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# EXISTENCE SOLUTIONS FOR A CLASS OF SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. The paper studies, existence solutions for second order differential inclusions with mixed semicontinuous maps, which are upper semicontinuous in some points and lower semicontinuous in remaining points.

## 1. Introduction

Existence solutions for second order differential inclusions of the form $\ddot{u}(t) \in F(t, u(t), \dot{u}(t))$, where $F : [0,1] \times E \times E \rightharpoonup E$ is a convex compact valued multifunction, Lebesgue-measurable on $[0,1]$ and upper semicontinuous on $E \times E$, have been studied where $E$ is a finite dimensional space by several authors (see [9], [10], [11]). Later, existence results for the above differential inclusion in the general context of Banach spaces has been proved by Azzam-Castaing-Thibault [3]. The aim of this paper is to provide new existence results for Problem (1), where $F$ is a mixed semicontinuous multifunction. Namely, we consider the differential inclusion in a separable Banach space, of the form

$$\begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)) \subset \Gamma(t) \text{ a.e in } [0,1]; \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $F : [0,1] \times E \times E \rightharpoonup E$ is a mixed semicontinuous multifunction with nonempty compact values and $\Gamma : [0,1] \rightharpoonup E$ is a nonempty convex compact valued Lebesgue-measurable and integrably bounded multifunction; that is, the scalar function $t \mapsto |\Gamma(t)| := \sup\{\|x\| : \ x \in \Gamma(t)\}$ is Lebesgue integrable on [0,1].

One of possible ways to obtain the existence solutions of the first order differential inclusion

$$\begin{cases} \dot{u}(t) \in F(t, u(t)) \text{ a.e in } [0,1]; \\ u(0) = u_0, \end{cases} \tag{2}$$

where $F$ is a mixed semicontinuous multifunction has been treated by Fryszkowski-Gorniewicz [7]. The authors have considered the mapping

$$K_F(s) = \{u \in \mathbf{L}^1([0,1]) : \ u(t) \in F(t, s(t)) \text{ a.e } t \in [0,1]\},$$

which is defined on $\mathbf{C}([0,1])$ and takes decomposable subsets of $\mathbf{L}^1([0,1])$ as values. They have constructed an u.s.c and convex multifunction $M : \mathbf{C}([0,1]) \rightharpoonup \mathbf{L}^1([0,1])$, which called a multiselection, such that $M(s) \subset K_F(s)$, and have concluded the existence of solutions to the cauchy problem (2) from the fixed point theorem.

*Key words and phrases.* Multifunction, mixed semicontinuous conditions, differential inclusions, second derivative, selections.

In this paper, we use this technique to prove the existence results for our problem (1).

## 2. Preliminaries and notations

Throughout, $(E, \|\cdot\|)$ is a separable Banach space and $E'$ is its topological dual, $\mathcal{L}([0,1])$ is the $\sigma$-algebra of Lebesgue-measurable sets of $[0,1]$, $\lambda = dt$ is the Lebesgue measure on $[0,1]$ and $\mathcal{B}(E)$ is the $\sigma$-algebra of Borel subsets of $E$. By $\mathbf{L}^1_E([0,1])$ we denote the space of all Lebesgue-Bochner integrable E-valued functions defined on $[0,1]$. Let $\mathbf{C}_E([0,1])$ be the Banach space of all continuous functions $u$ from $[0,1]$ into $E$, endowed with the sup-norm. By $\mathbf{W}^{2,1}_E([0,1])$ we denote the space of all continuous functions $u$ in $\mathbf{C}_E([0,1])$ such that their first derivatives are continuous and their second weak derivatives belong to $\mathbf{L}^1_E([0,1])$. For a set $A \subset E$, $\overline{co}A$ is its closed convex hull.

A set $K \subset \mathbf{L}^1_E([0,1])$ is said to be decomposable if and only if for every $u, v \in K$ and any $A \in \mathcal{L}([0,1])$ we have $u.\chi_A + v.(1-\chi_A) \in K$, where $\chi_A$ stands for the characteristic function of $A$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The family of all nonempty closed and decomposable sets in $\mathbf{L}^1_E([0,1])$ we will denote by $\mathbf{dec}_E([0,1])$. For given $K \in \mathbf{dec}_E([0,1])$ and $\chi \in \mathbf{L}^\infty_{\mathbb{R}}([0,1])$ let $K\chi$ stands for the set

$$K\chi = \{u \in \mathbf{L}^1_E([0,1]) : \ u = v.\chi \text{ and } v \in K\}.$$

Consider a multifunction $K : E \to \mathbf{dec}_E([0,1])$, for given $\chi : E \to \mathbf{L}^\infty_{\mathbb{R}}([0,1])$ denote by $K\chi(\cdot)$ the multifunction defined by $K\chi(x) = K(x)\chi(x)$.

we need the following results in [7].

**Definition 2.1.** The mapping $D : E \to \mathcal{L}([0,1])$ is said to be:

(a) lower semicontinuous with respect to $\lambda$ ($\lambda$-l.s.c.) at $x_0$ if and only if

$$\lim_{x \to x_0} \lambda[D(x_0) \setminus D(x)] = 0;$$

(b) upper semicontinuous with respect to $\lambda$ ($\lambda$-u.s.c.) at $x_0$ if and only if $G(s) = [0,1] \setminus D(s)$ is $\lambda$-l.s.c. at $x_0$;

(c) $\lambda$-l.s.c. ($\lambda$-u.s.c.) if it is $\lambda$-l.s.c. ($\lambda$-u.s.c.) at every point $x_0 \in E$;

(d) continuous with respect to $\lambda$ ($\lambda$-c.) at $x_0$ if and only if it is both $\lambda$-l.s.c. and $\lambda$-u.s.c. at $x_0$.

**Proposition 2.2.** *Notice that*

(a) *$\lambda$-c. of $D : E \to \mathcal{L}([0,1])$ is equivalent to the continuity of $x \mapsto \chi_{D(x)}$ from $E$ into $\mathbf{L}^1$.*

(b) *$\lambda$-l.s.c. of $D : E \to \mathcal{L}([0,1])$ is equivalent to the existence of $\lambda$-c. $D_k : E \to \mathcal{L}([0,1])$, $k = 1, 2, ....$, such that*

$$\chi_{D(x)} = \sup_k \chi_{D_k(x)} \quad \textit{for } x \in E.$$

(c) $\lambda$-*u.s.c. of* $D : E \to \mathcal{L}([0,1])$ *is equivalent to the existence of* $\lambda$-*c.* $D_k : E \to \mathcal{L}([0,1])$, $k = 1, 2, ....$, *such that*

$$\chi_{D(x)} = \inf_k \chi_{D_k(x)} \quad \text{for } x \in E.$$

**Lemma 2.3.** *Let* $U : I := [0,1] \to E$ *be a measurable multifunction with closed values for almost all* $t \in I$. *For any given* $x \in \mathbf{C}_E(I)$ *denote by* $D(x) = \{t : \ x(t) \notin U(t)\}$. *Then* $D(x)$ *is* $\lambda$-*l.s.c.*

**Definition 2.4.** Let $K : E \to \mathbf{dec}_E([0,1])$ be a multifunction satisfying

(i) $K(\cdot)$ is integrably bounded;

(ii) there is a $\lambda$-l.s.c. mapping $D : E \to \mathcal{L}([0,1])$ such that

(H1) $K\chi_D(\cdot)$ is l.s.c.;

(H2) for every $x_0 \in E$ the map $K(\cdot)(1 - \chi_{D(x_0)})$ is u.s.c. at $x_0$.

Then $K$ is called mixed semicontinuous (m.s.c.).

**Theorem 2.5.** *Let* $K : E \to \mathbf{dec}_E([0,1])$ *be a mixed semicontinuous multifunction. Then* $K(\cdot)$ *admits an u.s.c. multiselection* $M(\cdot)$. *This multiselection can be chosen in the form*

$$M(x) = l(x).\chi_{D(x)} + K_{(1-\chi_D)}(x)$$

*where* $l : E \to \mathbf{L}^1_E([0,1])$ *is a properly constructed Borel selection of* $K(\cdot)$.
*Moreover, if the sets* $K(x)(1 - \chi_{D(x)})$ *are convex, then the multiselection* $M(\cdot)$ *is convex valued.*

## 3. Existence results for second order differential inclusions

We begin by a lemma which summarizes some properties of some Hartman type function (see [2], [3], [9], [8], [10]). It is useful in the study of our boundary problem for differential inclusions.

**Lemma 3.1.** *Let* $E$ *be a separable Banach space and let* $G : [0,1] \times [0,1] \to \mathbb{R}$ *be the function defined by*

$$G(t,s) = \begin{cases} (t-1)s & \text{if } 0 \le s \le t \\ t(s-1) & \text{if } t \le s \le 1. \end{cases}$$

*Then the following assertions hold:*

1) *if* $u \in \mathbf{W}^{2,1}_E([0,1])$ *with* $u(0) = u(1) = 0$, *then*

$$u(t) = \int_0^1 G(t,s)\ddot{u}(s)ds, \forall t \in [0,1],$$

2) $G(\cdot, s)$ *is derivable on* $[0,1]$, *for every* $s \in [0,1]$, *its derivative is given by*

$$\frac{\partial G}{\partial t}(t,s) = \begin{cases} s & \text{if } 0 \le s < t \\ (s-1) & \text{if } t \le s \le 1. \end{cases}$$

3) $G(\cdot,\cdot)$ *and* $\dfrac{\partial G}{\partial t}(\cdot,\cdot)$ *satisfies*

$$\sup_{t,s\in[0,1]} |G(t,s)| \le 1, \qquad \sup_{t,s\in[0,1]} |\frac{\partial G}{\partial t}(t,s)| \le 1.$$

4) *Let* $f \in \mathbf{L}^1_E([0,1])$ *and let* $u_f : [0,1] \to E$ *be the function defined by*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1],$$

*then* $u_f(0) = u_f(1) = 0$.

*Further, the function* $u_f$ *is derivable, and its derivative* $\dot{u}_f$ *satisfies*

$$\lim_{h\to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds$$

*for all* $t \in [0,1]$. *Consequently* $\dot{u}_f$ *is a continuous mapping from* $[0,1]$ *into* $E$.

5) *The function* $\dot{u}_f$ *is scalary derivable, that is, for every* $x' \in E'$, *the scalar function* $\langle x', \dot{u}_f(\cdot)\rangle$ *is derivable, and its weak derivative* $\ddot{u}_f$ *is equal to* $f$ *a.e.*

Let us mention a useful consequence of Lemma 3.1.

**Proposition 3.2.** *Let* $E$ *be a separable Banach space and let* $f : [0,1] \to E$ *be a continuous mapping (respectively a mapping in* $\mathbf{L}^1_E([0,1])$*). Then the function*

$$u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1]$$

*is the unique* $\mathbf{C}^2_E([0,1])$*-solution (respectively* $\mathbf{W}^{2,1}_E([0,1])$*-solution) to the differential equation*

$$\begin{cases} \ddot{u}(t) = f(t) \ \ \forall t \in [0,1]; \\ u(0) = u(1) = 0. \end{cases}$$

Now, we are ready to prove the main existence theorem.

**Theorem 3.3.** *Let* $E$ *be a separable Banach space, and let* $F : [0,1] \times E \times E \rightharpoonup E$ *be a multifunction with nonempty compact values satisfying:*

(i) $F$ *is* $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$*-measurable;*
(ii) $F(t,\cdot,\cdot)$ *is upper semicontinuous for almost every* $t \in [0,1]$*;*
(iii) *for each* $(t,x,y) \in [0,1] \times E \times E$ *such that* $F(t,x,y)$ *is nonconvex the map* $F(t,\cdot,\cdot)$ *is lower semicontinuous at* $(x,y)$.

*Let* $\Gamma : [0,1] \rightharpoonup E$ *be an integrably bounded multifunction with nonempty convex compact values such that* $F(t,x,y) \subset \Gamma(t)$ *for every* $(t,x,y) \in [0,1] \times E \times E$. *Then the* $\mathbf{W}^{2,1}_E([0,1])$*-solutions set of the differential inclusion*

$$\begin{cases} \ddot{u}(t) \in F(t,u(t),\dot{u}(t)) \subset \Gamma(t) \ \ a.e \ in \ [0,1]; \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

*is nonempty and compact in the Banach space* $\mathbf{C}_E([0,1])$.

*Proof. Step 1.* Let us consider the differential inclusion

$$\begin{cases} \ddot{u}(t) \in \Gamma(t) \ \text{ a.e in } [0,1]; \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

We wish to show that the $\mathbf{W}_E^{2,1}([0,1])$-solutions set $\mathbf{X}_\Gamma$ of (3) is nonempty convex compact in the Banach space $\mathbf{C}_E([0,1])$ endowed with the topology of uniform convergence.

First, let us recall (see [6]), that the set $\mathbf{S}_\Gamma^1$ of all measurable selections of $\Gamma$ is convex and compact for the weak topology $\sigma(\mathbf{L}_E^1([0,1]), \mathbf{L}_{E'}^\infty([0,1]))$. Furthermore, the set-valued integral

$$\int_0^1 \Gamma(t)dt = \{\int_0^1 f(t)dt, \quad f \in \mathbf{S}_\Gamma^1\}$$

is convex and norm-compact. (See [4], [5], [6]) for a more general result.
In view of Lemma 3.1 and Proposition 3.2, the solutions set $\mathbf{X}_\Gamma$ is nonempty and characterized by

$$\mathbf{X}_\Gamma = \{u_f : [0,1] \to E : \quad u_f(t) = \int_0^1 G(t,s)f(s)ds, \ \forall t \in [0,1]; \ f \in \mathbf{S}_\Gamma^1\}.$$

Clearly $\mathbf{X}_\Gamma$ is convex. We claim that $\mathbf{X}_\Gamma$ is compact in $\mathbf{C}_E([0,1])$. Since

$$\begin{aligned}\|u_f(t) - u_f(\tau)\| &\leq \int_0^1 |G(t,s) - G(\tau,s)| \|f(s)\| ds \\ &\leq \int_0^1 |G(t,s) - G(\tau,s)| |\Gamma(s)| ds\end{aligned}$$

for all $f \in \mathbf{S}_\Gamma^1$ and for all $t, \tau \in [0,1]$, $\mathbf{X}_\Gamma$ is equicontinuous in $\mathbf{C}_E([0,1])$. Further the set $\mathbf{X}_\Gamma(t)$ is relatively compact in $E$ because it is included in the norm compact set $\int_0^1 G(t,s)\Gamma(s)ds$. Let $(f_n)_n$ be a sequence in $\mathbf{S}_\Gamma^1$. As $\mathbf{S}_\Gamma^1$ is weakly compact in $\mathbf{L}_E^1([0,1])$, we extract from $(f_n)$ a sequence $(f_m)$ such that $(f_m)$ converges $\sigma(\mathbf{L}_E^1([0,1]), \mathbf{L}_{E'}^\infty([0,1]))$ to a function $f \in \mathbf{S}_\Gamma^1$. The sequence $(u_{f_n})$ is relatively compact in $\mathbf{C}_E([0,1])$ by Arzelà-Ascoli's theorem the sequence $(u_{f_m})$ converges uniformly to a continuous function $\zeta \in \mathbf{C}_E([0,1])$. In particular for every $x' \in E'$ and for every $t \in [0,1]$, we have

$$(*) \qquad \begin{aligned}\lim_{m\to\infty} \int_0^1 \langle G(t,s)x', f_m(s)\rangle ds &= \lim_{m\to\infty} \langle x', \int_0^1 G(t,s) f_m(s) ds\rangle \\ \int_0^1 \langle G(t,s)x', f(s)\rangle ds &= \langle x', \int_0^1 G(t,s)f(s)ds\rangle.\end{aligned}$$

As the set-valued integral $\int_0^1 G(t,s)\Gamma(s)ds$ $(t \in [0,1])$ is norm compact, $(*)$ shows that the sequence $(u_{f_m}(\cdot)) = (\int_0^1 G(\cdot,s)f_m(s)ds)$ converges pointwise to $u_f(\cdot)$, for $E$ endowed with the strong topology, thus we get $\zeta = u_f$. This shows the compactness of $\mathbf{X}_\Gamma$ in $\mathbf{C}_E([0,1])$.

At this point, it is worth to mention that the sequence $(\dot{u}_{f_m}(\cdot)) = (\int_0^1 \frac{\partial G}{\partial t}(\cdot,s)f_m(s)ds)$ converges pointwise to $\dot{u}_f(\cdot)$, for $E$ endowed with the strong topology, using the weak convergence of $(f_m)$ and the norm compactness of the set-valued integral $\int_0^1 \frac{\partial G}{\partial t}(t,s)\Gamma(s)ds$ $(t \in [0,1])$.

*Step 2.* With problem (1) we shall associate the multifunction $K_F : \mathbf{X}_\Gamma \rightharpoondown \mathbf{L}^1_E([0,1]$ given as follows

$$K_F(u_f) = \{v \in \mathbf{L}^1_E([0,1] : \ \ v(t) \in F(t, u_f(t), \dot{u}_f(t)) \ \text{ a.e. in } [0,1]\}.$$

In view of the existence theorem of measurable selections (See [1], [6]) we deduce that $K_F(u_f) \neq \emptyset$ for every $u_f \in \mathbf{X}_\Gamma$. Moreover, $K_F(u_f)$ is closed decomposable for every $u_f \in \mathbf{X}_\Gamma$.

Now, we proceed to prove that $K_F$ is a mixed semicontinuous multifunction and consequently it has a convex-valued multiselection $M$.

The proof would be similar to the one for the first order problem in Theorem 3.9 example (3) in [7]. We include it for the convenience of the reader.

Let $D : \mathbf{X}_\Gamma \rightharpoondown \mathcal{L}([0,1])$, such that

$$D(u_f) = \{t \in [0,1] : \ \ F(t, u_f(t), \dot{u}_f(t)) \text{ is nonconvex}\}.$$

Denote by

$$A = \{(t,x,y) \in [0,1] \times E \times E : \ \ F(t,x,y) = \overline{co}F(t,x,y)\},$$

clearly $A$ is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable. Consider for every $t \in [0,1]$ the set

$$U(t) = \{(x,y) \in E \times E : \ \ (t,x,y) \in A\}$$

and observe that, for every $u_f \in \mathbf{X}_\Gamma$

$$D(u_f) = \{t \in [0,1] : \ \ (u_f(t), \dot{u}_f(t)) \notin U(t)\}.$$

Since $A$ is $\mathcal{L}([0,1]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable, the map $t \mapsto U(t)$ is $\mathcal{L}([0,1])$-measurable. We need to check that $D(\cdot)$ is $\lambda$-l.s.c. Using Lemma 2.3 we need to prove that the sets $U(t)$ are closed for all $t \in [0,1]$. Let us assume to a contrary that, for some $t \in [0,1]$ $U(t)$ is not closed. Then there exists a sequence $(x_n, y_n) \in E \times E$, $(x_n, y_n) \to (x_0, y_0)$ such that $(x_n, y_n) \in U(t)$ and $(x_0, y_0) \notin U(t)$. Therefore $F(t, x_n, y_n)$ are convex for all $n \in \mathbb{N}^*$ while $F(t, x_0, y_0)$ is not. Then there are $w, z \in F(t, x_0, y_0)$ and $\alpha \in ]0,1[$ such that $\alpha w + (1-\alpha)z \notin F(t, x_0, y_0)$. By the assumptions, the map $(x,y) \mapsto F(t,x,y)$ is continuous at $(x_0, y_0)$, then we can pick up $w_n, z_n \in F(t, x_n, y_n)$ such that $w_n \to w$ and $z_n \to z$. Observe that $\alpha w_n + (1-\alpha)z_n \in F(t, x_n, y_n)$ and $\alpha w_n + (1-\alpha)z_n \to \alpha w + (1-\alpha)z \notin F(t, x_0, y_0)$, against the upper semicontinuity of $F(t, \cdot, \cdot)$ at $(x_0, y_0)$. Then the sets $U(t)$ are closed for all $t \in [0,1]$. Consequently $D(\cdot)$ is $\lambda$-l.s.c.

Now, we prove that $D(\cdot)$ satisfies $(H1)$ and $(H2)$ of Definition 2.4. By the upper semicontinuity of $F(t, \cdot, \cdot)$ we get the u.s.c of $K_F$ and thus $(H2)$ holds. To see $(H1)$, fixe $u_{f_0} \in \mathbf{X}_\Gamma$, $v_0 \in K_F(u_{f_0})$ and take any sequence $(u_{f_n}) \subset \mathbf{X}_\Gamma$ such that $u_{f_n} \to u_{f_0}$ and thus $\dot{u}_{f_n} \to \dot{u}_{f_0}$. Let $v_n \in K_F(u_{f_n})$ be such integrable functions such that for a.e. $t \in [0,1]$ we have

$$d(v_n(t), F(t, u_{f_0}(t), \dot{u}_{f_0}(t))) = \|v_n(t) - v_0(t)\|.$$

For any $t \in D(u_{f_0})$, $F(t, \cdot, \cdot)$ is continuous at $(u_{f_0}(t), \dot{u}_{f_0}(t))$ and therefore

$$\|v_n(t) - v_0(t)\| \to 0 \ \text{ a.e. in } D(u_{f_0}).$$

The sequence $(v_n)$ is integrably bounded and thus

$$\lim_{n\to\infty}\int_{D(u_{f_0})}\|v_n(t)-v_0(t)\|d\lambda=0.$$

Denote by

$$A_m(u_f)=\bigcup_{k=1}^{m}D_k(u_f)$$

where $D_k(u_f)$ are as in Proposition 2.2. Observe that $(A_m(\cdot))$ is an increasing sequence of $\lambda$-continuous mapping such that

$$D(u_f)=\bigcup_{m=1}^{\infty}A_m(u_f).$$

Thus

$$\lim_{n\to\infty}\int_{[0,1]}v_n\chi_{A_m(u_{f_n})}d\lambda=\lim_{n\to\infty}\int_{A_m(u_{f_n})}v_nd\lambda=\sum_{k=1}^{m}\lim_{n\to\infty}\int_{D_k(u_{f_n})}v_nd\lambda$$
$$=\sum_{k=1}^{m}\int_{D_k(u_{f_0})}v_0d\lambda=\int_{A_m(u_{f_0})}v_0d\lambda=\int_{[0,1]}v_0\chi_{A_m(u_{f_0})}d\lambda$$

then, $v_n\chi_{A_m(u_{f_n})}\to v_0\chi_{A_m(u_{f_0})}$ in $\mathbf{L}^1([0,1])$. Consequently $K_F(u_f)\chi_{A_m(u_f)}$ are l.s.c, $m=1,2,....$, and thus

$$K_F(u_f)\chi_{D(u_f)}=\bigcup_{m=1}^{\infty}K_F(u_f)\chi_{A_m(u_f)}$$

is l.s.c. We concluded that $K_F$ is mixed semicontinuous. Then, there exists an u.s.c multiselection $M:\mathbf{X}_\Gamma\rightharpoonup\mathbf{L}^1_E([0,1])$ with closed convex values such that $M(u_f)\subset K_F(u_f)$ for every $u_f\in\mathbf{X}_\Gamma$.

*Step 3.* Taking the results obtained in *Step 1* account, a map $u:[0,1]\to E$ is a $\mathbf{W}^{2,1}_E([0,1])$-solution of the problem (1), iff there exists $f\in\mathbf{S}^1_\Gamma$ such that $u:=u_f\in\mathbf{X}_\Gamma$ and such that $f(t)\in F(t,u_f(t),\dot{u}_f(t))$ for almost every $t\in[0,1]$.
For any $f\in\mathbf{S}^1_\Gamma$ consider the set

$$\Phi(f)=\{g\in\mathbf{L}^1_E([0,1]):\quad g\in M(u_f)\text{ a.e. in }[0,1]\}.$$

From the existence theorem of measurable selections (see [1], [6]) one can deduce that $\Phi(f)$ are nonempty subsets of $\mathbf{L}^1_E([0,1])$. It is clear that $\Phi(f)$ are convex weakly compact subsets of $\mathbf{S}^1_\Gamma$. We need to check that $\Phi:\mathbf{S}^1_\Gamma\rightharpoonup\mathbf{S}^1_\Gamma$ is upper semicontinuous on the convex weakly compact metrizable set $\mathbf{S}^1_\Gamma$. Equivalently, we need to prove that the graph of $\Phi$ is sequentially weakly compact in $\mathbf{S}^1_\Gamma\times\mathbf{S}^1_\Gamma$. Let $(f_n)_n$ be a sequence in $\mathbf{S}^1_\Gamma$. By extracting a subsequence we may suppose that $(f_n)$ converges weakly to $f\in\mathbf{S}^1_\Gamma$. It follows that $(u_{f_n})$ converges pointwise to $u_f$, for $E$ endowed with the norm topology. Let $g_n\in\Phi(f_n)\subset\mathbf{S}^1_\Gamma$. We may suppose that $(g_n)$ converges weakly to some element $g\in\mathbf{S}^1_\Gamma$. As $g_n\in M(u_{f_n})$ using the upper semicontinuity of $M(\cdot)$ we can check that $g\in M(u_f)$. Thus the graph of $\Phi$ is weakly compact in the weakly compact set $\mathbf{S}^1_\Gamma\times\mathbf{S}^1_\Gamma$. Hence, the application of the

Kakutani-Ky Fan fixed point theorem to the multifunction $\Phi(\cdot)$, there exists $f_0 \in \mathbf{S}^1_\Gamma$ such that $f_0 \in M(f_0) \subset K_F(f_0)$, and so $f_0(t) \in F(t, u_{f_0}(t), \dot{u}_{f_0}(t))$ for almost every $t \in [0,1]$. Equivalently, (see Lemma 3.1) $\ddot{u}_{f_0}(t) \in F(t, u_{f_0}(t), \dot{u}_{f_0}(t))$ a.e. *in* $[0,1]$ with $u_{f_0}(0) = u_{f_0}(1)$, what, in turn, means that the mapping $u_{f_0}$ is a solution to our problem (1). □

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