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ε-OPTIMALITY WITHOUT CONSTRAINT QUALIFICATION FOR MULTIOBJECTIVE FRACTIONAL PROGRAM

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Dedicated to Professor Koei Sekigawa of Niigata University for his 60th birthday.

Abstract. In this paper, some ε-optimality results are extended to vector optimization problems with inequality, equality and abstract constraints via exact penalty functions. Subsequently, these results are utilized to derive KKT-type optimality conditions for ε-Pareto solutions of a multiobjective fractional programming problem without any constraint qualification.

1. Introduction

During the past few decades much attention has been paid to develop optimality conditions for approximate solutions of vector optimization problems under various assumptions. This is because of the fact that in most practical cases mathematical models formulated for real life problems are not the precise copies of the original problems. Moreover, as observed by Loridan [8], the exact optimal solution of neither the scalar nor one of the vector optimization problem are necessarily attained. Hence, it is interesting to have a theoretical analysis of approximate solutions.

Numerous research articles have appeared in this direction. For more details, see, Hiriart-Urruty [4], Liu [5], Loridan [7,8], Strodiot et al. [9], Yokoyama [10,11], and references cited therein.

In this paper, following the techniques of Hamel [3], Liu and Yokoyama [6] and Yokoyama and Shiraishi [12], we derive optimality conditions for ε-Pareto solutions of a nondifferentiable locally Lipschitz multiobjective fractional programming problem without any constraint qualification. The problem considered in the present paper involves inequality, equality and abstract constraints.

The organization of the paper is as follows. In Section 2, we give the preliminary terminologies. In Section 3, we use a vector exact penalty function to establish ε-optimality criteria for the multiobjective optimization problem by estimating the sizes of penalty parameters in terms of ε-optimal solutions of the dual problem. In Section 4, the results of Section 3 along with Ekeland's variational principle [2] are utilized to derive Karush-Kuhn-Tucker type conditions for ε-Pareto optimality of locally Lipschitz multiobjective fractional programming problems without any constraint qualification.

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2. Preliminaries

Consider the following vector minimization problem

$$\text{(P)} \qquad \begin{array}{rl} \text{Minimize} & \phi(x) = (\phi_1(x), \ldots, \phi_p(x)) \\ \text{subject to} & x \in S \end{array}$$

where $S = \{x \in \mathbb{R}^n : h_j(x) \leqq 0, 1 \leqq j \leqq m\,; q_r(x) = 0, 1 \leqq r \leqq k\,; x \in C\}$ is the set of feasible solutions of (P), ϕ_i $(1 \leqq i \leqq p)$, h_j $(1 \leqq j \leqq m)$, $q_r(1 \leqq r \leqq k) : \mathbb{R}^n \to \mathbb{R}$ and C is a closed subset of $\mathbb{R}^n$.

We associate p-scalar problems (P_i), $1 \leqq i \leqq p$ with (P) as

$$(\mathrm{P}_i) \qquad \begin{array}{rl} \text{Minimize} & \phi_i(x) \\ \text{subject to} & x \in S. \end{array}$$

For each $i = 1, \ldots, p$, the dual problem (D_i) associated with the i-th problem (P_i) is given by

$$(\mathrm{D}_i) \qquad \text{Maximize } w_i(\lambda, \mu, \nu)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$, $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{R}^k$, $\nu \in \mathbb{R}$ and

$$w_i(\lambda, \mu, \nu) = \inf_{x \in \mathbb{R}^n} L_i(x, \lambda, \mu, \nu)\,,$$

$$L_i(x, \lambda, \mu, \nu) = \begin{cases} \phi_i(x) + \displaystyle\sum_{j=1}^{m} \lambda_j\, h_j(x) + \sum_{r=1}^{k} \mu_r\, q_r(x) + \nu d_C(x), & \text{if } \lambda \geqq 0, \nu \geqq 0 \\ -\infty, & \text{otherwise}\,. \end{cases}$$

To transform the problem (P) into an unconstrained problem, we use a vector exact penalty function defined by

$$\sigma(x, \rho, s, \eta) = (\sigma_1(x, \rho, s, \eta), \ldots, \sigma_p(x, \rho, s, \eta))$$

where

$$\sigma_i(x, \rho, s, \eta) = \phi_i(x) + \rho \sum_{j=1}^{m} \max(0, h_j(x)) + s \sum_{r=1}^{k} \max(q_r(x), -q_r(x)) + \eta d_C(x)\,,$$

$$d_C(x) = \inf_{c \in C} \|x - c\|\,,$$

and ρ, s, η are positive reals.

Throughout the paper, we assume that the set $S \neq \phi$.

Also, for any $\xi \in \mathbb{R}^\ell$, $\|\xi\|_1 = \displaystyle\sum_{t=1}^{\ell} |\xi_t|$ and the duality gap between the problems (P_i) and (D_i) is denoted by r_i, i.e.,

$$r_i = \inf_{x \in S} \phi_i(x) - \sup_{(\lambda, \mu, \nu) \in \mathbb{R}^{m+k+1}} w_i(\lambda, \mu, \nu)\,, \quad 1 \leqq i \leqq p\,.$$

Further, let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p) > 0$ be the permissible vector perturbation. We now define ε-approximate solutions.

Definition 2.1. $\bar{x} \in S$ is called an ε-Pareto solution of (P) if there exists no $x \in S$ such that $\phi_i(x) \leqq \phi_i(\bar{x}) - \varepsilon_i$ for any $i = 1, \ldots, p$ with at least one strict inequality.

Definition 2.2. $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \mathbb{R}^{m+k+1}$ is called an ε_i-optimal solution of (D_i) if it satisfies $w_i(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \geqq \sup_{(\lambda,\mu,\nu)\in\mathbb{R}^{m+k+1}} (w_i(\lambda, \mu, \nu)) - \varepsilon_i$.

Definition 2.3. For $\rho, s, \eta > 0$, $\bar{x} \in \mathbb{R}^n$ is called an ε-Pareto solution of the unconstrained problem (UP),

$$\text{(UP)} \qquad \text{Minimize} \quad \sigma(x, \rho, s, \eta) = (\sigma_1(x, \rho, s, \eta), \ldots, \sigma_p(x, \rho, s, \eta))$$

if there exists no $x \in \mathbb{R}^n$ such that

$$\sigma_i(x, \rho, s, \eta) \leqq \sigma_i(\bar{x}, \rho, s, \eta) - \varepsilon_i$$

for any $i = 1, \ldots, p$ with at least one strict inequality.

3. Characterization of ε-Pareto Solution

In this section, we characterize ε-Pareto solutions of (P) by estimating the size of penalty parameters ρ, s, η in terms of ε-optimal solutions of (D_i).

The following proposition gives a necessary condition to obtain an ε-Pareto solution of (P).

Proposition 3.1. *Let $\bar{x}$ be an ε-Pareto solution of (P) and let*

$$\rho_0 = \max_{1\leqq i\leqq p} \|\bar{\lambda}^i\|_\infty, \quad s_0 = \max_{1\leqq i\leqq p} \|\bar{\mu}^i\|_\infty, \quad \eta_0 = \max_{1\leqq i\leqq p} |\bar{\nu}^i|,$$

where $(\bar{\lambda}^i, \bar{\mu}^i, \bar{\nu}^i)$ is an ε_i-optimal solution of (D_i), $1 \leqq i \leqq p$. Then, for any $(\rho, s, \eta) \geqq (\rho_0, s_0, \eta_0)$, $\bar{x}$ is a $(\beta+3\varepsilon)$-Pareto solution of (UP) where $\beta = (\beta_1, \ldots, \beta_p)$ and $\beta_i = \inf_{x\in S} \max_{1\leqq i\leqq p} \phi_i(x) - \sup_{(\lambda,\mu,\nu)\in\mathbb{R}^{m+k+1}} w_i(\lambda, \mu, \nu)$.

Proof. We prove the result by contradiction. Let there exist $(\rho, s, \eta) \geqq (\rho_0, s_0, \eta_0)$ such that $\bar{x}$ is not a $(\beta+3\varepsilon)$-Pareto solution of (UP). Then there exists $\hat{x} \in \mathbb{R}^n$ such that

$$\sigma_i(\hat{x}, \rho, s, \eta) \leqq \sigma_i(\bar{x}, \rho, s, \eta) - (\beta_i + 3\varepsilon_i)$$

for any $i = 1, \ldots, p$ with at least one strict inequality.

Since $\bar{x} \in S$, we have

$$\phi_i(\bar{x}) - \varepsilon_i \geqq \sigma_i(\hat{x}, \rho, s, \eta) + (\beta_i + 2\varepsilon_i) \tag{3.1}$$

for any $i = 1, \ldots, p$ with at least one strict inequality.

Now, we can estimate as follows.

$$\begin{aligned}
\sigma_i(\hat{x}, \rho, s, \eta) &= \phi_i(\hat{x}) + \rho \sum_{j=1}^{m} \max(0, h_j(\hat{x})) + s \sum_{r=1}^{k} \max(q_r(\hat{x}), -q_r(\hat{x})) + \eta d_C(\hat{x}) \\
&\geqq \phi_i(\hat{x}) + \rho_0 \sum_{j=1}^{m} \max(0, h_j(\hat{x})) + s_0 \sum_{r=1}^{k} \max(q_r(\hat{x}), -q_r(\hat{x})) + \eta_0 d_C(\hat{x}) \\
&\geqq \phi_i(\hat{x}) + \sum_{j=1}^{m} |\bar{\lambda}_j^i| \max(0, h_j(\hat{x})) + \sum_{r=1}^{k} |\bar{\mu}_r^i| \max(q_r(\hat{x}), -q_r(\hat{x})) \\
&\qquad + |\bar{\nu}^i| d_C(\hat{x})
\end{aligned}$$

$$\geqq \phi_i(\hat{x}) + \sum_{j=1}^{m} \bar{\lambda}_j^i h_j(\hat{x}) + \sum_{r=1}^{k} \bar{\mu}_r^i q_r(\hat{x}) + \bar{\nu}^i d_C(\hat{x})$$

which on using (3.1) yields

$$\phi_i(\bar{x}) - \varepsilon_i \geqq \phi_i(\hat{x}) + \sum_{j=1}^{m} \bar{\lambda}_j^i h_j(\hat{x}) + \sum_{r=1}^{k} \bar{\mu}_r^i q_r(\hat{x}) + \bar{\nu}^i d_C(\hat{x}) + \beta_i + 2\varepsilon_i$$

for any $i = 1, \ldots, p$ with at least one strict inequality.

Thus, we have

$$\begin{aligned}\phi_i(\bar{x}) - \varepsilon_i &\geqq \inf_{x \in \mathbb{R}^n} \left(\phi_i(x) + \sum_{j=1}^{m} \bar{\lambda}_j^i h_j(x) + \sum_{r=1}^{k} \bar{\mu}_r^i q_r(x) + \bar{\nu}^i d_C(x) \right) + \beta_i + 2\varepsilon_i \\ &= w_i(\bar{\lambda}^i, \bar{\mu}^i, \bar{\nu}^i) + \beta_i + 2\varepsilon_i \\ &\geqq \sup_{(\lambda,\mu,\nu) \in \mathbb{R}^{m+k+1}} w_i(\lambda, \mu, \nu) - \varepsilon_i + \beta_i + 2\varepsilon_i \,.\end{aligned}$$

By the choice of β_i, we get

$$\begin{aligned}\phi_i(\bar{x}) - \varepsilon_i &\geqq \inf_{x \in S} \max_{1 \leqq i \leqq p} \phi_i(x) + \varepsilon_i \\ &\geqq \inf_{x \in S} \max_{1 \leqq i \leqq p} \phi_i(x) + \min_{1 \leqq i \leqq p} \varepsilon_i \\ &= \inf_{x \in S} \max_{1 \leqq i \leqq p} \phi_i(x) + \varepsilon_0, \quad \varepsilon_0 = \min_{1 \leqq i \leqq p} \varepsilon_i > 0 \\ &\geqq \max_{1 \leqq i \leqq p} \phi_i(\tilde{x}) \quad \text{for some} \quad \tilde{x} \in S.\end{aligned}$$

Hence, for some $\tilde{x} \in S$,

$$\phi_i(\bar{x}) - \varepsilon_i \geqq \phi_i(\tilde{x})$$

for any $i = 1, \ldots, p$ with at least one strict inequality. This contradicts that $\bar{x}$ is an ε-Pareto solution of (P). Hence we get the result. □

The next theorem, giving another necessary condition for ε-Pareto solutions of (P), will be used as a principle tool in obtaining the main results in the sequel.

Theorem 3.1. *Let $\bar{x}$ be an ε-Pareto solution of (P) and ρ_0, s_0, η_0 be as same as in Proposition 3.1. Then for any $(\rho, s, \eta) \geqq (\rho_0, s_0, \eta_0)$, there exits $\bar{u} \in \mathbb{R}^p$ such that $\bar{u} > 0$ and*

$$\|\sigma(x, \rho, s, \eta) - \bar{\sigma}\|_{\bar{u}} + \left(\gamma + \sum_{i=1}^{p} \varepsilon_i \right) \geqq \|\sigma(\bar{x}, \rho, s, \eta) - \bar{\sigma}\|_{\bar{u}} \quad \textit{for any } x \in \mathbb{R}^n \tag{3.2}$$

where

$$\|y\|_{\bar{u}} = \max_{1 \leqq i \leqq p} \bar{u}_i |y_i| \quad \textit{for} \quad y \in \mathbb{R}^p$$

$$\gamma = p \left(\inf_{x \in S} \max_{1 \leqq i \leqq p} \phi_i(x) \right) - \sum_{i=1}^{p} \sup\{ w_i(\lambda, \mu, \nu) \,|\, (\lambda, \mu, \nu) \in \mathbb{R}^{m+k+1} \} \geqq 0,$$

$$\bar{\sigma}_i < \phi_i(x) - 1 \textit{ for any } \ x \in \mathbb{R}^n, \ 1 \leqq i \leqq p.$$

Proof. Let $(\rho, s, \eta) \geqq (\rho_0, s_0, \eta_0)$. Define $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_p) \in \mathbb{R}^p$ as

$$\bar{u}_i = (\sigma_i(\bar{x}, \rho, s, \eta) - \bar{\sigma}_i)^{-1}, \quad 1 \leqq i \leqq p\,.$$

Then, we have $\bar{u}_i = (\phi_i(\bar{x}) - \bar{\sigma}_i)^{-1} > 0$.

We now prove the result by contradiction. Suppose relation (3.2) does not hold. Then, there exists $y \in \mathbb{R}^n$ such that

$$\|\sigma(y, \rho, s, \eta) - \bar{\sigma}\|_{\bar{u}} \,<\, \|\sigma(\bar{x}, \rho, s, \eta) - \bar{\sigma}\|_{\bar{u}} - \left(\gamma + \sum_{i=1}^{p} \varepsilon_i\right).$$

The rest of the proof runs on similar lines as in the proof of [11, Theorem 3.1]. A contradiction will be obtained by using Proposition 3.1. □

4. ε-OPTIMALITY CONDITIONS

In this section, we consider the following multiobjective fractional programming problem

$$\text{(FP)} \qquad \text{Minimize} \quad \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right) \quad \text{subject to } x \in S$$

where

$$S = \{x \in \mathbb{R}^n : h_j(x) \leqq 0, 1 \leqq j \leqq m; q_r(x) = 0, 1 \leqq r \leqq k; x \in C\}$$

is a feasible set of (FP), and for $1 \leqq i \leqq p$, $1 \leqq j \leqq m$, $1 \leqq r \leqq k$, the functions $f_i, g_i, h_j, q_r : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz, $f_i, -g_i$ are bounded from below on $\mathbb{R}^n$ and $g_i(x) > 0$ for any $x \in S$. The set C is taken to be a closed subset of $\mathbb{R}^n$. Minimization is taken in terms of obtaining ε-Pareto solutions of (FP) defined as follows:

Definition 4.1. $\bar{x} \in S$ is called an ε-Pareto solution of (FP) if there is no $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$$

for any $i = 1, \ldots, p$ with at least one strict inequality.

Using parametric approach, we transform the problem (FP) into the multiobjective program $(\mathrm{MP})_v$, with parameter $v \in \mathbb{R}^p$,

$$(\mathrm{MP})_v \qquad \text{Minimize} \quad (f(x) - vg(x)) = (f_1(x) - v_1 g_1(x), \ldots, f_p(x) - v_p g_p(x)) \quad \text{subject to} \quad x \in S\,.$$

Following lemma relates the ε-Pareto solutions of (FP) and $(\mathrm{MP})_v$.

Lemma 4.1. *$\bar{x} \in S$ is an ε-Pareto solution of (FP) if and only if there exists $\bar{v} \in \mathbb{R}^p$, with $\bar{v}_i = \dfrac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$, $1 \leqq i \leqq p$, such that $\bar{x}$ is an $\bar{\varepsilon}$-Pareto solution of $(MP)_{\bar{v}}$ where $\bar{\varepsilon} = \varepsilon g(\bar{x}) = (\varepsilon_1 g_1(\bar{x}), \ldots, \varepsilon_p g_p(\bar{x})) > 0$.*

Proof. Let $\bar{x} \in S$ be an ε-Pareto solution of (FP). Then there is no $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$$

for any $i = 1, \ldots, p$ with at least one strict inequality. Thus, there is no $x \in S$ such that

$$f_i(x) - \bar{v}_i\, g_i(x) \leqq 0$$

for any $i = 1, \ldots, p$ with at least one strict inequality. Using the definition of $\bar{v}$, we get that there is no $x \in S$ such that

$$f_i(x) - \bar{v}_i\, g_i(x) \leqq f_i(\bar{x}) - \bar{v}_i\, g_i(\bar{x}) - \bar{\varepsilon}_i$$

for any $i = 1, \ldots, p$ with at least one strict inequality. Hence, $\bar{x}$ is an $\bar{\varepsilon}$-Pareto solution of $(\mathrm{MP})_{\bar{v}}$. The converse also follows on similar lines. □

For fixed $v \in \mathbb{R}^p$, the exact penalty function associated with the problem $(\mathrm{MP})_v$ is defined as

$$\begin{aligned}
\theta(x, \rho, s, \eta) &= (\theta_1(x, \rho, s, \eta), \ldots, \theta_p(x, \rho, s, \eta)), \\
\theta_i(x, \rho, s, \eta) &= f_i(x) - v_i g_i(x) + \rho \sum_{j=1}^{m} \max(0, h_j(x)) \\
&\quad + s \sum_{r=1}^{k} \max(q_r(x), -q_r(x)) + \eta d_C(x), 1 \leqq i \leqq p\,.
\end{aligned}$$

We now establish Karush-Kuhn-Tucker (KKT) necessary optimality conditions for ε-Pareto solutions of (FP).

Theorem 4.1. *Let $\bar{x}$ be an ε-Pareto solution of (FP). Then there exists an $x_\tau \in \mathbb{R}^n$ such that $\|x_\tau - \bar{x}\| \leqq 1$ and there exist $\lambda \in \mathbb{R}^p$, $\rho \in \mathbb{R}$, $s \in \mathbb{R}^k$, $\tau \in \mathbb{R}$, $\eta \in \mathbb{R}$ such that*

$$\begin{aligned}
&0 \in \sum_{i=1}^{p} \lambda_i \partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho \sum_{j=1}^{m} \partial h_j(x_\tau) + \sum_{r=1}^{k} s_r \partial q_r(x_\tau) + \tau B^* + \eta \partial_C(x_\tau), \\
&h_j(x_\tau) \leqq \sqrt{\varepsilon}\,, 1 \leqq j \leqq m\,, \\
&-\sqrt{\varepsilon} \leqq q_r(x_\tau) \leqq \sqrt{\varepsilon}\,, 1 \leqq r \leqq k\,, \\
&d_C(x_\tau) \leqq \sqrt{\varepsilon}\,, \\
&\lambda_i \geqq 0,\ 1 \leqq i \leqq p,\ \rho > 0,\ \tau > 0,\ \eta > 0,\ \sum_{i=1}^{p} \lambda_i = 1
\end{aligned}$$

where B^* is a unit ball in $\mathbb{R}^n$.

Proof. Since $\bar{x}$ is an ε-Pareto solution of (FP), hence by Lemma 4.1, there exists $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_p)$, $\bar{v}_i = \dfrac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$ $1 \leqq i \leqq p$ such that $\bar{x}$ is an $\bar{\varepsilon}$-Pareto solution of $(\mathrm{MP})_{\bar{v}}$ where $\bar{\varepsilon}_i = \varepsilon_i g_i(\bar{x})$.

It follows from Theorem 3.1, that there exist $\rho_0 > 0$, $s_0 > 0$, $\eta_0 > 0$, $\gamma \geqq 0$ such that for all $(\rho, s, \eta) \geqq (\rho_0, s_0, \eta_0)$, there exists $\bar{u} = (\bar{u}_1, \dots, \bar{u}_p) > 0$ satisfying

$$\|\theta(x,\rho,s,\eta) - \bar{\theta}\|_{\bar{u}} + \left(\gamma + \sum_{i=1}^{p} \bar{\varepsilon}_i\right) \geqq \|\theta(\bar{x},\rho,s,\eta) - \bar{\theta}\|_{\bar{u}} \quad \text{for any} \quad x \in \mathbb{R}^n$$

where $\bar{\theta} = (\bar{\theta}_1, \dots, \bar{\theta}_p)$ with

$$\bar{\theta}_i < f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) - 1, \qquad 1 \leqq i \leqq p$$

and

$$\gamma = \left(p \inf_{x \in S} \max_{1 \leqq i \leqq p} (f_i(x) - \bar{v}_i g_i(x)) - \sum_{i=1}^{p} \sup_{(\lambda,\mu,\nu)} \inf_{x} L_i(x,\lambda,\mu,\nu)\right).$$

From this setting, we get

$$\begin{aligned}(4.3) \quad & \max_{1 \leqq i \leqq p} \bar{u}_i \left(f_i(x) - \bar{v}_i g_i(x) + \rho \sum_{j=1}^{m} \max(0, h_j(x)) \right. \\ & \left. + s \sum_{r=1}^{k} \max(q_r(x), -q_r(x)) + \eta d_C(x) - \bar{\theta}_i \right) + \left(\gamma + \sum_{i=1}^{p} \bar{\varepsilon}_i\right) \\ & \geqq \max_{1 \leqq i \leqq p} \bar{u}_i \left(f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) + \rho \sum_{j=1}^{m} \max(0, h_j(\bar{x})) \right. \\ & \left. + s \sum_{r=1}^{k} \max(q_r(\bar{x}), -q_r(\bar{x})) + \eta d_C(\bar{x}) - \bar{\theta}_i \right) \quad \text{for any } x \in \mathbb{R}^n .\end{aligned}$$

Set, for $x \in \mathbb{R}^n$,

$$\begin{aligned}\psi(x) = & \max_{1 \leqq i \leqq p} \bar{u}_i \left(f_i(x) - \bar{v}_i g_i(x) + \rho \sum_{j=1}^{m} \max(0, h_j(x)) \right. \\ & \left. + s \sum_{r=1}^{k} \max(q_r(x), -q_r(x)) + \eta d_C(x) - \bar{\theta}_i \right)\end{aligned}$$

and $\tau = \gamma + \sum_{i=1}^{p} \bar{\varepsilon}_i > 0$.

Then, $\psi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. From (4.1), we have ψ is bounded from below and

$$\psi(x) + \tau \geqq \psi(\bar{x}) \qquad \text{for any } x \in \mathbb{R}^n .$$

Thus, $\bar{x}$ is a τ-optimal solution of the following unconstrained optimization problem

$$\operatorname*{Minimize}_{x \in \mathbb{R}^n} \psi(x) .$$

By Ekeland's variational principle [2], there exists an $x_\tau \in \mathbb{R}^n$ such that the following relations hold

(i) $\psi(x_\tau) \leqq \psi(x) + \tau$ for any $x \in \mathbb{R}^n$,
(ii) $\|\bar{x} - x_\tau\| \leqq 1$,

(iii) $\psi(x_\tau) \leqq \psi(x) + \tau\|x - x_\tau\|$ for any $x \in \mathbb{R}^n$.

From condition (iii) it follows that x_τ is an optimal solution of the problem

$$\underset{x\in\mathbb{R}^n}{\text{Minimize}}\ \psi(x) + \tau\|x - x_\tau\|\,.$$

Also, the function $\psi(\cdot) + \tau\|\cdot - x_\tau\|$ is locally Lipschitz. By Clarke's necessary optimality conditions [1],

$$0 \in \partial(\psi(\cdot) + \tau\|\cdot - x_\tau\|)(x_\tau)$$

where ∂ denotes the Clarke's generalized subgradient.
The above relation yields that

$$0 \in \partial\psi(x_\tau) + \tau B^* \tag{4.4}$$

where $B^* = \{y \in \mathbb{R}^n : \|y\| \leqq 1\}$ is a unit ball in $\mathbb{R}^n$.
Now, we have

$$\begin{aligned}\partial\psi(x_\tau) &= \partial \max_{1\leqq i\leqq p}\left(\bar{u}_i\left(f_i(\cdot) - \bar{v}_i g_i(\cdot) + \rho\sum_{j=1}^{m}\max(0, h_j(\cdot))\right.\right.\\ &\qquad \left.\left.+ s\sum_{r=1}^{k}\max(q_r(\cdot), -q_r(\cdot)) + \eta d_C(\cdot) - \bar{\theta}_i\right)\right)(x_\tau)\\ &\subseteq \sum_{i=1}^{p}\alpha_i\bar{u}_i\left(\partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho\sum_{j=1}^{m}\partial h_j(x_\tau)\right.\\ &\qquad \left.+ s\sum_{r=1}^{k}\beta_r\partial q_r(x_\tau) + \eta\partial_C(x_\tau)\right)\end{aligned}$$

where $\alpha_i \geqq 0$, $1 \leqq i \leqq p$, $\sum_{i=1}^{p}\alpha_i = 1$, $\beta_r \in \mathbb{R}$, $1 \leqq r \leqq k$.

Set $\alpha_i\bar{u}_i = \lambda_i \geqq 0$, $1 \leqq i \leqq p$ and at least one $\lambda_i > 0$. Then

$$\begin{aligned}\partial\psi(x_\tau) &\subseteq \sum_{i=1}^{p}\lambda_i\left(\partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho\sum_{j=1}^{m}\partial h_j(x_\tau)\right.\\ &\qquad \left.+ s\sum_{r=1}^{k}\beta_r\partial q_r(x_\tau) + \eta\partial d_C(x_\tau)\right)\\ &\subseteq \sum_{i=1}^{p}\lambda_i\,\partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho\sum_{i=1}^{p}\lambda_i\sum_{j=1}^{m}\partial h_j(x_\tau)\\ &\qquad + s\sum_{i=1}^{p}\lambda_i\sum_{r=1}^{k}\beta_r\partial q_r(x_\tau) + \eta\sum_{i=1}^{p}\lambda_i\partial d_C(x_\tau)\,.\end{aligned}$$

Since $\lambda_i \geqq 0$ with at least one strict inequality, hence, without loss of generality, we can take $\sum_{i=1}^{p} \lambda_i = 1$. Therefore

$$\partial\psi(x_\tau) \subseteq \sum_{i=1}^{p} \lambda_i \, \partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho \sum_{j=1}^{m} \partial h_j(x_\tau) + \sum_{r=1}^{k} s_r \partial q_r(x_\tau) + \eta \partial d_C(x_\tau)$$

where $s_r = s\beta_r \in \mathbb{R}$. Substituting in (4.2), we get

$$0 \in \sum_{i=1}^{p} \lambda_i \, \partial(f_i - \bar{v}_i g_i)(x_\tau) + \rho \sum_{j=1}^{m} \partial h_j(x_\tau) + \sum_{r=1}^{k} s_r \partial q_r(x_\tau) + \tau B^* + \eta \partial d_C(x_\tau)$$

where $\lambda_i \geqq 0$, $\sum_{i=1}^{p} \lambda_i = 1$, $\rho > 0$, $s_r \in \mathbb{R}$, $1 \leqq r \leqq k$, $\tau = \gamma + \sum_{i=1}^{p} \bar{\varepsilon}_i$, $\bar{\varepsilon}_i = \varepsilon_i g_i(\bar{x})$ and $\bar{v}_i = \dfrac{f_i(\bar{x})}{g_i(\bar{x})} - \varepsilon_i$, $1 \leqq i \leqq p$.

We now prove the complementarity conditions.

Let $\bar{v} \in \mathbb{R}^p$ be fixed. Setting $\max_{1 \leqq i \leqq p} \bar{u}_i(\theta_i(x_\tau, \rho, s, \eta) - \bar{\theta}_i) = \bar{u}_{i_1}(\theta_{i_1}(x_\tau, \rho, s, \eta) - \bar{\theta}_{i_1})$,

$$\begin{aligned}
&\inf_{x \in \mathbb{R}^n} \min_{1 \leqq i \leqq p} \bar{u}_i(f_i(x) - \bar{v}_i g_i(x) - \bar{\theta}_i) \\
&\leqq \bar{u}_{i_1}(f_{i_1}(x_\tau) - \bar{v}_{i_1} g_{i_1}(x_\tau) - \bar{\theta}_{i_1}) \\
&\leqq \bar{u}_{i_1}(f_{i_1}(x_\tau) - \bar{v}_{i_1} g_{i_1}(x_\tau) + \rho \sum_{j=1}^{m} \max(0, h_j(x_\tau)) \\
&\quad + s \sum_{r=1}^{k} \mid q_r(x_\tau) \mid + \eta d_C(x_\tau) - \bar{\theta}_{i_1}) \\
&= \max_{1 \leqq i \leqq p} \bar{u}_i(f_i(x_\tau) - \bar{v}_i g_i(x_\tau) + \rho \sum_{j=1}^{m} \max(0, h_j(x_\tau)) \\
&\quad + s \sum_{r=1}^{k} \mid q_r(x_\tau) \mid + \eta d_C(x_\tau) - \bar{\theta}_i) \\
&= \psi(x_\tau) \\
&\leqq \inf_{x \in \mathbb{R}^n} \psi(x) + \tau \text{ (from (i))} \\
&= \inf_{x \in \mathbb{R}^n} \max_{1 \leqq i \leqq p} \bar{u}_i(\theta_i(x, \rho, s, \eta) - \bar{\theta}_i) + \tau.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\bar{u}_{i_1}(\rho \sum_{j=1}^{m} \max(0, h_j(x_\tau)) + s \sum_{r=1}^{k} \mid q_r(x_\tau) \mid + \eta d_C(x_\tau)) \\
&\leqq \inf_{x \in \mathbb{R}^n} \max_{1 \leqq i \leqq p} \bar{u}_i(\theta_i(x, \rho, s, \eta) - \bar{\theta}_i) - \inf_{x \in \mathbb{R}^n} \min_{1 \leqq i \leqq p} \bar{u}_i(f_i(x) - \bar{v}_i g_i(x) - \bar{\theta}_i) + \tau,
\end{aligned}$$

that is,

$$\rho \sum_{j=1}^{m} \max(0, h_j(x_\tau)) + s \sum_{r=1}^{k} \mid q_r(x_\tau) \mid + \eta d_C(x_\tau)$$
$$\leqq (\inf_{x \in \mathbb{R}^n} \max_{1 \leqq i \leqq p} \bar{u}_i(\theta_i(x, \rho, s, \eta) - \bar{\theta}_i) - \inf_{x \in \mathbb{R}^n} \min_{1 \leqq i \leqq p} \bar{u}_i(f_i(x) - \bar{v}_i g_i(x) - \bar{\theta}_i) + \tau)/\bar{u}_{i_1}.$$

From (4.1), $\max_{1 \leqq i \leqq p} \bar{u}_i(\theta_i(x, \rho, s, \eta) - \bar{\theta}_i)$ is bounded from below. Since $f_i \geqq F_i$ and $0 > -g_i \geqq G_i$ for some F_i, G_i, we have

$$\begin{aligned} f_i(x) - \bar{v}_i g_i(x) &= f_i(x) - f_i(\bar{x}) g_i(x)/g_i(\bar{x}) + \varepsilon_i g_i(x) \\ &\geqq F_i + \mid f_i(\bar{x})/g_i(\bar{x}) \mid G_i + \varepsilon_i 0. \end{aligned}$$

So, $\min_{1 \leqq i \leqq p} \bar{u}_i(f_i(x) - \bar{v}_i g_i(x) - \bar{\theta}_i)$ is bounded from below.

Then, we have $\sum_{j=1}^{m} \max(0, h_j(x_\tau)) \to 0$ if $\rho \to +\infty$, $\sum_{r=1}^{k} \mid q_r(x_\tau) \mid \to 0$ if $s \to +\infty$ and $d_C(x_\tau) \to 0$ if $\eta \to +\infty$.

Thus, there exist $\rho(\epsilon)$, $s(\epsilon)$ and $\eta(\epsilon)$ such that $h_j(x_\tau) \leqq \sqrt{\epsilon}$ $(1 \leqq j \leqq m)$, $-\sqrt{\epsilon} \leqq q_r(x_\tau) \leqq \sqrt{\epsilon}$ $(1 \leqq r \leqq k)$ and $d_C(x_\tau) \leqq \sqrt{\epsilon}$. □

Concluding remarks

If the functions involved in the fractional programming problem (FP) are assumed to be differentiable and the set C is taken to be a convex set then the necessary optimality condition of Theorem 4.1 reduces to

$$\left\| \sum_{i=1}^{p} \lambda_i \nabla (f_i - \bar{v}_i g_i)(x_\tau) + \rho \sum_{j=1}^{m} \nabla h_j(x_\tau) + \sum_{r=1}^{k} s_r \nabla q_r(x_\tau) + \xi \right\| \leqq \tau$$

for some $\xi \in N_C(x_\tau)$, the normal cone to C at x_τ.

Thus, starting from any initial point, we can generate a sequence in the neighbourhood of the ε–optimal solution of (FP) at which the above stated optimality condition holds for some $(\lambda, \rho, s, \xi) \geqq 0, \sum_{i=1}^{p} \lambda_i = 1$. The solution thus obtained is correct up to τ-precision. The importance of this information can be judged from the fact that in most practical situations, the algorithms designed to solve nonlinear programming problems solve them only up to a given precision level. Moreover, unlike the paper of Liu and Yokoyama [6], we have not placed any convexity restriction on the functions involved in the problem. The functions are required to be locally Lipschitz only. Further, optimality conditions obtained in this paper yield additional information about the solution behaviour in its neighbourhood.

The above observations obviously indicate that the results of this research work are more general than the similar results established in earlier research articles.

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