end{cases}.$$

As before we consider the truncation map $\tau_1 : \mathbb{R} \to \mathbb{R}_+$ defined by $\tau_1(x) = x^+$ and we set

$$j_1(t,x) = j(t,\tau_1(x)).$$

Then we consider the locally Lipschitz energy functional $\varphi_1 : W_0^{1,p}(0,b) \to \mathbb{R}$ defined by

$$\varphi_1(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j_1(t,x(t))dt, \quad x \in W_0^{1,p}(0,b).$$

Proposition 4.8. *If hypotheses $H(j)_6$ hold, then φ_1 satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ be a sequence such that

$$|\varphi_1(x_n)| \le M_4 \ \text{ for some } M_4 > 0, \text{ all } n \ge 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty.$$

We can find $x_n^* \in \partial\varphi_1(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. We have

$$x_n^* = A(x_n) - u_n \ \text{ with } \ u_n \in L^1(T),\ u_n(t) \in \partial j_1(t,x_n(t)) \text{ a.e. on } T, n \ge 1.$$

As in the proof of Proposition 4.2, we can check that $\{x_n^-\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$ is bounded. In addition, from the mean value theorem for locally Lipschitz functions, hypothesis $H(j)_6(iii)$ and the fact that $j(t,0) = 0$ a.e. on T, for almost all $t \in T$ and all $x \in \mathbb{R}$, we have

$$|j_1(t,x)| \le \alpha(t) + c|x|^r, \quad \alpha \in L^\infty(T)_+,\ c > 0. \tag{4.7}$$

From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(0,b)$, we have

$$\eta\varphi_1(x_n^+) + \langle x_n^*, -x_n^+\rangle \le \eta M_1 + \varepsilon_n\|x_n^+\| \ \text{ with } \ \varepsilon_n \downarrow 0,$$

$$\Rightarrow \left(\frac{\eta}{p} - 1\right)\|(x_n^+)'\| - \int_0^b (u_n(t)(-x_n^+)(t) + \eta j(t,x_n^+(t)))dt \le \eta M_1 + \varepsilon_n\|x_n^+\|$$

(since $j(t,\cdot)|_{\mathbb{R}_+} = j_1(t,\cdot)$)

$$\Rightarrow \left(\frac{\eta}{p} - 1\right)\|(x_n^+)'\|_p^p + \int_0^b (-j^0(t,x_n^+(t);-x_n^+(t))dt - \eta j(t,x_n^+(t)))dt \le \eta M_1 + \varepsilon_n\|x_n^+\|. \tag{4.8}$$

We estimate the integral in the left hand side of (4.8). Thus we have

(4.9)

$$\begin{aligned}
&\int_0^b (-j^0(t,x_n^+(t);-x_n^+(t)) - \eta j(t,x_n^+(t)))dt \\
&= \int_{\{x_n^+\geq M\}} (-j^0(t,x_n^+(t);-x_n^+(t)) - \eta j(t,x_n^+(t)))dt \\
&+ \int_{\{0\leq x_n^+<M\}} (-j^0(t,x_n^+(t);-x_n^+(t)) - \eta j(t,x_n^+(t)))dt \\
&\geq \int_{\{0\leq x_n^+<M\}} (-j^0(t,x_n^+(t);-x_n^+(t)) - \eta j(t,x_n^+(t)))dt \ \text{ (see hypothesis } H(j)_6(iv)) \\
&\geq -\beta_3 \text{ for some } \beta_3>0 \text{ (see (4.7) and hypothesis } H(j)_6(iii))
\end{aligned}$$

Returning to (4.8) and using (4.9), we obtain

$$\begin{aligned}
&(\frac{n}{p}-1)\|(x_n^+)'\|_p^p \leq \beta_4 + \varepsilon_n\|x_n^+\| \ \text{ for some } \ \beta_4>0, \text{ all } n\geq 1, \\
&\Rightarrow \{x_n^+\}_{n\geq 1} \subseteq W_0^{1,p}(0,b) \quad \text{is bounded (by the Poincaré inequality)}, \\
&\Rightarrow \{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(0,b) \ \text{ is bounded.}
\end{aligned}$$

So we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(0,b)$ and $x_n \to x$ in $C(T)$ as $n\to\infty$. As in the proof of Proposition 3.6, we conclude that $x_n \to x$ in $W_0^{1,p}(0,b)$ and so φ_1 satisfies the nonsmooth PS-condition. □

Using this proposition we can establish the existence of a nontrivial nonnegative solution for problem (1.1).

Theorem 4.9. *If hypotheses $H(j)_6$ hold, then (1.1) has a solution $x\in C_0^1(T)$ such that $x\neq 0$ and $x(t)\geq 0$ for all $t\in T$.*

Proof. On $\mathbb{R}_+\backslash\{0\}$, the function $r \to \frac{1}{r^\eta}$ is continuous convex, thus locally Lipschitz. Then $r\to \frac{1}{r^\eta}j(t,rx)$ is locally Lipschitz for almost all $t\in T$ and all $x\in\mathbb{R}_+\backslash\{0\}$. As a result we have

$$\partial_r\left(\frac{1}{r^\eta}j(t,rx)\right) \subseteq -\frac{\eta}{r^{\eta+1}}j(t,rx) + \frac{1}{r^\eta}\partial_x j(t,rx)x$$

(see Denkowski-Migorski-Papageorgiou [6], p.612). Here by ∂_r(resp. ∂_x) we denote the generalized subdifferential with respect to $r\in\mathbb{R}_+\backslash\{0\}$ (resp. $x\in\mathbb{R}$). Using the mean value theorem for locally Lipschitz functions, we can find $\lambda\in(1,r)$ such that for almost all $t\in T$ and all $x\geq M$, we have

$$\begin{aligned}
&\frac{1}{r^\eta}j(t,rx) - j(t,x) \in \left(-\frac{\eta}{\lambda^{\eta+1}}j(t,\lambda x) + \frac{1}{\lambda^\eta}\partial_x j(t,\lambda x)x\right)(r-1), \\
&\Rightarrow \frac{1}{r^\eta}j(t,rx) - j(t,x) = \frac{r-1}{\lambda^{\eta+1}}\left(-\eta j(t,\lambda x) + \partial_x(t,\lambda x)\lambda x\right)
\end{aligned}$$

$$\geq \frac{r-1}{\lambda^{\eta+1}}\Big(-\eta j(t,\lambda x) - j^0(t,\lambda x;-\lambda x)\Big) \geq 0 \text{ (see hypothesis } H(j)_6(iv)),$$

$$\Rightarrow r^\eta j(t,x) \leq j(t,rx) \text{ for almost all } t \in T, \text{ all } x \geq M \text{ and all } r \geq 1. \tag{4.10}$$

From 4.7 and 4.10 it follows that for almost all $t \in T$ and all $x \geq 0$, we have

$$j_1(t,x) \geq \frac{j(t,M)}{M^\eta} x^\eta - \xi, \quad \text{for some } \xi > 0.$$

Since $j(\cdot, M) \in L^1(T)_+$ and $\eta > p$, for all $\theta > 0$ we have

$$\varphi_1(\theta u_1) \leq \frac{\theta^p \lambda_1}{p} \|u_1\|_p^p - \theta^\eta \beta_5 \|u_1\|_p^\eta + \beta_6$$

$$\text{for some } \beta_5, \beta_6 > 0 \text{ (recall } \|u_1'\|_p^p = \lambda_1 \|u_1\|_p^p)$$

$$\Rightarrow \varphi_1(\theta u_1) \to -\infty \text{ as } \theta \to +\infty.$$

Also from Proposition 4.4, we know that

$$\varphi_1(x) \geq \beta_7 \|x\|^p - \beta_8 \|x\|^\sigma$$

$$\text{for some } \beta_7, \beta_8 > 0, \text{ all } x \in W_0^{1,p}(0,b) \text{ and with } \sigma > p.$$

Therefore we can find $r > 0$ small and $\theta > 0$ large such that

$$\varphi_1(x) \geq \widehat{\beta} > 0 = \varphi(0) > \varphi(\theta u_1) \text{ for all } \|x\| = r.$$

Because of this and Proposition 4.8, we can apply Theorem 2.1 and obtain $x \in W_0^{1,p}(0,b)$ such that

$$0 \in \partial \varphi_1(x) \text{ and } \varphi_1(0) = 0 < \widehat{\beta} \leq \varphi_1(x), \text{ i.e. } x \neq 0.$$

As before we check that $x \in C_0^1(T)$, $x(t) \geq 0$ for all $t \in T$ and it solves problem (1.1). ☐

To have a positive solution, we need the unilateral growth condition on $\partial j(t,\cdot)$.

$\underline{H(j)_7}$**:** $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T and $(i),(ii),(iii),(iv),(v)$ are the same as hypotheses $H(j)_6(i),(ii),(iii),(iv),(v)$ respectively and

(vi) for almost all $t \in T$, all $x \geq 0$ and all $u \in \partial j(t,x)$, we have

$$-\widehat{c} x^{p-1} \leq u \text{ with } \widehat{c} > 0.$$

Theorem 4.10. *If hypotheses $H(j)_7$ hold, then problem (1.1) has a solution $x \in C_0^1(T)$ such that $x(t) > 0$ for all $t \in (0,b)$,*

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