Journal of Nonlinear and Convex Analysis
Volume 6, Number 2, 2005, 359–374

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

NONCONVEX SWEEPING PROCESS AND PROX-REGULARITY IN HILBERT SPACE

MESSAOUD BOUNKHEL AND LIONEL THIBAULT

Abstract. Several papers have been recently devoted to properties of prox-regular (or proximally smooth) nonconvex sets in Hilbert spaces. Motivated by the study of differential inclusions defined by nonconvex sweeping process, we establish new characterizations of prox-regular sets S in terms of the subdifferential of the distance function d_S associated with S. Using these characterizations we prove an existence result of the perturbed nonconvex sweeping process in Hilbert space.

1. Introduction

J.J. Moreau introduced and thoroughly studied in a series of papers (see [23, 24, 25] for references) the general class of differential inclusion known as the sweeping process and given by

$$\dot{x}(t) \in -N(C(t); x(t)), \quad x(0) = x_0 \in C(0). \tag{1.1}$$

There, $C(t)$ is a closed convex set moving in an absolutely continuous way in an infinite Hilbert space and $N(C(t); \cdot)$ denotes the usual normal cone. This evolution differential inclusion corresponds to several important mechanical problems.

Later, C. Castaing [6] introduced some new techniques from which many results can be derived, essentially the existence of a solution of (1.1) for $C(t) = S + v(t)$, where S is a fixed nonconvex closed set and v is a mapping with finite variation. The first general study of inclusion (1.1) with a nonconvex set $C(t)$ has been realized by M. Valadier [30, 31] who obtained, in the finite dimensional setting, existence of solution for (1.1) whenever the set-valued mapping $(t, x) \to N^C(C(t); x)$ has a closed graph. The problem (1.1) is then considered with the natural Clarke normal cone $N^C(C(t); \cdot)$. The main and rich example of such sets $C(t)$ provided in [31] is that of complements of open convex sets. Very recently, H. Benabdellah [2] and G. Colombo and V. V. Goncharov [13] independently proved the existence of a solution of (1.1) for general nonconvex sets moving in a Lipschitz way in a finite dimensional space. The connection of (1.1) with an appropriate differential inclusion, associated with the subdifferential of the distance function d_C, has been provided by L. Thibault [29]. That approach yields another proof of existence of (1.1) via a viable solution to some differential inclusion with convex compact values. In fact, that method provides an existence result for the more general differential inclusion

$$-\dot{x}(t) \in N^C(C(t); x(t)) + F(t, x(t)), \quad x(0) = x_0 \in C(0), \tag{1.2}$$

2000 *Mathematics Subject Classification.* 49J52, 46N10 58C20.

Key words and phrases. prox-regular set, nonconvex sweeping process.

where $F(t,.)$ is an upper semicontinuous set-valued mapping with convex compact values. The class of inclusions (1.2) appears in particular in mathematical economy. It corresponds for $C(t) = S$ (independent of t) to a modelisation of planning procedures introduced by C. Henry [20] for S convex and also considered by B. Cornet [15, 16] for S Clarke tangentially regular.

All the results recalled in the last paragraph above have been proved for H finite dimensional. The first existence result of (1.1) for nonconvex sets in infinite dimensional Hilbert space has been established by C. Colombo and V. V. Goncharov [13] for ϕ-convex sets $C(t)$. Here, we continue the study of reduction to a differential inclusion associated with the subdifferential of the distance function d_C as introduced by Thibault [29]. In the setting of infinite dimensional space, it appears that closedness properties with respect to the time variable t is essential. Those properties are strongly connected with the concept of prox-regularity for the sets $C(t)$. The main purpose of the paper is to show how such properties allow us to study the differential inclusion (1.2) in the general framework of infinite dimensional Hilbert space.

After recalling the needed concepts in the second section, the third section is then devoted to establish several characterizations of prox-regular (or proximally smooth) sets in terms of distance functions. Those characterizations (as it will be seen) are appropriate for moving sets and they enlarge the lists of characterizations of prox-regular sets given in F. H. Clarke et al. [11], H. Federer [19], R. A. Poliquin et al. [26]. In Section 4, we prove the closedness property of the subdifferential $\partial d_{C(t)}$ of the distance function $d_{C(t)}$ with respect to the parameter t. This result is then used to obtain an existence theorem for (1.2) in Hilbert space when the sets $C(t)$ are prox-regular.

2. Preliminaries

In all the paper H will be a real Hilbert space, and for a closed subset S of H, we will denote by $d_S(.)$ the usual distance function to the subset S, i.e., $d_S(x) := \inf_{u \in S} \|x - u\|$. First we need to recall, in this section, some notions that will be used in all the paper.

Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and let x be any point where f is finite. We recall that:

•) *The Clarke subdifferential* of f at x is defined by (see [27])

$$\partial^C f(x) = \{\xi \in H : \langle \xi, h \rangle \leq f^{\uparrow}(x; h), \quad \text{for all } \ h \in H\},$$

where $f^{\uparrow}(x; h)$ is *the generalized Rockafellar directional derivative* given by

$$f^{\uparrow}(x; h) := \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \inf_{h' \to h} t^{-1}\big[f(x' + th') - f(x')\big],$$

where $x' \longrightarrow^f x$ means $x' \longrightarrow x$ and $f(x') \longrightarrow f(x)$. We refer to [27] for the meaning of this mixed limit.

•) *The proximal subdifferential* $\partial^P f(x)$ is (see [12, 28]) the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\mathbb{B}$

$$\langle \xi, x' - x \rangle \leq f(x') - f(x) + \sigma\|x' - x\|^2.$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H.

Note that one always has $\partial^P f(x) \subset \partial^C f(x)$ and by convention we set $\partial^P f(x) = \partial^C f(x) = \emptyset$ if $f(x)$ is not finite. Note also that if f is locally Lipschitz around x, then the generalized Rockafellar directional derivative $f^{\uparrow}(x;h)$ coincides with *the Clarke directional derivative* $f^0(x;h)$ defined by

$$f^0(x;h) := \limsup_{\substack{x' \to x \\ t \downarrow 0}} t^{-1}\big[f(x' + th) - f(x')\big].$$

Let S be a nonempty closed subset of H and x be a point in S. Let us recall (see [10, 11, 27]) that *the Clarke normal cone* (resp. *the proximal normal cone*) of S at x is defined by $N^C(S;x) := \partial^C \psi_S(x)$ $\Big($resp.$N^P(S;x) := \partial^P \psi_S(x)\Big)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x') = 0$ if $x' \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P(S;x) := \{\xi \in H : \exists r > 0 \text{ s.t. } d_S(x + r\xi) = r\|\xi\|\}.$$

3. Further characterizations of prox-regular sets

First we begin by recalling that, for a given $r \in]0, +\infty]$, a subset S is uniformly r-prox-regular (see [26]) or equivalently r-proximally smooth (see [11]) if and only if every nonzero proximal normal to S can be realized by an r-ball. This means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S;\bar{x})$ one has

$$S \cap int(\bar{x} + r\frac{\xi}{\|\xi\|} + r\mathbb{B}) = \emptyset, \; i.e., \; \langle \frac{\xi}{\|\xi\|}, x - \bar{x} \rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ and we will just say in the sequel that S is *r-prox-regular.* Recall that for $r = +\infty$, the r-prox-regularity of S is equivalent to the convexity of S. The following proposition summarizes some important consequences of the prox-regularity (or proximal smoothness) property needed in the sequel of the paper. For the proof of these results we refer the reader to [11, 26]. We will denote by $Proj_S(x)$ the set of nearest points of x in S. When this set has a unique point, we will use the notation $proj_S(x)$.

Proposition 3.1. *Let S be a nonempty closed subset in H and let $r > 0$. If the subset S is r-prox-regular, then the following hold:*

i) *For all $x \in H$ with $d_S(x) < r$, $proj_S(x)$ exists;*
ii) *For every $r' \in]0, r[$, the enlarged subset $Enl\,(S, r') := \{x \in H : d_S(x) \leq r'\}$ is $(r - r')$-prox-regular;*
iii) *The Clarke and the proximal subdifferentials of d_S coincide at all points $x \in H$ with $d_S(x) < r$.*

For several other important geometric concepts of regularity in nonsmooth analysis, we refer to [4, 10, 14].

The following proposition shows that, in the inequality above characterizing the prox-regularity, one may use the proximal subdifferential of the distance function in place of the proximal normal cone. For a given subset S in H and a given $r > 0$ we will set

$$(P_r) \qquad \begin{cases} \text{for all } \bar{x} \in S \text{ and all } 0 \neq \xi \in N^P(S;\bar{x}) \text{ one has} \\ \big\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \big\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2 \quad \text{for all } x \in S \end{cases}$$

and

$$(P'_r) \qquad \begin{cases} \text{for all } \bar{x} \in S \text{ and all } \xi \in \partial^P d_S(\bar{x}) \text{ one has} \\ \big\langle \xi, x - \bar{x} \big\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2 \quad \text{for all } x \in S. \end{cases}$$

Proposition 3.2. *Let S be a nonempty closed subset in H and let $r > 0$. Then $(P_r) \Leftrightarrow (P'_r)$.*

Proof. ($(P_r) \Rightarrow (P'_r)$). Assume that S satisfies (P_r). The property (P'_r) obviously holds for $\xi = 0$. Let $\bar{x} \in S$ and $0 \neq \xi \in \partial^P d_S(\bar{x}) \subset N^P(S;\bar{x})$. Then by (P_r) one has for all $x \in S$

$$\big\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \big\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

and hence

$$\big\langle \xi, x - \bar{x} \big\rangle \leq \frac{\|\xi\|}{2r}\|x - \bar{x}\|^2 \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

because $\|\xi\| \leq 1$. The property (P'_r) then holds.

($(P'_r) \Rightarrow (P_r)$). Now assume that S satisfies (P'_r). Let $\bar{x} \in S$ and $0 \neq \xi \in N^P(S;\bar{x})$. Then by Theorem 4.1 in Bounkhel and Thibault [4] one has $\frac{\xi}{\|\xi\|} \in \partial^P d_S(\bar{x})$ and hence one gets (by (P'_r))

$$\big\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \big\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2$$

for all $x \in S$. This completes the proof of the second implication and so the proof of the proposition is finished. □

The following lemma has been established in Bounkhel and Thibault [4] in the context of a general normed vector space. It will be used in the proof of the next theorem. For the convenience of the reader, we show how the Hilbert norm allows us to give another simple proof. The reader will also note that the arguments work for any Kadec norm of a reflexive Banach space (see for example, [17, 18] for the definition and properties of Kadec norms).

Lemma 3.3. *Let S be a nonempty closed subset in H and let $r > 0$. Then for all $x \notin Enl\,(S, r)$ one has*

$$d_{Enl\,(S,r)}(x) = d_S(x) - r. \tag{3.1}$$

Proof. As the set $\{x \notin Enl\,(S, r) : \ Proj_S(x) \neq \emptyset\}$ is dense in $X \setminus Enl\,(S, r)$ by [21], and as the functions d_S and $d_{Enl\,(S,r)}$ are continuous, it is enough to prove (3.1) only for points $x \notin Enl\,(S, r)$ satisfying $Proj_S(x) \neq \emptyset$. Fix any such point x and fix also p in S such that $d_S(x) = \|x - p\|$. Set

$$u := p + \Big(\frac{r}{\|x - p\|}\Big)(x - p). \tag{3.2}$$

We observe that u is in $Enl\,(S,r)$ because (3.2) and the relation $p \in S$ ensure $d_S(u) \leq \|u - p\| = r$.

Now let us prove that $u \in Proj_{Enl\,(S,r)}(x)$. Consider any $y \in Enl\,(S,r)$, that is, $d_S(y) \leq r$, and fix any positive number ϵ. We may choose some $y_\epsilon \in S$ satisfying

$$\|y - y_\epsilon\| \leq d_S(y) + \epsilon \leq r + \epsilon.$$

Consequently

$$\|y - x\| \geq \|y_\epsilon - x\| - \|y_\epsilon - y\| \geq \|x - p\| - r - \epsilon = \|x - u\| - \epsilon,$$

and this yields $d_{Enl\,(S,r)}(x) \geq \|x - u\| - \epsilon$. As this holds for all $\epsilon > 0$, we have $d_{Enl\,(S,r)}(x) \geq \|x - u\|$ and hence $d_{Enl\,(S,r)}(x) = \|x - u\|$ because u is in $Enl\,(S,r)$ as observed above. According to (3.2), we obtain

$$d_{Enl\,(S,r)}(x) = \|x - u\| = \|x - p\| - r = d_S(x) - r,$$

and hence the proof of the lemma is finished. □

Now we establish the main result of this section from which some new characterizations of r-prox-regular sets will be derived. Here the point where the proximal subdifferential of d_S is considered is not required to stay in S contrarily to Proposition 3.1.

Theorem 3.4. *Let S be a nonempty closed subset in H and let $r > 0$. Assume that S is r-prox-regular. Then the following hold:*

(a)

$$(P''_r) \qquad \begin{cases} \text{for all } x \in H, \text{ with } d_S(x) < r, \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\ \langle \xi, x' - x \rangle \leq \dfrac{8}{r - d_S(x)}\|x' - x\|^2 + d_S(x') - d_S(x), \\ \text{for all } x' \in H \text{ with } d_S(x') \leq r. \end{cases}$$

(b)

$$(P'''_r) \qquad \begin{cases} \text{for all } x \in S \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\ \langle \xi, x' - x \rangle \leq \dfrac{2}{r}\|x' - x\|^2 + d_S(x'), \\ \text{for all } x' \in H \text{ with } d_S(x') < r. \end{cases}$$

Proof. I) We begin with the proof of (b). Fix any $x \in S$ and any $\xi \in \partial^P d_S(x)$. Fix also any $z \in H$ satisfying $d_S(z) < r$. As S is r-prox-regular one can find some $y_z \in Proj_S(z) \neq \emptyset$, that is, y_z is in S and

$$\|z - y_z\| = d_S(z). \tag{3.3}$$

Then

$$\|y_z - x\| \leq \|y_z - z\| + \|z - x\| \leq 2\|z - x\|,$$

and hence by (P'_r) (see Proposition 3.2) and the inequality $\|\xi\| \le 1$, and also by the equality (3.3) one gets

$$\begin{aligned}\langle \xi, z - x\rangle &= \langle \xi, y_z - x\rangle + \langle \xi, z - y_z\rangle \\ &\le \frac{1}{2r}\|y_z - x\|^2 + \|\xi\|\|y_z - z\| \\ &\le \frac{2}{r}\|z - x\|^2 + d_S(z) - d_S(x).\end{aligned}$$

This completes the proof of (P'''_r).

II) Note that (see Proposition 3.1) for every $0 < r' < r$ the enlarged set $Enl\,(S, r')$ is $(r - r')$-prox-regular. Further, for any $u' \in H$ it can be seen that the inequality $d_{Enl\,(S,r')}(u') < r - r'$ holds if and only if $d_S(u') < r$. Indeed, if we suppose that $d_{Enl\,(S,r')}(u') < r - r'$, then there exists some z in H with $d_S(z) \le r'$ and $\|u' - z\| < r - r'$, and hence

$$d_S(u') \le d_S(z) + \|u' - z\| < r.$$

Now Suppose that $d_S(u') < r$. In the case $u' \in Enl\,(S, r')$, we can write $d_{Enl\,(S,r')}(u') = 0 < r - r'$. In the other case $u' \notin Enl\,(S, r')$, we have according to Lemma 3.1

$$d_{Enl\,(S,r')}(u') = d_S(u') - r' < r - r'.$$

The equivalence then holds, and hence the property $(P'''_{(r-r')})$, which holds because of (I), may be written as

$$(P'''_{(r-r')}) \qquad \begin{cases} \text{for all } u \in Enl\,(S, r'), \text{ and all } \zeta \in \partial^P d_{Enl\,(S,r')}(u) \text{ one has} \\ \langle \zeta, u' - u\rangle \le \dfrac{2}{(r - r')}\|u' - u\|^2 + d_{Enl\,(S,r')}(u'), \\ \text{for all } u' \in H \text{ with } d_S(u') < r. \end{cases}$$

Now, fix any $x \in H$ with $d_S(x) < r$ and any $\xi \in \partial^P d_S(x)$. We distinguish two cases:

case 1) If $x \in S$, then by (P'''_r) one obtains for all $x' \in H$ with $d_S(x') < r$

$$\langle \xi, x' - x\rangle \le \frac{2}{r}\|x' - x\|^2 + d_S(x') - d_S(x). \tag{3.4}$$

case 2) If $x \notin S$, we put $r' := d_S(x) > 0$ in this case. Firstly, one observes that $\xi \in \partial^P d_{Enl\,(S,r')}(x)$. Indeed, one knows by Theorems 4.1 and 4.3 in Bounkhel and Thibault [4] (see also Theorem 3.2 in [11] for the equality in the following relation) that

$$\partial^P d_S(x) = N^P(Enl\,(S, r'), x) \cap \{\zeta : \|\zeta\| = 1\} \subset \partial^P d_{Enl\,(S,r')}(x),$$

and hence as ξ is fixed in $\partial^P d_S(x)$, one then gets $\xi \in \partial^P d_{Enl\,(S,r')}(x)$. Applying $(P'''_{(r-r')})$ in the form obtained above one gets for any $x' \in H$ with $d_S(x') < r$

$$\langle \xi, x' - x\rangle \le \frac{2}{r - r'}\|x' - x\|^2 + d_{Enl\,(S,r')}(x').$$

Consequently, for any $x' \in H$ satisfying $d_S(x') < r$ and $x' \notin Enl\,(S, r')$ (that is, $r' < d_S(x') < r$) one gets according to Lemma 3.1

$$\langle \xi, x' - x \rangle \leq \frac{2}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x). \tag{3.5}$$

Now fix any $x' \in H$ satisfying $d_S(x') < r$ and $x' \in Enl\,(S, r')$. We begin by noting that $(P'''_{(r-r')})$ ensures that the inequality

$$\langle \xi, y - x \rangle \leq \frac{2}{r - r'} \|y - x\|^2 \tag{3.6}$$

holds for all $y \in H$ with $d_S(y) \leq r'$. Now choose according to the equality $\|\xi\| = 1$, some $u \in H$ with $\|u\| = 1$ and such that $\langle \xi, u \rangle = 1$. Put $t := d_S(x) - d_S(x') \geq 0$, the number t being nonnegative because $x' \in Enl\,S(, r')$. Then $x' + tu \in Enl\,(S, r')$, because $d_S(x' + tu) \leq d_S(x') + t = d_S(x) = r'$. Therefore (3.6) allows us to write

$$\begin{aligned} \langle \xi, x' - x \rangle &= \langle \xi, x' + tu - x \rangle - \langle \xi, tu \rangle \\ &\leq \frac{2}{r - r'} \|x' + tu - x\|^2 - t. \end{aligned} \tag{3.7}$$

Observing that

$$\|x' + tu - x\| \leq \|x' - x\| + t \leq 2\|x' - x\|,$$

we deduce from (3.7)

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x).$$

It then follows from (3.4), (3.5) and the last inequality that one has for all $x \in H$ with $d_S(x) < r$ and all $\xi \in \partial^P d_S(x)$

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x) \text{ for all } x' \in H \text{ with } d_S(x') < r.$$

Taking the continuity of both members of that inequality with respect to x' into account, we may replace the requirement $d_S(x') < r$ by $d_S(x) \leq r$. The proof of the theorem is then complete. □

The following corollary of the theorem above adds some further characterizations of prox-regular sets to the lists in Clarke et al. [11] and Poliquin et al. [26]. For the concepts of Fréchet and limiting subdifferentials, we refer to [22].

Corollary 3.5. *Let S be a nonempty closed subset of H and let $r > 0$. Then, the following assertions are equivalent:*

(a) *S is r-prox-regular;*
(b) *the property (P''_r) holds for the proximal subdifferential of d_S;*
(c) *the property (P''_r) holds for the Fréchet subdifferential of d_S;*
(d) *the property (P''_r) holds for the limiting subdifferential of d_S;*
(e) *the property (P''_r) holds for the Clarke subdifferential of d_S.*

Proof. The implication (a)⇒(b) follows from Theorem 3.1 and (b)⇒(c) holds because any $\xi \in \partial^F d_S(x)$ is the weak limit of a sequence $(\xi_n)_n$ such that $\xi_n \in \partial^P d_S(x_n)$ and $(x_n)_n$ converges to x. In the same way, the implication (c)⇒(d) is true. The implication (d)⇒(e) can be seen easily as a consequence of the definition of (P''_r)

and of the formula (see for example [10]) characterizing the Clarke subdifferential of a Lipschitz function as the closed convex hull of its limiting subdifferential. So, it remains to see (e)⇒(a). We know that $\partial^C d_S(x)$ is nonempty at any x (see [10]). Suppose that (e) holds. This property tells us that any Clarke subgradient is a proximal subgradient. Therefore, for any $x \in H$ with $d_S(x) < r$ we have $\partial^P d_S(x) \neq \emptyset$. The implication is thus a consequence of Corollary 4.3 in [26] or Theorem 4.1 in [11]. □

Observe that the assertion (e) in the corollary entails that the Clarke and proximal (and hence also the Fréchet) subdifferentials of d_S coincide at all points $x \in H$ with $d_S(x) < r$ provided that S is r-prox-regular. In fact, it is easily seen that this equality property of these subdifferentials characterizes the r-prox-regularity of sets.

4. Nonconvex sweeping process

Our purpose, in this section, is to show how our results established in the previous section allow us to study the differential inclusion (1.2) for prox regular sets in Hilbert space.

Let C be a set-valued mapping from an interval $I \subset \mathbb{R}$ into closed subsets of H satisfying for any $y \in H$ and any $t, t' \in I$

$$|d_{C(t)}(y) - d_{C(t')}(y)| \leq |v(t) - v(t')|, \tag{4.1}$$

where $v : I \to \mathbb{R}$ is a continuous function. This means that C has a continuous variation with respect to the Hausdorff distance. We start with an important result of closedness of the proximal subdifferential of the distance function to images of set-valued mappings whose images are prox-regular. It has its own interest.

Proposition 4.1. *Let $r > 0$. Assume that $C(t)$ is r-prox-regular for all t in the interval I . For a given $0 < \delta < r$, the following closedness property of the proximal subdifferential of the distance function holds:*

"for any $\bar{t} \in I$, $\bar{x} \in C(\bar{t}) + (r - \delta)\mathbb{B}$, $x_n \to \bar{x}$, $t_n \to \bar{t}$ with $t_n \in I$, (x_n is not necessarily in $C(t_n)$) and $\xi_n \in \partial^P d_{C(t_n)}(x_n)$ with $\xi_n \to^w \bar{\xi}$, one has $\bar{\xi} \in \partial^P d_{C(\bar{t})}(\bar{x})$." Here $\to^w$ means the weak convergence in H.

Proof. Fix $\bar{t} \in I$ and $\bar{x} \in C(\bar{t}) + (r - \delta)\mathbb{B}$. As $x_n \to \bar{x}$ one gets for n sufficiently large $x_n \in \bar{x} + \frac{\delta}{4}\mathbb{B}$. On the other hand, since the subset $C(\bar{t})$ is r-prox-regular, one can choose (by Proposition 3.1) a point $\bar{y} \in C(\bar{t})$ with $d_{C(\bar{t})}(\bar{x}) = \|\bar{y} - \bar{x}\|$. So, for every n large enough one can write by (4.1),

$$|d_{C(t_n)}(x_n) - d_{C(\bar{t})}(\bar{y}))| \leq |v(t_n) - v(\bar{t})| + \|x_n - \bar{y}\|,$$

and hence the continuity of v yields for n large enough

$$d_{C(t_n)}(x_n) \leq \frac{\delta}{4} + \|x_n - \bar{x}\| + \|\bar{x} - \bar{y}\| \leq \frac{\delta}{4} + \frac{\delta}{4} + r - \delta = r - \frac{\delta}{2} < r.$$

Therefore, for any n large enough, we apply the property (P''_r) in Theorem 3.1 with $\xi_n \in \partial^P d_{C(t_n)}(x_n)$ to get

$$\langle \xi_n, u - x_n \rangle \leq \frac{8}{r - d_{C(t_n)}(x_n)} \|u - x_n\|^2 + d_{C(t_n)}(u) - d_{C(t_n)}(x_n), \tag{4.2}$$

for all $u \in H$ with $d_{C(t_n)}(u) < r$. This inequality still holds for all $u \in \bar{x} + \delta'\mathbb{B}$ with $0 < \delta' < \dfrac{\delta}{4}$ because for such u one has

$$d_{C(t_n)}(u) \leq \|u - \bar{x}\| + \|\bar{x} - x_n\| + d_{C(t_n)}(x_n) \leq \delta' + \frac{\delta}{4} + r - \frac{\delta}{2} < r.$$

Consequently, by the continuity (because of (4.1)) of the distance function with respect to (t, x), the inequality (4.2) gives, by letting $n \to +\infty$,

$$\langle \bar{\xi}, u - \bar{x} \rangle \leq \frac{8}{r - d_{C(\bar{t})}(\bar{x})} \|u - \bar{x}\|^2 + d_{C(\bar{t})}(u) - d_{C(\bar{t})}(\bar{x}) \ \text{ for all } \ u \in \bar{x} + \delta'\mathbb{B}.$$

This ensures that $\bar{\xi} \in \partial^P d_{C(\bar{t})}(\bar{x})$ and so the proof of the proposition is complete. $\square$

Remark 4.1. One obtains the same result if $C(t)$ is $r(t)$-prox-regular with either $r(t)$ is bounded below by a positive number $\alpha > 0$ (i.e., $r(t) > \alpha > 0$, for all $t \in I$) or $r(\cdot)$ is a continuous positive function at $\bar{t}$.

Now we recall some notation needed in the next theorem. Let $T > 0$ and put $I := [0, T]$. A *solution* $x(\cdot)$ of the sweeping process (SP) (see below) is taken to mean an absolutely continuous mapping $x(\cdot) : I \to H$ satisfying (together with $\dot{x}(\cdot)$, its derivative with respect to t) the following:

$$(SP) \qquad \begin{cases} \dot{x}(t) \in -N^C\big(C(t); x(t)\big) \text{ a.e. } t \in I \\ x(0) = x_0 \in C(0). \end{cases}$$

By the definition of the Clarke normal cone, any solution of (SP) must satisfy $x(t) \in C(t)$ for all $t \in I$ whenever (4.1) holds.

The following existence theorem establishes our main result in this section. The result is proved by showing that the Moreau catching-up algorithm (introduced for convex sets in [25]) still converges for prox-regular sets. More generally, the catching-up algorithm will be used to get solution of the sweeping process with a perturbation set-valued mapping. For the concept of measurability of set-valued mappings, we refer to [9].

Theorem 4.2. *Let H be a separable Hilbert space, $T > 0$, and $r > 0$. Assume that $C(t)$ is r-prox-regular for every $t \in I := [0, T]$ and that the assumption (4.1) holds with an absolutely continuous function v. Let $F : I \times H \to H$ be a set-valued mapping with closed convex values in H such that $F(t, \cdot)$ is upper semicontinuous on H for any fixed $t \in I$ and $F(\cdot, x)$ admits a measurable selection on I for any fixed $x \in H$. Assume that, for some fixed convex compact set $\mathcal{K} \subset H$, one has $F(t, x) \subset \mathcal{K}$ for all $(t, x) \in I \times H$. Then, for any $x_0 \in C(0)$, the sweeping process (SPP) with the perturbation F has at least one absolutely continuous solution, that is, there exists an absolutely continuous mapping $x : I \to H$ such that*

$$(SPP) \qquad \begin{cases} -\dot{x}(t) \in N^C\big(C(t); x(t)\big) + F(t, x(t)) \ \textit{a.e. } t \in I \\ x(0) = x_0 \in C(0). \end{cases}$$

Proof. We may suppose that the convex compact set $\mathcal{K}$ is symmetric.

I) First we assume that F is globally upper semicontinuous on $I \times H$ and we prove the conclusion of the theorem.

Step 1. Observing that (4.1) ensures for $t \le t'$

$$|d_{C(t')}(y) - d_{C(t)}(y)| \le \int_t^{t'} |\dot{v}(s)| ds,$$

we may suppose (replacing $\dot{v}$ by $|\dot{v}|$ if necessary) that $\dot{v}(t) \ge 0$ for all $t \in I$. Consider for every $n \in \mathbb{N}$, the following partition of I:

$$t_{n,i} := \frac{iT}{2^n} \ (0 \le i \le 2^n) \quad \text{and} \quad I_{n,i} :=]t_{n,i}, t_{n,i+1}] \text{ if } 0 \le i \le 2^n - 1. \tag{4.3}$$

Put

$$\mu_n := \frac{T}{2^n}, \ \epsilon_{n,i} := \int_{t_{n,i}}^{t_{n,i+1}} \dot{v}(s) ds, \ \text{and } \epsilon_n := \max_{0 \le i < 2^n} \{\mu_n + \epsilon_{n,i}\}. \tag{4.4}$$

As $\epsilon_n \to 0$, we can fix $n_0 \ge 1$ satisfying for every $n \ge n_0$

$$2\mu_n < \frac{r}{(2l+1)} \text{ and } 2\epsilon_n < \min\{1, \frac{r}{(4l+3)}\}, \tag{4.5}$$

where l is a positive number satisfying $\mathcal{K} \subset l\mathbb{B}$ (because $\mathcal{K}$ is a compact set in H). For every $n \ge n_0$, we choose by induction:

$$u_{n,0} := x_0; \quad z_{n,0} \in F(t_{n,0}, u_{n,0});$$

$$z_{n,i} \in F(t_{n,i}, u_{n,i});$$

$$u_{n,i+1} := proj_{C(t_{n,i+1})}(u_{n,i} - \mu_n z_{n,i}). \tag{4.6}$$

This last equality is well defined. Indeed, by (4.1) one has for all $t \in I$

$$d(u_{n,0} - \mu_n z_{n,0}, C(t)) \le l\mu_n + v(t) - v(t_{n,0}).$$

Then for $t := t_{n,1}$ one gets (by (4.4) and(4.5))

$$d(u_{n,0} - \mu_n z_{n,0}, C(t_{n,1})) \le l\mu_n + v(t_{n,1}) - v(t_{n,0}) \le (l+1)\epsilon_n \le \frac{r}{2} < r$$

and hence as C has r-prox-regular values, by Proposition 3.1 one can choose the point $u_{n,1} := proj_{C(t_{n,1})}(u_{n,0} - \mu_n z_{n,0})$. Similarly, we can define, by induction, the points $(u_{n,i})_{0 \le i \le 2^n}$ and $(z_{n,i})_{0 \le i \le 2^n}$. From (4.6) and (4.1) one deduces for every $0 \le i < 2^n$

$$\|u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}\| \le l\mu_n + \epsilon_{n,i} \le (l+1)(\mu_n + \epsilon_{n,i}). \tag{4.7}$$

For every $n \ge n_0$, these $(u_{n,i})_{0 \le i \le 2^n}$ and $(z_{n,i})_{0 \le i \le 2^n}$ are used to construct two mappings z_n and u_n from I to H by defining their restrictions to each interval $I_{n,i}$ as follows: for $t = 0$, set $z_n(0) := z_{n,0}$ and $u_n(0) := u_{n,0} = x_0$, and for all $t \in I_{n,i}$ $(0 \le i \le 2^n)$, set $z_n(t) := z_{n,i}$ and

$$u_n(t) := u_{n,i} + \frac{a(t) - a(t_{n,i})}{\epsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - (t - t_{n,i}) z_{n,i}, \tag{4.8}$$

where $a(t) := v(t) + t$ for all $t \in I$. Hence for every t and t' in $I_{n,i}$ $(0 \le i \le 2^n)$ one has

$$u_n(t') - u_n(t) = \frac{a(t') - a(t)}{\epsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - (t' - t) z_{n,i}.$$

Thus, in view of (4.7), if $t, t' \in I_{n,i}$ $(0 \le i < 2^n)$ with $t \le t'$, one obtains

$$\|u_n(t') - u_n(t)\| \le (2l+1)(a(t') - a(t)), \tag{4.9}$$

and, by addition this also holds for all $t, t' \in I$ with $t \le t'$. This inequality entails that u_n is absolutely continuous.

Coming back to the definition of u_n in (4.8), one observes that for $0 \le i < 2^n$

$$\dot{u}_n(t) = \frac{\dot{a}(t)}{\epsilon_{n,i} + \mu_n}(u_{n,i+1} - u_{n,i} + \mu_n z_{n,i}) - z_{n,i} \text{ for } \text{ a. e. } t \in I_{n,i}.$$

Then one obtains, in view of (4.7), for a. e. $t \in I$

$$\|\dot{u}_n(t) + z_n(t)\| \le (l+1)(\dot{v}(t) + 1). \tag{4.10}$$

Now, let θ_n, ρ_n be defined from I to I by $\theta_n(0) = 0$, $\rho_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \rho_n(t) = t_{n,i} \quad \text{if } t \in I_{n,i} \ (0 \le i < 2^n). \tag{4.11}$$

Then, by (4.6), the construction of u_n and z_n, and the properties of proximal normal cones to subsets, we have for a. e. $t \in I$

$$z_n(t) \in F(\rho_n(t), u_n(\rho_n(t))) \text{ and } \dot{u}_n(t) + z_n(t) \in -N^P(C(\theta_n(t)); u_n(\theta_n(t))). \tag{4.12}$$

This last inclusion, relation (4.10), and Theorem 4.1 in [4] entail for a. e. $t \in I$

$$\dot{u}_n(t) + z_n(t) \in -(l+1)\dot{a}(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))). \tag{4.13}$$

Step 2. Now, let us define $Z_n(t) := \displaystyle\int_0^t z_n(s)ds$. Observe that for all $t \in I$ the set $\{Z_n(t) : \ n \ge n_0\}$ is contained in the strong compact set $T\mathcal{K}$ and so it is relatively strongly compact in H. Then by Arzela-Ascoli's theorem we get the relative strong compactness of the set $\{Z_n : \ n \ge n_0\}$ with respect to the uniform convergence in $C(I, H)$ and so we may assume without loss of generality that (Z_n) converges uniformly to some mapping Z. As $\|z_n(t)\| \le l$, we may suppose that (z_n) converges weakly in $L^1(I, H, dt)$ to some mapping z. Then, for all $t \in I$,

$$Z(t) = \lim_n Z_n(t) = \lim_n \int_0^t z_n(s)ds = \int_0^t z(s)ds,$$

which gives that Z is absolutely continuous and $\dot{Z}(t) = z(t)$ for almost all $t \in I$.

Step 3. Now let us show that the sequence $(u_n)_n$ satisfies the Cauchy property in the space of continuous mappings $\mathcal{C}(I, H)$ endowed with the norm of uniform convergence. Fix $m, n \in \mathbb{N}$ such that $m \ge n_0$, $n \ge n_0$ and fix also $t \in I$ with $t \ne t_{m,i}$ for $i = 0, \ldots, 2^m$ and $t \ne t_{n,j}$ for $j = 0, \ldots, 2^n$. Observe by (4.1), (4.4), and (4.9) that

$$\begin{aligned} d_{C(\theta_n(t))}(u_m(t)) &= d_{C(\theta_n(t))}(u_m(t)) - d_{C(\theta_m(t))}(u_m(\theta_m(t))) \\ &\le |v(\theta_n(t)) - v(\theta_m(t))| + \|u_m(\theta_m(t)) - u_m(t)\| \\ &\le |\int_{\theta_m(t)}^{\theta_n(t)} \dot{v}(s)ds| + (2l+1)[\int_t^{\theta_m(t)} \dot{v}(s)ds + (\theta_m(t) - t)] \\ &\le \epsilon_m + \epsilon_n + (2l+1)\epsilon_m \end{aligned}$$

and hence, by (4.5) $d_{C(\theta_n(t))}(u_m(t)) < r$. Set $\delta(t) := (l+1)\dot{a}(t)$. Then, (4.13) and (P_r''') in Theorem 3.1 and also the last inequality above entail

$$\begin{aligned}
&\big\langle \dot{u}_n(t) + z_n(t), u_n(\theta_n(t)) - u_m(t)\big\rangle \\
&\le \frac{2\delta(t)}{r}\|u_n(\theta_n(t)) - u_m(t)\|^2 + \delta(t) d_{C(\theta_n(t))}(u_m(t)) \\
&\le \frac{2\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + \|u_n(\theta_n(t)) - u_n(t)\|\Big]^2 + \delta(t)(\epsilon_n + (2l+2)\epsilon_m),
\end{aligned}$$

and this yields by (4.4) and (4.9)

$$\begin{aligned}
&\big\langle \dot{u}_n(t) + z_n(t), u_n(\theta_n(t)) - u_m(t)\big\rangle \\
&\qquad \le \frac{2\delta(t)}{r}\Big[\|u_n(t) - u_m(t)\| + (2l+1)\epsilon_n\Big]^2 + \delta(t)(2l+2)(\epsilon_n + \epsilon_m).
\end{aligned} \tag{4.14}$$

Now Put $w_n(t) := u_n(t) + Z_n(t)$ for all $n \ge n_0$ and all $t \in I$ and put $\eta_n := \max\{\epsilon_n, \|Z_n - Z\|_\infty\}$. Then by (4.10) and (4.14) one gets

$$\begin{aligned}
&\big\langle \dot{w}_n(t), w_n(\theta_n(t)) - w_m(t)\big\rangle \\
&= \big\langle \dot{w}_n(t), u_n(\theta_n(t)) - u_m(t)\big\rangle + \big\langle \dot{w}_n(t), Z_n(\theta_n(t)) - Z_m(t)\big\rangle \\
&\le \frac{2\delta(t)}{r}\Big[\|w_n(t) - w_m(t)\| + \|Z_n(t) - Z_m(t)\| + (2l+1)\epsilon_n\Big]^2 \\
&\quad + \delta(t)(2l+2)(\epsilon_n + \epsilon_m) + \delta(t)\|Z_n(\theta_n(t)) - Z_m(t)\| \\
&\le \frac{2\delta(t)}{r}\Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n\Big]^2 \\
&\quad + 2\delta(t)(2l+2)(\eta_n + \eta_m).
\end{aligned}$$

This last inequality ensures by (4.10)

$$\begin{aligned}
&\big\langle \dot{w}_n(t), w_n(t) - w_m(t)\big\rangle \\
&\le \big\langle \dot{w}_n(t), w_n(t) - w_n(\theta_n(t))\big\rangle + 2\delta(t)(2l+2)(\eta_n + \eta_m) \\
&\quad + \frac{2\delta(t)}{r}\Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n\Big]^2 \\
&\le 3\delta(t)(2l+2)(\eta_n + \eta_m) \\
&\quad + \frac{2\delta(t)}{r}\Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_n\Big]^2.
\end{aligned}$$

In the same way, we also have

$$\begin{aligned}
\big\langle \dot{w}_m(t), w_m(t) - w_n(t)\big\rangle &\le 3\delta(t)(2l+2)(\eta_n + \eta_m) \\
&\qquad + \frac{2\delta(t)}{r}\Big[\|w_n(t) - w_m(t)\| + (\eta_n + \eta_m) + (2l+1)\eta_m\Big]^2.
\end{aligned}$$

It then follows from both last inequalities that we have for some positive constant α independent of m, n, and t (note that $\|w_n(t)\| \le lT + \|x_0\| + \int_0^T \dot{v}(s)ds$)

$$2\big\langle \dot{w}_m(t) - \dot{w}_n(t), w_m(t) - w_n(t)\big\rangle \le \alpha\delta(t)(\eta_n + \eta_m) + 8\frac{\delta(t)}{r}\|w_m(t) - w_n(t)\|^2,$$

and so, for some positive constants β and γ independent of m, n, and t

$$\frac{d}{dt}\Big(\|w_m(t) - w_n(t)\|^2\Big) \le \beta\dot{a}(t)\|w_m(t) - w_n(t)\|^2 + \gamma\dot{a}(t)(\eta_n + \eta_m).$$

As $\|w_m(0) - w_n(0)\|^2 = 0$, the Gronwall inequality yields for all $t \in I$

$$\|w_m(t) - w_n(t)\|^2 \le \gamma(\eta_n + \eta_m)\int_0^t [\dot{a}(s)\exp(\beta\int_s^t \dot{a}(u)\,du)]ds$$

and hence for some positive constant K independent of m, n, and t we have

$$\|w_m(t) - w_n(t)\|^2 \le K(\eta_n + \eta_m).$$

The Cauchy property in $\mathcal{C}(I, H)$ of the sequence $(w_n)_n = (u_n + Z_n)_n$ is thus established and hence this sequence converges uniformly to some mapping w. Therefore the sequence $(u_n)_n$ converges uniformly to $u := w - Z$. Furthermore, (4.10) ensures that a subsequence of $(\dot{u}_n)_n$ may be extracted to converge in the weak topology of $L^1(I, H, dt)$. Without loss of generality, we may suppose that this subsequence is $(\dot{u}_n)_n$. Denote by p its weak limit in $L^1(I, H, dt)$. Then, for all $t \in I$

$$u(t) = \lim_{n\to\infty} u_n(t) = x_0 + \lim_{n\to\infty}\int_0^t \dot{u}_n(s)ds = x_0 + \int_0^t p(s)ds,$$

which gives that u is absolutely continuous and $\dot{u}(t) = p(t)$ for a. e. $t \in I$.

Moreover, for a.e. $t \in I$, by the definition (4.11) of $\theta_n(t)$ one has $|\theta_n(t) - t| \le \dfrac{T}{2^n}$ and (by (4.9) and (4.4))

$$\begin{aligned}\|u_n(\theta_n(t)) - u(t)\| &\le \|u_n(t) - u(t)\| + (2l+1)(a(\theta_n(t)) - a(t))\\ &\le \|u_n(t) - u(t)\| + (2l+1)\epsilon_n.\end{aligned}$$

So,

$$\lim_{n\to\infty}\theta_n(t) = t \ \text{ and } \lim_{n\to\infty} u_n(\theta_n(t)) = u(t). \tag{4.15}$$

As $u_n(\theta_n(t)) \in C(\theta_n(t))$, it follows from (4.1)

$$d_{C(t)}(u_n(\theta_n(t))) \le v(\theta_n(t)) - v(t)$$

and hence, by (4.15), one obtains $u(t) \in C(t)$, because the set $C(t)$ is closed.

Step 4. Now we proceed to prove that $\dot{u}(t) + z(t) \in -N^C(C(t); u(t))$ for almost all $t \in I$. We know by (4.13) that we have for almost all $t \in I$

$$\dot{u}_n(t) + z_n(t) \in -\delta(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))). \tag{4.16}$$

We can thus apply Castaing techniques (see [6]). The weak convergence in $L^1(I, H, dt)$ of $(\dot{u}_n)_n$ and $(z_n)_n$ to $\dot{u}$ and z respectively entail for almost all $t \in I$ (by Mazur's lemma)

$$\dot{u}(t) + z(t) \in \bigcap_n \overline{co}\{\dot{u}_k(t) + z_k(t) : k \ge n\}.$$

Here $\overline{co}$ denotes the closed convex hull. Fix any such $t \in I$ and consider any $\xi \in H$. The last relation above yields

$$\big\langle \xi, \dot{u}(t) + z(t)\big\rangle \le \inf_n \sup_{k\ge n}\big\langle \xi, \dot{u}_k(t) + z_k(t)\big\rangle,$$

and hence according to (4.16)

$$\begin{aligned}\langle \xi, \dot{u}(t) + z(t) \rangle &\leq \limsup_n \sigma(-\delta(t)\partial^P d_{C(\theta_n(t))}(u_n(\theta_n(t))); \xi) \\ &\leq \sigma(-\delta(t)\partial^P d_{C(t)}(u(t)); \xi),\end{aligned}$$

where the second inequality follows from the closedness property in Proposition 4.1 because (4.15) holds and $u(t) \in C(t)$. As the set $\partial^P d_{C(t)}(u(t))$ is closed and convex (see Proposition 3.1), we obtain

$$\dot{u}(t) + z(t) \in -\delta(t)\partial^P d_{C(t)}(u(t)) \subset -N^P(C(t); u(t)).$$

By the global upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that $z(t) \in F(t, u(t))$ and so we get

$$-\dot{u}(t) \in N^P(C(t); u(t)) + F(t, u(t)),$$

which completes the proof of the first step. Note also by (4.10) that

$$\|\dot{u}(t)\| \leq (2l + 1)(\dot{v}(t) + 1) \quad \text{for a. e.} \quad t \in I.$$

II) Now, we assume that F satisfies the hypothesis in the statment of the theorem.

According to the proof of Theorem 2.1 in [7] (see [8, 3] for more details concerning the existence of such approximation and their properties), there exists a sequence $(F_n)_n$ of globally u.s.c. set-valued mappings on $I \times H$ with convex compact values in H with $F_n(t, x) \subset T\mathcal{K}$ for all $(t, x) \in I \times H$ and satisfying : For any sequence (x_n) of Lebesgue measurable mappings from I to H which converges pointwise to a Lebesgue measurable mapping x and any sequence (z_n) converging weakly to z in $L^1(I, H, dt)$ and such that $z_n(t) \in F_n(t, x_n(t))$ a.e. on I, one has

$$z(t) \in F(t, x(t)), \quad \text{a. e. on } I.$$

Since F_n satisfies the hypothesis of the first step, for every $n \geq 1$, there exists an absolutely continuous mapping $x_n : I \to H$ and a Lebesgue measurable mapping $z_n : I \to H$ satisfying $z_n(t) \in F_n(t, x_n(t)) \subset T\mathcal{K}$ for a.e. $t \in I$ and

$$\dot{x}_n(t) + z_n(t) \in -N^C(C(t); x_n(t)) \text{ a. e. on } I,$$

with $x_n(0) = x_0 \in C(0)$ and $\|\dot{x}_n(t)\| \leq (2lT+1)(\dot{v}(t)+1)$ for a.e. $t \in I$. Observe that (z_n) admits a subsequence (that we do not relabel) converging weakly in $L^1(I, H, dt)$ to some mapping z. So, by the property of the sequence (F_n) stated above we conclude that $z(t) \in F(t, x(t))$ for a.e. $t \in I$. Now, with the same techniques as in the first step, we prove easily the uniform convergence of the sequence (x_n) to some absolutely continuous mapping x and that

$$\dot{x}(t) + z(t) \in -N^C(C(t); x(t)) \quad \text{a. e. on } I.$$

Thus, we get $-\dot{x}(t) \in N^C(C(t); x(t)) + F(t, x(t))$, for a.e. $t \in I$. This ends the proof of the theorem. □

The following corollary is a direct consequence of Theorem 4.1. A similar result is also established by Colombo and Goncharov [13] where the set-valued mapping C is assumed to be Lipschitz with ϕ-convex values.

Corollary 4.3. *Let H be any Hilbert space, $T > 0$, and $r > 0$. Assume that $C(t)$ is r-prox-regular for every $t \in I := [0, T]$ and that the assumption (4.1) holds with a nondecreasing absolutely continuous function v. Then the sweeping process (SP) associated with the set-valued mapping C has one and only one absolutely continuous solution.*

Proof. The existence follows from Theorem 4.1 since for $F = 0$ the separability of H is not needed as it is easily seen in the proof of the first step of Theorem 4.1. The uniqueness part follows from the proof of Corollary 5.1 in Thibault [29]. □

References

[1] J. P. Aubin and A. Cellina, *Differential inclusions-Set-valued maps and viability theory*, Springer-Verlag, Berlin, 1984.

[2] H. Benabdellah, *Existence of solutions to the nonconvex sweeping process*, J. Differential Equations **164** (2000), 286-295.

[3] H. Benabdellah, C. Castaing and A. Salvadori, *Compactness and discretization methods for differential inclusions and evolution problems*, Atti. Semi. Mat. Fis. Modena **XLV** (1997),. 9-51 .

[4] M. Bounkhel and L. Thibault, *On various notions of regularity of sets*, Nonlinear Anal.TMA. **48** (2002), 223-246.

[5] A. Canino, *On p-convex sets and geodesics*, J. Differential Equations **75** (1988), 118-157.

[6] C. Castaing, *Equation differentielle multivoque avec contrainte sur l'état dans les espaces de Banach*, Sem. Anal. Convexe Montpellier (1978) exposé no. **13**.

[7] C. Castaing and M.D.P. Monteiro-Marques, *Evolution problems associated with nonconvex closed moving sets with bounded variation*. Portugal. Math. **53** (1996), 73-87.

[8] C. Castaing, M. Moussaoui, and A. Syam, *Multivalued differential Equations on closed convex sets in Banach spaces*, Set-valued Analysis **1** (1994), 329-353.

[9] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, 1977.

[10] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[11] F. H. Clarke, R. J. Stern and P. R. Wolenski, *Proximal smoothness and the lower C^2 property*, J. Convex Anal. **2** (1995), 117-144.

[12] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag New York, 1998.

[13] G. Colombo and V. V. Goncharov, *The sweeping processes without convexity*, Set-valued Anal. **7** (1999), 357-374.

[14] G. Colombo and V. V. Goncharov, *Variational inequalities and regularity of closed sets in Hilbert spaces* J. Convex Anal. **8** (2001), 197-221.

[15] B. Cornet, *Contributions à la théorie mathématique des mécanismes dynamiques d'allocation de ressources*, thèse de doctorat d'état, Université Paris-Dauphine, 1981.

[16] B. Cornet, *Existence of slow solutions for a class of differential inclusions, J. Math. Anal. Appl.* **96** (1983), 130-147.

[17] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renorming in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Math., Vol **64**, Longman, 1993.

[18] J. Diestel, *Geometry of Banach spaces. Selected topics*, Lecture Notes in Math. **485**, Spring-Verlag, 1975.

[19] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418-491.

[20] C. Henry, *An existence theorem for a class of differential inclusions with multivalued right-hand side*, J. Math. Anal. Appl. **41** (1973), 179-186.

[21] K-S. Lau, *Almost Chebychev subsets in reflexive Banach spaces*, Indiana Univ. Math. J. **2** (1978), 791-795.

[22] B. Mordukhovich and Y. Shao, *Nonsmooth sequential analysis in Asplund spaces*, Trans. Amer. Math. Soc. **348** (1996), 1235-1280.

[23] J.J. Moreau, *Râfle par un convexe variable I*, Sem. Anal. Convexe Montpellier (1971) exposé no **15**.
[24] J.J. Moreau, *Râfle par un convexe variable II*, Sem. Anal. Convexe Montpellier (1972) exposé no **3**.
[25] J.J. Moreau, *Evolution problem associated with a moving convex set in Hilbert space*, J. Differential Equations **26** (1977), 347-374.
[26] R. A. Poliquin, R. T. Rockafellar and L. Thibault, *Local differentiability of distance functions*, Trans. Amer. Math. Soc. **352** (2000), 5231-5249.
[27] R. T. Rockafellar, *Directionally Lipschitzian functions and subdifferential calculus*, Proc. London Math. Soc. **39** (1979), 331-355.
[28] R. T. Rockafellar and R. Wets, *Variational Analysis*, Springer Verlag, Berlin, 1998.
[29] L. Thibault, *Sweeping process with regular and nonregular sets*, J. Differential Equations **193** (2003), 1-26.
[30] M. Valadier, *Entrainement unilateral, lignes de descente, fonctions lipschitziennes non pathologiques*, C. R. Acad. Sci. Paris **308** Series I (1989), 241-244.
[31] M. Valadier, *Quelques problèmes d'entrainement unilatéral en dimension finie*, Sem. Anal. Convexe. Montpellier (1988) exposé no. **8**.

Manuscript received February 12, 2005
revised July 16, 2005

Messaoud Bounkhel
Department of mathematics, College of Sciences King Saud University, Riyadh 11451, Saudi Arabia
E-mail address: bounkhel@ksu.edu.sa

Lionel Thibault
Département de mathématiques, C.C.051, Université Montpellier II, 34095 Montpellier, France
E-mail address: thibault@math.univ-montp2.fr