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VOLTERRA OPERATOR: BACK TO THE FUTURE

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Abstract. The notion of *Volterra operator* is central in many considerations regarding differential, integral, and functional-differential equations. Its history traces back to a Volterra's paper of 1913, where he studied an integro-differential equation with the integral operator

$$(Kx)(t) = \int_a^t K(t,s)x(s)ds.$$

Afterwards operators of such type appeared in a more general form in works by Tonelli (1929) and Tikhonov (1938). The definition of *Volterra operator* introduced by Tikhonov is very easy to grasp:

An operator is Volterra if any two functions coinciding on an interval $[a,t]$ *have equal images on* $[a,t]$, $\quad t \in [a,b]$.

This innocently looking definition leads however to a wide variety of far-reaching consequences. Inspired by the success of the notion of Volterra operator in study of equations e.g. in the space of continuous functions, researchers dealing with equations in abstract spaces were tempted to introduce some equivalent of this definition into their considerations. This led to a series of works providing a notion reminiscent of the Volterra operator in each particular situation.

We introduce here a new definition of Volterra operator. Our approach stems mainly from the initial considerations of Volterra-Tonelli-Tikhonov, i.e. uses mainly the evolutionary nature of the Volterra operator. By going **back** to the origins of the theory, we are freeing **future** research from cumbersome conditions and notions.

Basing on the notions of operator's *memory* and *chain* we single out a class of operators possessing the evolutionary property, which we call Volterra. We define properties of nilpotentness, quasi-nilpotentness and compactness of linear Volterra operators in some functional spaces. We also derive conditions for solvability of some functional equations with linear and non-linear Volterra operators in certain complete metric spaces.

1. Introduction

In 1913 in his seminal work "Lectures in Integral and Integro-Differential Equations" [31] Volterra considered the following integro-differential equation

$$\dot{x}(t) = \int_a^t K(t,s)x(s)ds + f(t)$$

with the operator

$$(Kx)(t) = \int_a^t K(t,s)x(s)ds.$$

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Afterwards the notion of Volterra operator appeared simultaneously in several areas of mathematics: integral operators, spectral theory, general theory of systems, functional-differential equations, etc. This notion was studied independently in the framework of the aforementioned theories. Therefore there is a great deal of repeating results, and there is no common accepted body of definitions and terminology. Even the term itself - Volterra operator - is not always used, other ones are: Volterra type, delay, hereditary, causal, non-anticipative operator, etc. which are attributed to different classes of operators with similar properties. Usually these definitions are based on such important properties of the Volterra operator as evolutionariness, compactness and quasi-nilpotence.

The singled out classes of Volterra type operators were based on one of the mentioned above properties, or on their combination. Mainly, some of the authors addressed the compactness and quasi-nilpotence properties of this operator, while others concentrated on its evolutionary side. However, all these classes preserved the name "Volterra" operator (or one of the equivalent terms as it has been mentioned above).

Let us give a brief review of results concerning the operators in question (see also [3], §2.4). L. Tonelli (1929) [28] was the first to single out a class of Volterra type operators, namely, such integral operators K that the equality $x(s) = y(s)$ for $s \leq t$ yields $(Kx)(s) = (Ky)(s)$ for $s \leq t$. His followers, D. Graffi (1931) [14] and S. Cinquini (1933) [6] derived first results in the theory of Volterra operators. An abstract theory of integral Volterra operators basing on a particular system of axioms was constructed by A. D. Myshkis and his apprentices [4], [13], [22]. Let us also note a series of papers by A. Ponosov on non-linear stochastic Volterra operators [20], [21] and the bibliography therein. In 1938 a definition of operator of Volterra type appeared in a paper by A. N. Tikhonov (1938) [27] devoted to problems in mathematical physics. This widely acclaimed work initiated a further development of the theory of Volterra operators in functional spaces, and the operators satisfying the Tikhonov's definition sometimes are called "Volterra operators according to Tikhonov". The property which is basic for the Tikhonov's definition is as follows: for functions x and y coinciding on $[a, t]$, $t \in [a, b]$, the restrictions of Fx and Fy on $[a, t]$ also coincide. This definition is used in research on functional-differential equations with Volterra operators according to Tikhonov (e.g. Azbelev et al. (1991) [2], Corduneanu (1991) [7]).

Several papers devoted to generalizations of the notion of Volterra operator according to Tikhonov appeared recently. V.I. Sumin (1989, 1992) [24], [25] proposed a generalization of Tikhonov's definition to the space of summable functions. An operator $F : L_p^m(M) \to S^\ell(M)$ is called Volterra on a system of sets Θ, Θ belongs to σ-algebra of measurable subsets of M, if the equality of functions x and y on a set $G \in \Theta$ yields the coincidence of Fx and Fy on G. Here $L_p^m(M)$ is the Lebesgue space of m-dimensional vector-functions, defined on a bounded measurable subset $M \subset R^n$ with the usual norm, $1 \leq p \leq \infty$, $S^l(M)$ is the space of l-dimensional vector-functions, component-wise measurable and almost everywhere finite on M. An analogous definition of generalized Volterra operators acting in the space $L_p[a, b]$, where all sorts of systems of subsets from $[a, b]$ ordered by inclusion with the measure continuously changing from 0 to $b-a$, was introduced and studied by E.S. Zhukovskii (1989,1994) [34], [35]. S. A. Gusarenko (1987) [15] and M. Vâth (1998) [30] used chains of ordered projectors for another generalization.

Alternatively, P.P. Zabrejko (1967) [32], [33] proposed a generalization of the notion of integral Volterra operator based on properties of its kernel. These properties provided existence of a chain of invariant subspaces of the operator. Zabrejko derived an expression for the spectral radius and proved that the equality to 0 of the spectral radius follows from a property due to T. Andô (1957) [1]. I.Z. Gohberg and M. G. Krein (1967) [12] defined an abstract Volterra operator in Hilbert space as completely continuous linear operator with zero spectral radius. A.L. Buhgeim (1983) [5] extended this definition to Banach spaces. He used for it a notion of a special chain of projectors. A similar construction is due to V.G. Kurbatov (1975,1990) [17], [18] based on a chain of embedded subspaces. Notice, that a basic theory of the translations of abstract causal operators was developed by G. Karakostas in [16].

The theory of Volterra operators in Hilbert spaces attracted a great deal of attention. D.C. Youla et al. (1959) [29] were the first to emphasize the importance of a Volterra notion in the general theory of systems. The main properties of Volterra operators were first stated for L_2, and then for general Hilbert spaces. According to A. Feintuch and R. Saeks (1982) [11] a linear ordered and closed family of orthogonal projectors P in space H is called "expansion of unity" if $P^\tau H \subset P^\theta H$ when $0 \le \tau \le \theta \le 1$ and $P^0 = 0$, $P^1 = I$. An operator is called Volterra ("causal" in the source), if it possesses the property: $P^\tau x = P^\tau y$ yields $P^\tau Fx = P^\tau Fy$. In these papers also notions of anti-Volterra (anticausality) and absolute Volterra (memoryless) were introduced. The problems which were mainly studied for linear operators, are operators' expansion, factorization, and invertibility.

This is just a non-complete sample of attempts generalize the notion. Nevertheless, all the mentioned approaches are deficient in the following sense:

- They use essentially the topology of underlying space, e.g. employ completeness, convergence, etc.
- Being formulated for a particular space, they do not cover all the situations described by the original definition (for example, operator $(Fx)(t) = x(t/2)$ is not Volterra according to [12], but is Volterra in the original sense for the interval $[0, a], \quad a \in R_+$).
- Apparently, there is no clear way of extending the introduced notions to spaces other than the ones under the particular consideration.

In the paper we attempt to provide a new definition of Volterra operators on abstract spaces avoiding the aforementioned shortcomings. The definition we give requires only existence of a σ-algebra on a metric space. Notice, that being applied to such spaces it covers most of the results achieved with the former definitions and essentially extends them to a wider classes of operators. It allows to relate different properties of the **Volterra type** operators.

Our definition of the **Volterra** operator is based on the notion of the *memory* of operator. Roughly speaking, the *memory* is an information about the preimages the operator is able to remember given some information about the images. In Section 2 we recall the definitions related to the notion of *memory* [10]. In Section 3, basing on the notions of operator's memory and chain, we single out a class of Volterra operators. In Section 4 we study linear Volterra operators, and derive properties of nilpotentness, quasi-nilpotentness and compactness of such operators. Section 5 is devoted to functional equations with Volterra operators. Conditions for solvability of some functional equations with linear and non-linear Volterra operator in certain complete metric spaces are derived. In Section 6 we formulate a statement on representation of a Volterra operator involved in the theory of functional differential equations with delayed argument.

2. Notation and preliminaries

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces, and $\Sigma_1^0 \subset \Sigma_1$, $\Sigma_2^0 \subset \Sigma_2$ be the σ-ideals of μ_1- and μ_2-nullsets respectively. We denote by $\tilde{\Sigma}_i := \Sigma_i/\Sigma_i^0$, $i = 1, 2$, the respective *measure algebras* (see § 42 of [23]). For the elements of $\tilde{\Sigma}_i$ (i.e. the equivalence classes of sets) will be denoted $\tilde{e}_i$ or $[e_i]$, $i = 1, 2$. Further on we will however frequently abuse the notation and identify the elements of the measure algebras $\tilde{\Sigma}_i$ with the elements of the respective original σ-algebras of sets Σ_i.

A measure space (Ω, Σ, μ) is called standard, if Ω is a Polish space, Σ is either the Borel σ-algebra or its completion with respect to finite or σ-finite Borel measure μ.

By $X(\Omega, \Sigma, \mu; \mathcal{Y})$ we will understand a linear space of measurable functions, defined on Ω and taking values in $\mathcal{Y}$. A topology in X will be defined explicitly depending on the particular problem under consideration.

Further, the notation $L^p(\Omega, \Sigma, \mu; \mathcal{Y})$, where $\mathcal{Y}$ is a separable Banach space, will stand, as usual, for the classical Lebesgue space of $\mathcal{Y}$-valued functions measurable with respect to Σ and μ-summable with power p (if $p \in [1, +\infty)$) or μ-essentially bounded (if $p = +\infty$). These spaces are silently assumed to be equipped with their strong topologies. If $\mathcal{Y}$ is a separable metric space, then $L^0(\Omega, \Sigma, \mu; \mathcal{Y})$ stands for the metric space of $\mathcal{Y}$-valued functions measurable with respect to Σ equipped with the topology of convergence in measure.

Whenever there is no possibility of confusion, the references to $\mathcal{Y}$, Ω, Σ and/or μ will be omitted. We will also omit in sequel the sign $\tilde{(\cdot)}$, assuming that all the considerations are modulo the equivalence classes of sets.

Let $X_i := X(\Omega_i, \Sigma_i, \mu_i; \mathcal{Y}_i)$, $i = 1, 2$. Consider an operator $T\colon X_1 \to X_2$. In what follows the notation $T\colon X_1 \to X_2$ assumes that the domain of T coincides with the whole space X_1. Following [10] (see also [8], [9]) we introduce now the concept of *memory* and the related concept of *comemory*.

Definition 2.1. We call the **memory** of an operator $T\colon X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ on a set $e_2 \in \Sigma_2$ the family of all possible $e_1 \in \Sigma_1$ such that for any $x, y \in X_1$ satisfying $x\,|_{e_1} = y\,|_{e_1}$ it follows that $T(x)\,|_{e_2} = T(y)\,|_{e_2}$. In other words,

$$\mathrm{Mem}_T(e_2) := \{e_1 \in \Sigma_1 \,:\, x\,|_{e_1} = y\,|_{e_1} \;\Rightarrow\; T(x)\,|_{e_2} = T(y)\,|_{e_2}\}\,.$$

Similarly, the **comemory** of operator T on a set $e_1 \in \Sigma_1$ is the family

$$\mathrm{Comem}_T(e_1) := \{e_2 \in \Sigma_2 \,:\, x\,|_{e_1} = y\,|_{e_1} \;\Rightarrow\; T(x)\,|_{e_2} = T(y)\,|_{e_2}\}\,.$$

Recall that according to our convention all the equalities in the above definition should be understood in almost everywhere sense.

It is clear from the definitions that

$$e_1 \in \mathrm{Mem}_T(e_2) \Longleftrightarrow e_2 \in \mathrm{Comem}_T(e_1). \tag{2.1}$$

The properties of *memory* and *comemory* along with some examples helping to understand deeper the definitions given above, could be found in [10]

3. Volterra Operator

In this section, basing on the notions of *chain* and *memory*, we single out a class of operators possessing the evolutionary property.

Definition 3.1. A collection of subsets $\{e_\nu\}$, $e_\nu \in \Sigma$, $\nu \in [0, \infty]$, in a measurable space (Ω, Σ, μ) is said to be **chain** if the following conditions are satisfied:

(1) $\mu(e_0) = 0$;
(2) $e_{\nu_1} \subset e_{\nu_2}$ if $\nu_1 \leq \nu_2$;
(3) for every $\alpha \in (0, \mu(\Omega))$ there exists a set $e_\beta \in \{e_\nu\}$ such that $\mu(e_\beta) = \alpha$.

Example 3.1.

(1) $\Omega = [0, 1], \{e_\nu\} = \{[0, \nu]\}, \nu \in [0, 1]$.
(2) $\Omega = [0, 1], \{e_\nu\} = \{[1 - \nu, 1]\}, \nu \in [0, 1]$.
(3) $\Omega = [0, 1], \{e_\nu\} = \{[\frac{1}{2} - \nu, \frac{1}{2} + \nu]\}, \nu \in [0, \frac{1}{2}]$.
(4) $\Omega = [0, 1], \{e_\nu\} = \{[0, \nu] \cup [1 - \nu, 1]\}, \nu \in [0, \frac{1}{2}]$.

Definition 3.2. An operator

$$T : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2),$$

is called **Volterra** (this will be denoted $T \in V$), if there exists a pair of chains $\{e^1_\nu\} \subset \Sigma_1, \{e^2_\lambda\} \subset \Sigma_2$, such that for every member e^2_α of the chain $\{e^2_\lambda\}$, the corresponding element e^1_α of the chain $\{e^1_\nu\}$ satisfies

$$e^1_\alpha \in \mathrm{Mem}_T(e^2_\alpha) \tag{3.1}$$

The correspondence between the pair of chains here is provided by the same lower index α.

Remark 3.1. Taking into account (2.1), the inclusion $e^1_\alpha \in \mathrm{Mem}_T(e^2_\alpha)$ in the above definition can be replaced by an equivalent one:

$$e^2_\alpha \in \mathrm{Comem}_T(e^1_\alpha) \tag{3.2}$$

Remark 3.2. Let operator $T_1 : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ be Volterra with respect to the pair of chains $\{e^1_\nu\} \subset \Sigma_1$, $\{e^2_\lambda\} \subset \Sigma_2$, and operator $T_2 : X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2) \to X_3(\Omega_3, \Sigma_3, \mu_3; \mathcal{Y}_3)$ be Volterra with respect to the pair of chains $\{e^2_\lambda\} \subset \Sigma_2$, $\{e^3_\delta\} \subset \Sigma_3$. **Then** evidently operator $T_2T_1 : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_3(\Omega_3, \Sigma_3, \mu_3; \mathcal{Y}_3)$ is Volterra with respect to the pair of chains $\{e^1_\nu\} \subset \Sigma_1$, $\{e^3_\delta\} \subset \Sigma_3$.

Remark 3.3. In [8], [9], [10] the notion of the local operator is given in terms of the *memory* of operator. Namely, operator $N : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$ is local if

$$\forall e \in \Sigma \Rightarrow e \in \mathrm{Mem}_N(e).$$

Taking into account **Definition 3.2** one can conclude that $N : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$ is Volterra with respect to any chain $\{e_\nu\} \subset \Sigma$. Thus, if operator $T : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ is Volterra with respect to the pair of chains $\{e^1_\nu\} \subset \Sigma_1$, $\{e^2_\lambda\} \subset \Sigma_2$, then operator $NT : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$, where $N : X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ is a local operator, also Volterra with respect to the same pair of chains. Moreover, operator $TN : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$, where $N : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1)$ is a local operator, also Volterra with respect to the same pair of chains.

Example 3.2.

(1) $T_i : X_1([0,1], \Sigma, m; \mathcal{Y}_1) \to X_2([0,1], \Sigma, m; \mathcal{Y}_2), \quad i = 1,2,3,4,5,6$, where

$$(T_1x)(t) = \int_0^t Q(t,s,x(s))ds, \quad t \in [0,1];$$

$$(T_2x)(t) = \int_{1-t}^1 Q(t,s,x(s))ds, \quad t \in [0,1];$$

$$(T_3x)(t) = f(t, x(g(t))), \quad t \in [0,1], g(t) \le t, \quad x(\zeta) = \varphi(\zeta) \text{ if } \zeta < 0;$$

$$(T_4x)(t) = f(t, x(\tau(t))), \quad t \in [0,1], \tau(t) \ge t, \quad x(\zeta) = \psi(\zeta) \text{ if } \zeta > 1;$$

$$(T_5x)(t) = f(t, x(1-t)), \quad t \in [0,1];$$

$$(T_6x)(t) = \int_0^t Q(t,s,x(1-s))ds, \quad t \in [0,1].$$

Here the operators T_1 and T_3 are Volterra with respect to the chain pair $\{[0,t]\}$ and $\{[0,t]\}$; T_2 and T_4 are Volterra with respect to $\{[1-t,1]\}$ and $\{[1-t,1]\}$, and, finally, T_5 and T_6 are Volterra with respect to $\{[1-t,1]\}$ and $\{[0,t]\}$.

(2) $S_i : X_1([0,1], \Sigma_1, m; \mathcal{Y}_1) \to X_2([0, \frac{1}{2}], \Sigma_2, m; \mathcal{Y}_2), \quad i = 1,2;$

$$(S_1x)(t) = \int_{\frac{1}{2}-t}^{\frac{1}{2}+t} Q(t,s,x(s))ds, \quad t \in [0, \frac{1}{2}];$$

$$(S_2x)(t) = \int_0^t Q_1(t,s,x(s))ds + \int_{\frac{1}{2}}^{\frac{1}{2}+t} Q_2(t,s,x(s))ds, \quad t \in [0, \frac{1}{2}].$$

Here S_1 is Volterra with respect to the pair of chains $\{[\frac{1}{2}-t, \frac{1}{2}+t]\} \in \Sigma_1$ and $\{[0,t]\} \in \Sigma_2$, and S_2 is Volterra with respect to the pair of chains $\{[0,t] \cup [\frac{1}{2}, \frac{1}{2}+t]\} \subset \Sigma_1$ and $\{[0,t]\} \subset \Sigma_2$.

Let us single out from the class of Volterra operators, $T : X_1(\Omega, \Sigma, \mu; \mathcal{Y}_1) \to X_2(\Omega, \Sigma, \mu; \mathcal{Y}_2)$, a subclass which will be called the class of **conventional** Volterra operators.

Definition 3.3. An operator

$$T : X_1(\Omega, \Sigma, \mu; \mathcal{Y}_1) \to X_2(\Omega, \Sigma, \mu; \mathcal{Y}_2),$$

is called **conventional Volterra** (this will be denoted $T \in CV$), if there exists a chain $\{e_\nu\} \subset \Sigma$ such that every element of the chain, $e_\alpha \in \{e_\nu\}$, satisfies

$$e_\alpha \in \mathrm{Mem}_T(e_\alpha) \tag{3.3}$$

Evidently the integral Volterra operators (of type T_1 in Example 3.2) belong to the class of conventional Volterra. The operator of type T_3 from Example 3.2 is called Volterra according to the definition by A.N.Tikhonov [27], and is also from the class of conventional Volterra operators. Note that the operators of type T_2 and T_4 from Example 3.2 and the local operators [10] also belong to this class.

Remark 3.4. Note that if an operator $T : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$ is conventional Volterra with respect to some chain, then every degree of T, $T^k, k = 1, 2, ...$ ($T^k = TT^{k-1}, T^0$-the identity operator) is conventional Volterra with respect to the same chain.

4. Linear Volterra operators

In this section we derive the properties of nilpotentness, quasi-nilpotentness and compactness of linear Volterra operators in some functional spaces.

Consider a linear operator

$$\mathcal{L} : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y}).$$

Definition 4.1. $\mathcal{L}$ is called **nilpotent** if there exist an integer $k > 0$, such that $\mathcal{L}^k : X \to X$ is zero operator.

Theorem 4.1. *Let a linear, continuous in measure, operator $\mathcal{L} : X \to X$, be conventional Volterra with respect to chain $\{e_\nu\} \subset \Sigma$, and*

(1) $\mu(\Omega) < \infty$;
(2) *for every element of the chain satisfying*

$$\mu(\inf \mathrm{Mem}_{\mathcal{L}}(e_\alpha)) > 0, \tag{4.1}$$

the following holds:

$$\mu(e_\alpha \setminus \inf \mathrm{Mem}_{\mathcal{L}}(e_\alpha)) \geq \delta > 0. \tag{4.2}$$

Then $\mathcal{L}$ is nilpotent.

Proof. First of all notice that for every set $e \in \Sigma$:

$$\mathrm{Mem}_{\mathcal{L}^2}(e) = \mathrm{Mem}_{\mathcal{L}}(\inf \mathrm{Mem}_{\mathcal{L}}(e)). \tag{4.3}$$

By the assumptions,

$$\mu(\inf \mathrm{Mem}_{\mathcal{L}}(\Omega)) \leq \mu(\Omega) - \delta.$$

Thus, having in mind (4.3), there exists an integer k such that

$$\mu(\inf \mathrm{Mem}_{\mathcal{L}^k}(\Omega)) = 0.$$

Thus the claim follows from linearity and continuity in measure of $\mathcal{L}$. □

In the following example we illustrate the necessity of imposed conditions.

Example 4.1. Consider operator $T : L_\infty([0, 1], \Sigma, m; R) \to L_\infty([0, 1], \Sigma, m; R)$. Here Σ is the σ-algebra on subsets of $[0, 1]$, Lebesgue measurable, and m is the Lebesgue measure. Let T be defined by one of the following equalities:

(1) $(Tx)(t) = x(t - \frac{1}{2}), \quad t \in [0, 1], x(\zeta) = 1$, if $\zeta < 0$;
(2) $(Tx)(t) = x(\frac{1}{2}t), \quad t \in [0, 1]$;

(3) $(Tx)(t) = \lim_{\delta \to 0} \sup \frac{1}{2\delta} \int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} x(s)ds \cdot 1(t)$.

It is easy to see that in each one of the aforementioned cases the conventional Volterra operators are not nilpotent.

Indeed, in the first case all the conditions of Theorem 1 but linearity are fulfilled. In the second case T is linear and continuous in the measure, but condition (4.2) is not satisfied. It follows from

$$\mu([0,\alpha] \setminus \inf \mathrm{Mem}_T([0,\alpha])) = \frac{\alpha}{2}$$

for any $\alpha \in (0,1]$. Finally, in the third case the conditions of continuity in the measure and linearity are violated. Notice, that

$$(\forall e \in \Sigma) \quad \mu(e \setminus \inf \mathrm{Mem}_T(e)) = \mu(e).$$

Definition 4.2. We say that a space X possesses property $\mathcal{X}$ (and write $X \in \mathcal{X}$) if $(\forall e \in \Sigma)(\forall x \in X)$, the function x_e defined by

$$x_e(t) = \chi_e(t)x(t), \quad t \in \Omega, \tag{4.4}$$

also belongs to X. Here χ_e is the characteristic function of e.

Let $X \in \mathcal{X}$, $e \in \Sigma$. Choose a subspace X_e of X as follows: to every function $x \in X$ we correspond the function $x_e \in X_e$ defined by (4.4). Let us define by $\mathcal{L}_e$ the reduction of $\mathcal{L} : X_1 \to X_2$ to the subspace X_{1e}.

Lemma 4.1. *Let a space $X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \in \mathcal{X}$ and a linear operator $\mathcal{L} : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ be Volterra with respect to the couple of chains $\{e^1_\nu\} \subset \Sigma_1, \{e^2_\nu\} \subset \Sigma_2$. Then for every collection $e^1_{\alpha_i} \in \{e^1_\nu\}, i = 1, ..., k; e^1_{\alpha_i} \subset e^1_{\alpha_j}$, whenever $i < j$, we have*

$$\mathcal{L}_{e^1_{\alpha_k}} = \mathcal{L}_{e^1_{\alpha_1}} + \mathcal{L}_{e^1_{\alpha_2} \setminus e^1_{\alpha_1}} + ... + \mathcal{L}_{e^1_{\alpha_k} \setminus e^1_{\alpha_{k-1}}} \tag{4.5}$$

Proof. Since $X_1 \in \mathcal{X}$, for every function $x \in X_1$ the function $x_{e^1_{\alpha_k}}$ also belongs to this space. Evidently we have,

$$x_{e^1_{\alpha_k}}(t) = x_{e^1_{\alpha_1}}(t) + x_{e^1_{\alpha_2} \setminus e^1_{\alpha_1}}(t) + ... + x_{e^1_{\alpha_k} \setminus e^1_{\alpha_{k-1}}}(t), \quad t \in \Omega_1. \tag{4.6}$$

To accomplish the proof now we apply $\mathcal{L}$ to both sides of (4.6), and take into account that it is linear and Volterra. ⋄ □

Definition 4.3. A linear operator $\mathcal{L} : X \to X$, is called **quasi-nilpotent** if the spectral radius of this operator is zero.

Theorem 4.2. *Let for a linear operator $\mathcal{L} : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$, where X is a Banach space possessing property $\mathcal{X}$, the following conditions hold:*

1. *Operator $\mathcal{L}$ is conventional Volterra;*
2. $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall e \in \Sigma) : \mu(e) < \delta \Rightarrow \|\mathcal{L}_e\|_{X_e \to X} < \varepsilon$.

Then $\mathcal{L}$ is quasi-nilpotent.

Proof. Seeking contradiction, let us assume that λ, $|\lambda| > 0$, is an eigenvalue of $\mathcal{L}$, and the corresponding eigenfunction is $x : \Omega \to \mathcal{Y}$, $\|x\|_X > 0$. Clearly x is a solution of

$$\lambda x(t) = (\mathcal{L}x)(t), \quad t \in \Omega. \tag{4.7}$$

Denote by $\{e_\nu\}$ the chain, with respect to which T is conventional Volterra. By the definition (of conventional Volterra operators) for every element e_α of the chain $\{e_\nu\}$, the function x_{e_α} is a solution of

$$\lambda x_{e_\alpha}(t) = (\mathcal{L}_{e_\alpha} x_{e_\alpha})(t), \quad t \in e_\alpha. \tag{4.8}$$

Thus for every k, $k = 1, 2, ...$, we have

$$\lambda^k x_{e_\alpha}(t) = (\mathcal{L}^k_{e_\alpha} x_{e_\alpha})(t), \quad t \in e_\alpha, k = 1, 2, ... \tag{4.9}$$

By Condition 2 of the theorem, it is possible to find e_α, $\mu(e_\alpha) > 0$, such that the following inequality is valid:

$$|\lambda|^{-1} \|\mathcal{L}_{e_\alpha}\|_{X_{e_\alpha} \to X} < 1. \tag{4.10}$$

Since (4.9) holds for every k, $k = 1, 2, ...$, (4.10) yields that the eigenfunction x satisfies

$$x(t) = 0, \quad t \in e_\alpha. \tag{4.11}$$

Furthermore, let us choose a set e_β from the chain $\{e_\nu\}$ such that the following relations hold:

$$e_\alpha \subset e_\beta, \quad \mu(e_\beta \setminus e_\alpha) = \mu(e_\alpha). \tag{4.12}$$

By Lemma 4.1 we have $\mathcal{L}_{e_\beta} = \mathcal{L}_{e_\alpha} + \mathcal{L}_{e_\beta \setminus e_\alpha}$. Applying $\mathcal{L}_{e_\beta}$ to the eigenfunction and taking into account (4.11), we get

$$(\mathcal{L}_{e_\beta} x)(t) = (\mathcal{L}_{e_\beta \setminus e_\alpha} x)(t), \quad t \in e_\beta \setminus e_\alpha. \tag{4.13}$$

Therefore, for every k, $k = 1, 2, ...$, we have for the eigenfunction x:

$$\lambda^k x_{e_\beta \setminus e_\alpha}(t) = (\mathcal{L}^k_{e_\beta \setminus e_\alpha} x_{e_\beta \setminus e_\alpha})(t), \quad t \in e_\beta \setminus e_\alpha,$$

and

$$|\lambda|^{-1} \|\mathcal{L}_{e_\beta \setminus e_\alpha}\|_{X_{e_\beta \setminus e_\alpha} \to X} < 1. \tag{4.14}$$

Thus

$$x(t) = 0, \quad t \in e_\beta.$$

Continuing this process, in a finite number of steps when $\mu(\Omega) < \infty$ or applying induction when the measure is σ-finite, we arrive at a contradiction:

$$x(t) = 0, \quad t \in \Omega. \qquad \square$$

The case of operators in Banach spaces of measurable real-valued functions was studied in [26].

Example 4.2. Consider operator $S : L_2([0,1],\Sigma,m;R) \to L_2([0,1],\Sigma,m;R)$, defined by

$$(Sx)(t) = \int_0^1 K(t,s)x(s)ds, \quad 0 \le s,t \le 1. \tag{4.15}$$

Assume that $K : [0,1]\times[0,1] \to R$, the kernel of S, is measurable and satisfies:

$$\int_0^1 \int_0^1 |K(t,s)|^2 dsdt < \infty. \tag{4.16}$$

As usual, we call the operator of type (4.15) **Fredholm** and, if $K(\cdot,\cdot)$ satisfies (4.16) it is called **Hilbert-Schmidt**, and

$$\|S\|_{L_2\to L_2} \le (\int_0^1 \int_0^1 |K(t,s)|^2 dsdt)^{\frac{1}{2}}. \tag{4.17}$$

It is easy to see that for the Hilbert-Schmidt operator defined by (4.15) Condition 2 of Theorem 4.2 is satisfied.

If an extra assumption,

$$K(t,s) = 0, \quad 0 < t < s < 1,$$

holds, the operator S has the form

$$(\mathcal{L}x)(t) = \int_0^t K(t,s)x(s)ds, \quad 0 \le s \le t \le 1 \tag{4.18}$$

For the operator $\mathcal{L}$ all the conditions of Theorem 4.2 are already fulfilled. As a corollary of Theorem 4.2 we derive the well-known fact that $\mathcal{L}$ defined by (4.18) is quasi-nilpotent.

Theorem 4.3. *Let for a linear continuous operator*

$$\mathcal{L} : X(\Omega_1,\Sigma_1,\mu_1;\mathcal{Y}_1) \to L^p(\Omega_2,\Sigma_2,\mu_2;\mathcal{Y}_2), \quad \mu_2(\Omega_2) < \infty, \quad 1 \le p \le \infty,$$

where X is a Banach space satisfying the property $\mathcal{X}$ the following conditions hold:

1. *There exists a pair of chains $\{e^1_\nu\} \subset \Sigma_1, \{e^2_\nu\} \subset \Sigma_2$, such that $\mathcal{L}$ is Volterra with respect to it;*
2. $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall e \in \Sigma_1) : \quad \mu_1(e) < \delta \Rightarrow \|\mathcal{L}_e\|_{X_e\to L^p} < \varepsilon$;
3. $(\forall e^1_\alpha \in \{e^1_\nu\})(\forall t_1,t_2 \in \Omega_2 \setminus e^2_\alpha)(\forall x \in X) : \quad (\mathcal{L}_{e^1_\alpha} x_{e^1_\alpha})(t_1) = (\mathcal{L}_{e^1_\alpha} x_{e^1_\alpha})(t_2)$.

Then $\mathcal{L}$ is compact.

Proof. 1. First let us consider the case $1 \le p < \infty$. To prove the theorem it is enough to establish integral equicontinuity of the image of any bounded set from X. Fix ε, $\varepsilon > 0$, and using Condition 2 pick sets $e^1_{\alpha_i}, i = 1,2,...,k$, belonging to the chain $\{e^1_\nu\} \subset \Sigma_1$, in such a way that

$$\|\mathcal{L}_{e^1_{\alpha_i}\setminus e^1_{\alpha_{i-1}}}\|_{X_{e^1_{\alpha_i}\setminus e^1_{\alpha_{i-1}}}\to L^p} < \varepsilon, \quad i = 1,2,...,k. \tag{4.19}$$

Here we denote: $e^1_{\alpha_0} = e^1_0, e^1_{\alpha_k} = \Omega_1$, and it is assumed that $e^1_{\alpha_i} \subset e^1_{\alpha_j}$ if $0 \le i \le j \le k$. Let us correspond to each point $t \in \Omega_2$ a point $t' \in \Omega_2$ satisfying the following condition: $t \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}}$ implies $t' \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}}, \quad i = 1,2,...,k$. Let us estimate

$$\|(\mathcal{L}x)(t') - (\mathcal{L}x)(t)\|_{\mathcal{Y}_2}, \quad t \in \Omega_2,$$

for an arbitrary $x \in X$. Let $t \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}}$. Since $X \in \mathcal{X}$,

$$(\mathcal{L}x)(t) = [\mathcal{L}(x_{e^1_{\alpha_{i-1}}} + x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}} + x_{e^1_{\alpha_k} \setminus e^1_{\alpha_i}})](t).$$

By the conditions of the theorem:

$$(\mathcal{L}x_{e^1_{\alpha_{i-1}}})(t) = (\mathcal{L}x_{e^1_{\alpha_{i-1}}})(t'), \quad t \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}},$$

$$(\mathcal{L}x_{e^1_{\alpha_k} \setminus e^1_{\alpha_i}})(t) = (\mathcal{L}x_{e^1_{\alpha_k} \setminus e^1_{\alpha_i}})(t') = 0, \quad t, t' \in e^2_{\alpha_i}.$$

Therefore,

$$\begin{aligned} (4.20) \quad & \|(\mathcal{L}x)(t') - (\mathcal{L}x)(t)\|_{\mathcal{Y}_2} = \|(\mathcal{L}x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}})(t') - (\mathcal{L}x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}})(t)\|_{\mathcal{Y}_2} = \\ & = \|(\mathcal{L}_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}} x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}})(t') - (\mathcal{L}_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}} x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}})(t)\|_{\mathcal{Y}_2} \leq 2\varepsilon \|x\|_X. \end{aligned}$$

Thus

$$\int_{\Omega_2} \|(\mathcal{L}x)(t') - (\mathcal{L}x)(t)\|^p_{\mathcal{Y}_2} d\mu_2(t) =$$

$$= \sum_{i=1}^{k} \int_{e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}}} \|(\mathcal{L}x)(t') - (\mathcal{L}x)(t)\|^p_{\mathcal{Y}_2} d\mu_2(t) \leq$$

$$\leq (2\varepsilon \|x\|_X)^p \mu_2(\Omega_2).$$

To accomplish the proof we use the compactness conditions in $L^p, \quad 1 \leq p < \infty$.

2. Now let us consider the case $p = \infty$. Taking into account (4.20), we obtain

$$\begin{aligned} & \operatorname*{essup}_{\substack{t,t' \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}} \\ i=1,2,\dots,k}} \|(\mathcal{L}x)(t') - (\mathcal{L}x)(t)\|_{\mathcal{Y}_2} = \\ = & \max_{i \in (1,\dots,k)} \operatorname*{essup}_{t,t' \in e^2_{\alpha_i} \setminus e^2_{\alpha_{i-1}}} \|(\mathcal{L}_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}} x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}}))(t') - (\mathcal{L}_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}} x_{e^1_{\alpha_i} \setminus e^1_{\alpha_{i-1}}})(t)\|_{\mathcal{Y}_2} \\ & \leq 2\varepsilon \|x\|_X. \end{aligned}$$

Last estimate means that every equivalence class $\mathcal{L}x$ contains the continuous function. Since for a bounded set from X this estimate holds equicontinuously, then for completeness of the proof it is enough to refer to the compactness conditions in the space of continuous functions. □

Remark 4.1. In order to clarify the conditions of Theorem 4.3 let us note the following. Operators T_1, T_2 from Example 3.2 satisfy Condition 3 of the theorem if we assume for the function $Q : [0,1] \times [0,1] \times \mathcal{Y}_1 \to \mathcal{Y}_2$ the validity of the equality $Q(t, s, 0) = 0, \quad t, s \in [0,1]$. Operators T_3 and T_4 from the same example in general do not satisfy the Condition 3 of the theorem.

5. Functional Equations with Volterra Operator

In this section conditions for solvability of some functional equations with linear and non-linear Volterra operator in certain complete metric spaces are derived. A generalization of the step method, well-known in the theory of functional differential equations (see, for example, [19]), is proposed.

Theorem 5.1. *Let $X(\Omega, \Sigma, \mu; \mathcal{Y})$ be a complete metric space. Moreover, let $X \in \mathcal{X}$. Let for a linear operator $T : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$ the following conditions*

1. *T is conventional Volterra with respect to some chain $\{e_\nu\}$;*

2. *For any two elements e_α, e_β of the chain $\{e_\nu\}$, $e_\alpha \subset e_\beta$, and for any x', $x'' \in X$ condition $\mu(e_\beta \setminus e_\alpha) \le \delta$, $\delta > 0$, implies*

$$\rho(Tx'_{e_\beta \setminus e_\alpha}, Tx''_{e_\beta \setminus e_\alpha}) \le \sigma \rho(x'_{e_\beta \setminus e_\alpha}, x''_{e_\beta \setminus e_\alpha}), \quad 0 < \sigma < 1.$$

hold.

Then *for any function $f \in X$ there exists the unique solution to the equation*

$$x(t) - (Tx)(t) = f(t), \quad t \in \Omega, \tag{5.1}$$

which could be found by the method of successive approximations.

Proof. Let us pick a set $e^1 \in \{e_\nu\}$ such that $0 < \mu(e_1) \le \delta$. Then, in virtue of the theorem's conditions, for any x', $x'' \in X$ the following inequality holds:

$$\rho(Tx'_{e^1}, Tx''_{e^1}) \le \sigma \rho(x'_{e^1}, x''_{e^1}).$$

According to the Banach principle there exists the unique solution $\bar{x}_{e^1}$ to the equation

$$x_{e^1}(t) - (T_{e^1} x_{e^1})(t) = f_{e^1}(t), \quad t \in e^1, \tag{5.2}$$

which could be found by the method of successive approximations. Furthermore, let us pick a set $e^2 \in \{e_\nu\}$ such that $0 < \mu(e^2 \setminus e^1) \le \delta$. Then for any $x', x'' \in X$ the following inequality takes place:

$$\rho(Tx'_{e^2 \setminus e^1}, Tx''_{e^2 \setminus e^1}) \le \sigma \rho(x'_{e^2 \setminus e^1}, x''_{e^2 \setminus e^1}).$$

Applying again the Banach principle, we conclude that there exists the unique solution $\bar{x}_{e^2 \setminus e^1}$ to the equation

$$x_{e^2 \setminus e^1}(t) - (T_{e^2 \setminus e^1} x_{e^2 \setminus e^1})(t) = f_{e^2 \setminus e^1}(t) + (T_{e^1} \bar{x}_{e^1})(t), \quad t \in e^2 \setminus e^1, \tag{5.3}$$

which also could be found by the method of successive approximations. Taking into account Lemma 4.1, one can conclude that the function $\bar{x}_{e^2} = \bar{x}_{e^1} + x_{e^2 \setminus e^1}$ is the unique solution to the equation

$$x_{e^2}(t) - (T_{e^2} x_{e^2})(t) = f_{e^2}(t), \quad t \in e^2, \tag{5.4}$$

which could be found by the method of successive approximations.

Continuing this process, after finite number of steps in the case $\mu(\Omega) < \infty$, or using the method of mathematical induction in the case of σ - finite μ, we will achieve the statement of the theorem. □

Theorem 5.2. *(Step method) Let operator* $F : X(\Omega, \Sigma, \mu; \mathcal{Y}) \to X(\Omega, \Sigma, \mu; \mathcal{Y})$ *satisfy the following conditions:*

1. *F is Volterra operator with respect to some pair of chains* $\{e^1_\nu\} \subset \Sigma, \quad \{e^2_\nu\} \subset \Sigma$. *Moreover, for any element* $e^2_\alpha \in \{e^2_\nu\}, \quad \mu(e^2_\alpha) > 0$, *the following inclusion*

$$e^1_\alpha \subset e^2_\alpha, \quad \mu(e^2_\alpha \setminus e^1_\alpha) > 0 \tag{5.5}$$

takes place.

2. *There exists a set* $e^1_\alpha \in \{e^1_\nu\}, \quad \mu(e^1_\alpha) > 0$ *such that the equation*

$$x(t) - (Fx)(t) = f(t), \quad t \in e^1_\alpha, \tag{5.6}$$

has a solution $y \in X$ on this set.

Then *for any function* $\varphi \in X, \quad \varphi(t) = f(t), \quad t \in e^1_\alpha$, *there exists a function* $z \in X, \quad z(t) = y(t), \quad t \in e^1_\alpha$, *such that it converts the equation*

$$x(t) - (Fx)(t) = \varphi(t), \quad t \in \Omega, \tag{5.7}$$

into identity on each set $e^2_\gamma \in \{e^2_\nu\}, \quad \mu(\Omega \setminus e^2_\gamma) > 0$, *i.e. the function z is a solution to (5.7) on the set* e^2_γ.

Proof. In virtue of the conditions function $\bar{x}_{e^2_\alpha} \in X$, defined as follows

$$\begin{aligned} \bar{x}_{e^2_\alpha}(t) &= y(t), \quad t \in e^1_\alpha \\ \bar{x}_{e^2_\alpha}(t) &= (Fy)(t) + \varphi(t), \quad t \in e^2_\alpha \setminus e^1_\alpha, \end{aligned} \tag{5.8}$$

is a solution to (5.7) on the set e^2_α or on the set $e^1_\beta = e^2_\alpha$. Then function $\bar{x}_{e^2_\beta}$ defined by the following equalities

$$\bar{x}_{e^2_\beta}(t) = \bar{x}_{e^2_\alpha}(t), \quad t \in e^1_\beta,$$

$$\bar{x}_{e^2_\beta}(t) = (Fx_{e^1_\beta})(t) + \varphi(t), \quad t \in e^2_\beta \setminus e^1_\beta,$$

satisfies (5.7) on e^2_β. Continuing this process (and taking into account that $\mu(\Omega) < \infty$), for each set $e^2_\gamma \in \{e^2_\nu\}, \quad \mu(\Omega \setminus e^2_\gamma) > 0$, one can find a function $z \in X$ satisfying (5.7) on this very set. □

Remark 5.1. If additionally to the conditions of Theorem 5.2 one assumes that

$$\mu(\inf \mathrm{Mem}_F(\Omega)) < \mu(\Omega), \tag{5.9}$$

then there exists a function $z \in X$, satisfying (5.7) on the whole set Ω.

Let us emphasize the essence of the condition on finiteness of the measure μ in Theorem 5.2. Namely, consider the following

Example 5.1. Let us consider the equation

$$x(t) - (Tx)(t) = 1, \quad t \in [0, \infty), \tag{5.10}$$

with continuous operator $T : L_\infty([0,\infty), \Sigma, m; R) \to L_\infty([0,\infty), \Sigma, m; R)$ defined as follows:

$$(Tx)(t) = \begin{cases} x(t-1), & t \geq 1, \\ 0, & t < 1. \end{cases}$$

A solution to (5.10) is defined by the equality

$$x(t) = n, \quad t \in [n-1, n), \quad n = 1, 2, ...,$$

and, evidently, does not belong to the space $L_\infty([0,\infty),\Sigma,m;R)$.

Remark 5.2. Note, that in Theorem 5.2 we do not concretize a topology on $X(\Omega,\Sigma,\mu;\mathcal{Y})$ as well as there is no assumption on the completeness of the space.

Corollary 5.1. *Let $X(\Omega,\Sigma,\mu;\mathcal{Y})$ be a complete metric space, $X\in\mathcal{X}$ and $\mu(\Omega)<\infty$. Let operator $F: X(\Omega,\Sigma,\mu;\mathcal{Y})\to X(\Omega,\Sigma,\mu;\mathcal{Y})$ satisfies the following conditions:*

1. *Condition* **1** *of Theorem 5.2 is satisfied.*

2. *There exists a set $e^1_\alpha\in\{e^1_\nu\}$, $\mu(e^1_\alpha)>0$, such that on this set for any x', x'' holds:*

$$\rho(Fx'_{e^1_\alpha},Fx''_{e^1_\alpha})\le\sigma\rho(x'_{e^1_\alpha},x''_{e^1_\alpha})\quad 0<\sigma<1.$$

Then *for any function $f\in X$ and for any set $e^2_\gamma\in\{e^2_\nu\}$, $\mu(\Omega\setminus e^2_\gamma)>0$, there exists the unique solution to the equation*

$$x(t)-(Fx)(t)=f(t),\quad t\in e^2_\gamma, \tag{5.11}$$

which can be found by the method of successive approximations.

Proof. In virtue of the conditions and using the Banach principle one can obtain that that on the set e^1_α, $\mu(e^1_\alpha)>0$, for any function $f\in X$ there exists the unique solution to (5.6), which can be found via the method of successive approximations. Furthermore, in virtue of Theorem 5.2, this solution can be extended on any set $e^2_\gamma\in\{e^2_\nu\}$, $\mu(\Omega\setminus e^2_\gamma)>0$, in such a way, that it will satisfy (5.11). □

The following example emphasizes the essence of the condition (5.9).

Example 5.2. Let us consider the scalar equation

$$x(t)-(Tx)(t)=\frac{1}{2},\quad t\in[0,1], \tag{5.12}$$

where operator $T: L_\infty([0,1],\Sigma,m;R)\to L_\infty([0,1],\Sigma,m;R)$ is defined as follows

$$(Tx)(t)=a(t)x(\tau(t)),\quad t\in[0,1]. \tag{5.13}$$

Here

$$a(t)=\begin{cases}\frac{1}{2}, & t\in[0,\frac{1}{2}],\\ 1, & t\in[\frac{1}{2},1],\end{cases}$$

$$\tau(t)=\begin{cases}\frac{1}{2}t, & t\in[0,\frac{1}{2}],\\ t-\frac{1}{2^{n+1}}, & t\in(\sum_{k=0}^{n}\frac{1}{2^{k+1}},\sum_{k=0}^{n+1}\frac{1}{2^{k+1}}],\quad n=0,1,....\end{cases}$$

For (5.12) in the space $L_\infty([0,1],\Sigma,m;R)$ all the conditions of Corollary 5.1 are fulfilled. Thus, on any segment $[0,\gamma]$, $\gamma<1$, there exists the unique solution to (5.12):

$$x(t)=\begin{cases}1, & t\in[0,\frac{1}{2}],\\ 1+\frac{n+1}{2}, & t\in(\sum_{k=0}^{n}\frac{1}{2^{k+1}},\sum_{k=0}^{n+1}\frac{1}{2^{k+1}}],\quad n=0,1,....\end{cases} \tag{5.14}$$

We just want to point out that the function x defined by (5.14) is not bounded on $[0,1]$.

6. Representation Theorem

Here we will formulate a statement on representation of a Volterra operator involved in the theory of functional differential equations with delayed argument.

Let us first recall few necessary definitions from [10].

Here again $X_i := X_i(\Omega_i, \Sigma_1, \mu_i; \mathcal{Y}_i), \quad i = 1, 2.$

Definition 6.1. Operator $T : X_1 \to X_2$ is called **full comemory**, if for any collection $\{e_{1i} \in \Sigma_1\}$, satisfying condition

$$\Omega_1 = \cup_i e_{1i}, \tag{6.1}$$

there exists a collection $\{e_{2i} \in \Sigma_2\}, \quad e_{2i} \in \mathrm{Comem}_T(e_{1i})$, such that the following equality

$$\Omega_2 = \cup_i e_{2i}, \tag{6.2}$$

holds.

Let $E_j \in \Sigma_j, \quad j = 1, 2$. Let us define by $\Sigma_j(E_j)$ a restriction of Σ_j on E_j, $j = 1, 2$. For any $e_1 \in \Sigma_1$ define a family $\mathrm{Comem}(E_2)_T(e_1)$ as follows:

$$\mathrm{Comem}(E_2)_T(e_1) = \{e_2 \in \Sigma_2(E_2) : x|_{e_1} = y_{e_1} \Rightarrow Tx|_{e_2} = Ty|_{e_2}\}. \tag{6.3}$$

If $E_2 = \Omega_2$, then E_2 in (6.3) will be omitted.

Definition 6.2. We say that operator $T : X_1 \to X_2$ satisfies **I**-condition ($T \in \mathbf{I}$) if there exists a collection of disjoint sets $E_{2i}, \quad E_{2i} \in \Sigma_2, \quad i = 1, 2, \ldots$, such that

$$\Omega_2 = \bigsqcup_i E_{2i},$$

and for any set $e_2 \in \Sigma_2$ the following equalities hold:

$$\max \mathrm{Comem}(E_{2i})_T[inf \mathrm{Mem}_T(e_2 \cap E_{2i})] = e_2 \cap E_{2i}, \quad i = 1, 2, \ldots. \tag{6.4}$$

Theorem 6.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be standard measure spaces. Let for a continuous operator $F : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_2(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$ the following conditions are fulfilled:*

1. *There exists a pair of chains $\{e^1_\nu\} \subset \Sigma_1, \quad \{e^2_\nu\} \subset \Sigma_2$ such that F is Volterra with respect to it.*

2. *Opertator F is full comemory.*

3. *$\mathrm{Comem}_F(\emptyset_{\mu_1}) = \emptyset_{\mu_2}$.*

4. *$F \in \mathbf{I}$.*

Then *F can be represented as*

$$(Fx)(t) = f(t, x(g(t))) \quad \text{for} \quad \mu_2 - a.e. \quad t \in \Omega_2 \tag{6.5}$$

for some Carathéodory function $f : \Omega_2 \times \mathcal{Y}_1 \to \mathcal{Y}_2$, a measurable function $g : \Omega_2 \to \Omega_1$, satisfying the conditions:

i). $(\forall e_1) \in \Sigma_1 \quad \mu_2(g^{-1}(e_1)) = 0$ *when* $\mu_1(e_1) = 0$,

ii). $(\forall e^2_\alpha \in \{e^2_\nu\}) \quad \mu_2(e^2_\alpha \setminus g^{-1}(e^1_\alpha)) = 0$

Proof. The proof of the theorem is based on the following statement proved in [10]. Here we will quote it as a lemma, using the notions of the present paper.

Lemma 6.1. [10] *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be standard measure spaces. Any continuous full comemory operator* $F : X_1(\Omega_1, \Sigma_1, \mu_1; \mathcal{Y}_1) \to X_1(\Omega_2, \Sigma_2, \mu_2; \mathcal{Y}_2)$, *satisfying Conditions 3 and 4 of the theorem, can be represented as*

$$(Fx)(t) = f(t, x(g(t)) \quad \textit{for} \quad \mu_2 - a.c \quad t \in \Omega_2 \tag{6.6}$$

for some Carathéodory function $f : \Omega_2 \times \mathcal{Y}_1 \to \mathcal{Y}_2$, *a measurable function* $g : \Omega_2 \to \Omega_1$, *satisfying the Condition* **i)** *of the theorem, and every* $x \in X_1$.

To complete the proof it is enough to note that the validity of Condition **ii)** is a corollary of the assumption on Volterra property of F with respect to the pair of the chains $\{e^1_\nu\} \subset \Sigma_1, \quad \{e^2_\nu\} \subset \Sigma_2$. □

The notions of **full comemory** operator and **I**-condition were studied in detail in our paper [10]. One can also find there some illustrative examples.

In conclusion we would like to call attention to Condition 3 in Definition 3.1. This condition means that only spaces with continuous measure are under consideration. Evidently, including into examination spaces with a more general measure, one has to revise the mentioned condition and thus generalize the notion of chain.

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