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UNIFORM SMOOTHNESS AND U-CONVEXITY OF ψ-DIRECT SUMS

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Abstract. We study the ψ-direct sum, introduced by K.-S. Saito and M. Kato, of U-spaces, introduced by K. S. Lau. For Banach spaces X and Y and a continuous convex function ψ on the unit interval $[0,1]$ satisfying certain conditions, let $X \oplus_\psi Y$ be the ψ-direct sum of X and Y equipped with the norm associated with ψ. We first show that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is isometric to the space $X^* \oplus_\varphi Y^*$ for some continuous convex function φ satisfying the same conditions as of ψ. We introduce the so-called u-spaces and show that: (1) $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function. We also show that (2) $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function. As consequences, using the notion of ultrapower, we obtain : (3) $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function, and (4) $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

1. Introduction

For every continuous convex function ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ $(0 \leq t \leq 1)$, there corresponds a unique absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^2$ (see Bonsall and Duncan [3], also [19]). Recently, in [16] the authors introduced the ψ-direct sums $X \oplus_\psi Y$ of Banach spaces X and Y equipped with the norm associated with ψ, and proved that $X \oplus_\psi Y$ is uniformly convex if and only if X, Y are uniformly convex and ψ is strictly convex. We write $X \simeq Y$ to indicate that X and Y are isometric (or Banach isomorphism, see [12]).

The purposes of this paper are to characterize uniform smoothness and U-convexity of $X \oplus_\psi Y$. In Section 2 we shall recall some fundamental facts on the ψ-direct sums of Banach spaces and introduce the dual function φ of ψ so that the dual space $(X \oplus_\psi Y)^*$ of $X \oplus_\psi Y$ is $X^* \oplus_\varphi Y^*$. In Section 3 we shall show that the ultrapower of $X \oplus_\psi Y$ is the ψ-direct sum of the ultrapowers of X and of Y. In Section 4 we shall prove that $X \oplus_\psi Y$ is a smooth space if and only if X, Y are smooth spaces and ψ is a smooth function, and by using the ultrapower technique we obtain that $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y are uniformly smooth and ψ is a smooth function. In Section 5 we introduce new spaces, namely u-spaces, and prove that $X \oplus_\psi Y$ is a u-space if and only if X, Y are u-spaces and ψ is a u-function, and again by using the ultrapowers we have $X \oplus_\psi Y$ is a U-space if and only if X, Y are U-spaces and ψ is a u-function.

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2. The ψ-direct sums

Let X be a Banach space. Throughout this paper, let X^* be the dual space of X, $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \le 1\}$, and for $x \neq 0$, $\nabla_x = \{f \in S_{X^*} : f(x) = \|x\|\}$. In this section we shall recall the definition of the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y. A norm on $\mathbb{C}^2$ is called *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $(z,w) \in \mathbb{C}^2$ and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The set of all absolute normalized norms on $\mathbb{C}^2$ is denoted by N_a. The l_p-norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ on $\mathbb{C}^2$ are examples of such norms, and for any norm $\|\cdot\| \in N_a$,

$$\|\cdot\|_\infty \le \|\cdot\| \le \|\cdot\|_1.$$

Let Ψ be the set of all continuous convex functions ψ on $[0,1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. N_a and Ψ are in one-to-one correspondence under the following equations. For each $\|\cdot\| \in N_a$, the function ψ defined by $\psi(t) = \|(1-t,t)\|$ $(0 \le t \le 1)$ belongs to Ψ. Conversely, for each $\psi \in \Psi$, let $\|(0,0)\|_\psi = 0$, and $\|(z,w)\|_\psi = (|z| + |w|)\psi(\frac{|w|}{|w|+|z|})$ for $(z,w) \neq (0,0)$ and this norm belongs to N_a (see [3] and [19]). For Banach spaces X and Y, we denote by $X \oplus_\psi Y$ the direct sum $X \oplus Y$ equipped with the norm

$$\|(x,y)\| = \|(\|x\|, \|y\|)\|_\psi \text{ for } (x,y) \in X \oplus Y.$$

Thus, under this norm, $X \oplus_\psi Y$, which will be called the ψ-direct sum of X and Y, is a Banach space and for all $(x,y) \in X \oplus Y$ we also have (see [16])

$$\|(x,y)\|_\infty \le \|(x,y)\|_\psi \le \|(x,y)\|_1.$$

Saito et al. [16] extended the concept to absolute normalized norm on $\mathbb{R}^n$. The corresponding set of all continuous convex functions on the $(n-1)-$simplex $\{(s_1,...,s_{n-1}) \in \mathbb{R}_+^{n-1} : s_1 + ... + s_{n-1} \le 1\}$ will be denoted by Ψ_n.

Now we show that the dual space of this ψ-direct sum is a direct sum $X \oplus_\varphi Y$ of the same kind for some $\varphi \in \Psi$. We first define

$$\varphi_\psi(s) = \varphi(s) := \sup_{t\in[0,1]} \frac{st + (1-s)(1-t)}{\psi(t)}$$

for $s \in [0,1]$. We show that $\varphi \in \Psi$ and call it *the dual function* of ψ.

Proposition 1. *The above function φ is continuous, convex on $[0,1]$ and satisfies $\varphi(s) \ge \max\{s, 1-s\}$ for all $s \in [0,1]$.*

Proof. It is easy to see that $\varphi(\cdot)$ is continuous. To show that φ is convex, we let $s_1, s_2 \in [0,1]$ and consider

$$\begin{aligned}\varphi\Big(\frac{s_1+s_2}{2}\Big) &= \sup_{t\in[0,1]} \frac{\frac{s_1+s_2}{2}t + (1-\frac{s_1+s_2}{2})(1-t)}{\psi(t)} \\ &= \sup_{t\in[0,1]} \frac{1}{2}\frac{s_1t + s_2t + (1-s_1)(1-t) + (1-s_2)(1-t)}{\psi(t)} \\ &\le \frac{1}{2}(\varphi(s_1) + \varphi(s_2)),\end{aligned}$$

which verifies the convexity of $\varphi(\cdot)$. Next we prove the last assertion. Since $\psi(t) \le 1$ for all $t \in [0,1]$,

$$\varphi(s) \ge \sup_{t\in[0,1]} \{st + (1-s)(1-t)\} \ge \max\{s, 1-s\}$$

for all $s \in [0,1]$, and the proof is complete. □

Theorem 2. *The dual space $(X \oplus_\psi Y)^*$ is isometric to $X^* \oplus_\varphi Y^*$, where φ is the dual function of ψ. Moreover, each bounded linear functional F in $(X \oplus_\psi Y)^*$ can be (uniquely) represented by (f, g) where $f \in X^*$ and $g \in Y^*$ and*

$$F(x, y) = f(x) + g(y)$$

for all $(x, y) \in X \oplus_\psi Y$. In this case, $\|F\| \le \|(f, g)\|_\varphi \|(x, y)\|_\psi$.

Proof. Define $T : X^* \oplus_\varphi Y^* \to (X \oplus_\psi Y)^*$ by

$$T(f, g)(x, y) = f(x) + g(y)$$

where $f \in X^*$, $g \in Y^*$, $x \in X$, and $y \in Y$. It is easy to see that T is linear. Moreover, by the definition of φ, we have, recalling that the norm of each nonzero element (f, g) of the φ-direct sum $X^* \oplus_\varphi Y^*$ is defined by

$$\|(f, g)\|_\varphi = (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$\begin{aligned}
|T(f, g)(x, y)| &\le \|f\|\|x\| + \|g\|\|y\| \\
&= (\|f\| + \|g\|)(\|x\| + \|y\|)\frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\| + \|g\|)(\|x\| + \|y\|)} \\
&\le (\|f\| + \|g\|)\varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big)(\|x\| + \|y\|)\psi\Big(\frac{\|y\|}{\|x\| + \|y\|}\Big) \\
&= \|(f, g)\|_\varphi \|(x, y)\|_\psi,
\end{aligned}$$

for all nonzero (f, g). Thus, $T(f, g)$ is actually an element of $(X \oplus_\psi Y)^*$. For each $F \in (X \oplus_\psi Y)^*$, $F(\cdot, 0)$ and $F(0, \cdot)$ are bounded linear functionals on X and Y, respectively. Put $f(x) = F(x, 0)$ and $g(y) = F(0, y)$, then $T(f, g) = F$ and the surjectivity of T is proved.

Finally we prove that T is an isometry, i.e., $\|T(f, g)\| = \|(f, g)\|_\varphi$. From the above calculation, we always have $\|T(f, g)\| \le \|(f, g)\|_\varphi$. Now we prove the reverse inequality. We choose sequences $\{t_n\} \subset [0, 1]$, $\{x_n\} \subset S_X$, and $\{y_n\} \subset S_Y$ so that

$$\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)\|f\|}{\|f\| + \|g\|} + \frac{t_n\|g\|}{\|f\| + \|g\|}\Big) \to \varphi\Big(\frac{\|g\|}{\|f\| + \|g\|}\Big),$$

$$f(x_n) \to \|f\|, \quad \text{and} \quad g(y_n) \to \|g\| \quad \text{as } n \to \infty.$$

Therefore, since $\frac{1}{\psi(t_n)}((1-t_n)x_n, t_n y_n) \in S_{X\oplus_\psi Y}$,

$$\begin{aligned}
\|T(f, g)\| &\ge \frac{1}{\psi(t_n)}\Big(f((1-t_n)x_n) + g(t_n y_n)\Big) \\
&= (\|f\| + \|g\|)\frac{1}{\psi(t_n)}\Big(\frac{(1-t_n)f(x_n)}{\|f\| + \|g\|} + \frac{t_n g(y_n)}{\|f\| + \|g\|}\Big).
\end{aligned}$$

The last expression tends to $\|(f,g)\|_\varphi$ as $n\to\infty$, proving that $\|T(f,g)\| \geq \|(f,g)\|_\varphi$ and this completes the proof. □

Our first application of Theorem 2 is to show that reflexivity is preserved under the ψ-direct sums.

Corollary 3. *For each $\psi \in \Psi$, $X \oplus_\psi Y$ is reflexive if and only if X and Y are reflexive.*

Proof. We only proof the sufficiency. We first show, without using reflexivity, that $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\psi Y^{**}$, i.e., they are isometric. For this, we let φ and then θ be the dual functions of ψ and of φ, respectively. Thus $(X \oplus_\psi Y)^* \simeq X^* \oplus_\varphi Y^*$ by the isometry T where $TF = (F_1, F_2)$, $F_1 = F(\cdot, 0)$ and $F_2 = F(0,\cdot)$; and $(X^* \oplus_\varphi Y^*)^* \simeq X^{**} \oplus_\theta Y^{**}$ by the isometry U where $UG = (G_1, G_2)$, $G_1 = G(\cdot, 0)$ and $G_2 = G(0,\cdot)$. Hence $(X \oplus_\psi Y)^{**} \simeq X^{**} \oplus_\theta Y^{**}$ via the isometry which maps $L \in (X \oplus_\psi Y)^{**}$ to $ULT^{-1} = (LT^{-1}(\cdot,0), LT^{-1}(0,\cdot)) \in X^{**} \oplus_\theta Y^{**}$ so that $ULT^{-1}(x^*, y^*) = (LT^{-1}(x^*,0), LT^{-1}(0,y^*)) = (L(x^*,0), L(0,y^*)) = (L_1(x^*), L_2(y^*))$. In particular, when $L = L_{(x,y)}$, the evaluation map at (x,y), i.e., $L_{(x,y)}(F) = F(x,y) = F_1(x) + F_2(y)$ for $F \in (X \oplus_\psi Y)^*$, $UL_{(x,y)}T^{-1}(x^*, y^*) = x^*(x) + y^*(y) = L_x(x^*) + L_y(y*) = (L_x, L_y)(x^*, y^*)$. This shows that $\|(x,y)\|_\psi = \|L_{(x,y)}\| = \|(L_x, L_y)\|_\theta$ for $(x,y) \in X \oplus Y$. Therefore, $\psi(\frac{\|y\|}{\|x\|+\|y\|}) = \theta(\frac{\|L_y\|}{\|L_x\|+\|L_y\|}) = \theta(\frac{\|y\|}{\|x\|+\|y\|})$ for $\|x\| + \|y\| \neq 0$. From this we can easily see that $\psi = \theta$.

Now suppose that X and Y are reflexive. Thus elements in X^{**} and Y^{**} are of the form L_x and L_y for some $x \in X$ and $y \in Y$. To show that $(X \oplus_\psi Y)^{**}$ is reflexive, let $L \in (X \oplus_\psi Y)^{**}$ and consider, for each $F \in (X \oplus Y)^*$, $L(F) = L(F_1, 0) + L(0, F_2) = L_x(F_1) + L_y(F_2) = F_1(x) + F_2(y) = L_{(x,y)}(F)$, for some $x \in X$ and $y \in Y$. That is $L = L_{(x,y)}$ showing that $X \oplus_\psi Y$ is reflexive and the proof is complete. □

We observe that $X \oplus_\psi Y$ is super-reflexive when (and only when) X and Y are super-reflexive. By Henson and Moore [7], this is equivalent to showing that the ultrapower $\widetilde{X \oplus_\psi Y}$ is reflexive. But this follows from Remark 5 below and Corollary 3.

3. Ultrapowers of the ψ-direct sums

The ultrapower of a Banach space is proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. In this section we prove that every ultrapower of a ψ-direct sum is isometric to the ψ-direct sum of their ultrapowers. First we recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I}X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from remark (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [17]).

Following T. Landes [11], a normed space Z is a *substitution* space (with index $I \neq \emptyset$ with any cardinality) whenever Z has a (Shauder) basis $(e_i)_{i\in I}$ (unconditional if I is uncountable) and the norm of Z is *monotone*, i.e., $\|z\| \leq \|z'\|$ whenever $0 \leq z_i \leq z'_i$ for all $i \in I$ $(z, z' \in Z)$, where we write $z = \sum_{i\in I} z_i e_i$ for $z \in Z$. Given a family $(X_i)_{i\in I}$ of normed spaces, then the Z *direct sum* $(\bigoplus_{i\in I} X_i)_Z$ of the family (X_i) is defined to be the space $\{x = (x_i)_{i\in I} \in \prod_{i\in I} X_i : \sum_{i\in I} \|x_i\| e_i \in Z\}$ endowed with the norm $\| \sum_{i\in I} \|x_i\| e_i \|_Z$. $\psi-$ direct sums are examples of $Z-$direct sums.

A property P defined for normed spaces is said to be preserved under the Z-direct-sum-operation, if the Z-direct sums of a family $(X_i)_{i\in I}$ of normed spaces satisfies P whenever all X_i do so.

The following proposition shows that, under some conditions, "normal structure" is preserved under the Z-direct-sum-operation. This result improves the first permanence result for normal structure obtained by Belluce, Kirk, and Steiner [2].

Proposition 4. [11, Theorem 2, Corollary 3 and Corollary 4] *Let Z be a substitution space with index set $I = \{1, \ldots, N\}$ such that*

$$\|z + z'\| < 2 \textit{ whenever } \|z\| = \|z'\| = 1, z_i \geq 0, z'_i \geq 0 \textit{ for all } i \in I,$$
$$\textit{and } z_i = z'_i \textit{ only for these } i \in I \textit{ for which } z_i = z'_i = 0.$$

Thus, normal structure is preserved under the Z-direct-sum-operation. In particular, if Z is strictly convex or $Z = l_p^N$ for any p with $1 < p \leq \infty$.

In case $I = \{1, ..., N\}$ and ψ is strictly convex, it follows from [9] that the norm $\|\cdot\|_\psi$ is monotone and strictly convex on $\mathbb{C}^N$. We note in passing that this result actually holds for $Z-$direct sum : The $Z-$ direct sums $(\bigoplus_i \mathrm{X}_i)_Z$ is uniformly convex if and only if Z and each of the Banach space X_i are uniformly convex with a common modulus of convexity (see Dowling [5]).

Remark 5. It is easy to see that the ultrapower of $Z-$direct sum $(\bigoplus_i \mathrm{X}_i)_Z$ is isometric to the $Z-$direct sum $(\bigoplus_i \widetilde{X}_i)_Z$ of ultrapowers. Thus in particular,

$(X_1 \widetilde{\oplus \cdots \oplus} X_N)_\psi \simeq (\widetilde{X_1} \oplus \cdots \oplus \widetilde{X_N})_\psi$. This follows from the fact that the $Z-$norm is monotone and from the continuity of norms.

It is known that X is uniformly convex if and only if $\widetilde{X}$ is strictly convex (see [17]). Combining these results and Remark 5 gives

Corollary 6. [9] *Let $X_1, ..., X_N$ be Banach spaces and $\psi \in \Psi_N$. Then $(X_1 \oplus \cdots \oplus X_N)_\psi$ is uniformly convex if and only if $X_1, ...X_N$ are uniformly convex and ψ is strictly convex.*

Thus, in the light of super-reflexivity, we can extend "normal structure" to "uniform normal structure" for ψ-direct sums whenever ψ is strictly convex.

Corollary 7. *Let $X_1, ..., X_N$ be super-reflexive Banach spaces and Z be uniformly convex. Then, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure if and only if $X_1, ..., X_N$ have uniform normal structure.*

Proof. Note that, by Khamsi [10], it suffices to show that the ultrapower $(X_1 \widetilde{\oplus \cdots \oplus} X_N)_Z$ has normal structure. But this is an immediate consequence of Remark 5 together with Proposition 4. □

It is well-known that every uniformly nonsquare space is super-reflexive (see [8]). Thus, Corollary 7 and [4, Corollary 3.7] give

Corollary 8. *Let $X_1, ..., X_N$ be Banach spaces and Z be uniformly convex. Then, if $C_{\rm NJ}(1, X_i) < 2$ for $i = 1, 2, ..., N$, the Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ has uniform normal structure.*

It is interesting to see if we can conclude that $C_{\rm NJ}(1, (X_1 \oplus \cdots \oplus X_N)_Z) < 2$ in Corollary 8.

4. Smoothness of the ψ-direct sums

A Banach space X is said to be *smooth* if for any $x \in S_X$, ∇_x is a singleton. We recall that a continuous convex function ψ has left and right derivatives ψ'_L, ψ'_R. Let G be defined on $[0, 1]$ by

$$G(0) = [-1, \psi'_R(0)],\ G(1) = [\psi'_L(1), 1],$$
$$G(t) = [\psi'_L(t), \psi'_R(t)]\ \ (0 < t < 1).$$

Given $\psi \in \Psi$, $t \in [0, 1]$, let

$$x(t) = \frac{1}{\psi(t)}(1 - t, t)$$

so that $\|x(t)\|_\psi = 1$. In [3], the authors identified the dual of $(\mathbb{C}^2, \|.\|_\psi)$ with $\mathbb{C}^2$ and used this fact to provide a proof of the following lemma.

Lemma 9. [3, Lemma 4] *For ψ, G, and x defined above,*
(1) $\nabla_{x(t)} = \{(\psi(t) - t\gamma, \psi(t) + (1 - t)\gamma) : \gamma \in G(t)\}$ *for* $0 < t < 1$,
(2) $\nabla_{x(0)} = \{(1, z(1 + \gamma)) : \gamma \in G(0), |z| = 1\}$, *and*
(3) $\nabla_{x(1)} = \{(z(1 - \gamma), 1) : \gamma \in G(1), |z| = 1\}$.

In general, using Theorem 2 and Lemma 9, we have the following:

Lemma 10. *Let* $(x,y) \in S_{X\oplus_\psi Y}$ *and* $t = \frac{\|y\|}{\|x\|+\|y\|}$. *Thus*

(1) $\nabla_{(x,y)} = \{((\psi(t)-t\gamma)f, (\psi(t)+(1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ *and* $g \in \nabla_{y/\|y\|}\}$ *for* $0 < t < 1$,

(2) $\nabla_{(x,y)} = \{(f, (1+\gamma)g) : \gamma \in G(0), f \in \nabla_x$ *and* $g \in S_{Y^*}\}$ *for* $t = 0$, *and*

(3) $\nabla_{(x,y)} = \{((1-\gamma)f, g) : \gamma \in G(1), g \in \nabla_y$ *and* $f \in S_{X^*}\}$ *for* $t = 1$.

Proof. We prove (1). Let $F = (f,g) \in \nabla_{(x,y)}$, then

$$\begin{aligned} F((x,y)) &= f(x) + g(y) \\ &\le \|f\|\|x\| + \|g\|\|y\| \\ &= \frac{\|f\|\|x\| + \|g\|\|y\|}{(\|f\|+\|g\|)(\|x\|+\|y\|)}(\|f\|+\|g\|)(\|x\|+\|y\|) \\ &\le \varphi\left(\frac{\|g\|}{\|f\|+\|g\|}\right)\psi\left(\frac{\|y\|}{\|x\|+\|y\|}\right)(\|f\|+\|g\|)(\|x\|+\|y\|) \\ &= \|F\|_\varphi\|(x,y)\|_\psi = 1. \end{aligned}$$

Thus, we have $\|f\|\|x\| + \|g\|\|y\| = 1$ and $f(x) = \|f\|\|x\|$ $g(y) = \|g\|\|y\|$, hence $(\|f\|, \|g\|) \in \nabla_{(\|x\|,\|y\|)}$ and $\frac{f}{\|f\|} \in \nabla_{\frac{x}{\|x\|}}$, $\frac{g}{\|g\|} \in \nabla_{\frac{y}{\|y\|}}$. We observe that $(\|x\|, \|y\|) = \frac{1}{\psi(t)}(1-t, t)$, thus it follows from Lemma 9 that

$$\|f\| = \psi(t) - t\gamma \text{ and } \|g\| = \psi(t) + (1-t)\gamma, \text{ for some } \gamma \in G(t).$$

Consequently, we have $(f,g) = (\|f\|\frac{f}{\|f\|}, \|g\|\frac{g}{\|g\|}) = ((\psi(t)-t\gamma)\frac{f}{\|f\|}, (\psi(t)+(1-t))\gamma\frac{g}{\|g\|})$. Thus, we have proved that $\nabla_{(x,y)} \subset \{((\psi(t)-t\gamma)f, (\psi(t)+(1-t)\gamma)g) : \gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}\}$. On the other hand, let $F = ((\psi(t)-t\gamma)f, (\psi(t)+(1-t)\gamma)g)$ where $\gamma \in G(t), f \in \nabla_{x/\|x\|}$ and $g \in \nabla_{y/\|y\|}$. Consider, by using Lemma 9,

$$\begin{aligned} F((x,y)) &= (\psi(t)-t\gamma)f(x) + (\psi(t)+(1-t)\gamma)g(y) \\ &= (\psi(t)-t\gamma)\|x\| + (\psi(t)+(1-t)\gamma)\|y\| \\ &= (\|x\|+\|y\|)\big((\psi(t)-t\gamma)(1-t) + (\psi(t)+(1-t)\gamma)t\big) \\ &= \frac{1}{\psi(t)}\big((\psi(t)-t\gamma)(1-t) + (\psi(t)+(1-t)\gamma)t\big) \\ &= 1. \end{aligned}$$

Hence, (1) has been proved. The proof of (2) and (3) can be proceeded similarly. □

We say that a function ψ is *smooth* if the following conditions hold:

(1) ψ is *smooth* at every $t \in (0,1)$, i.e., the derivative of ψ exists at t,
(2) the right derivative of ψ at 0 is -1, and
(3) the left derivative of ψ at 1 is 1.

Theorem 11. *Let* X *and* Y *be Banach spaces and* $\psi \in \Psi$. *Then* $X \oplus_\psi Y$ *is smooth if and only if* X *and* Y *are smooth and* ψ *is smooth.*

Proof. Necessity. Assume that $X \oplus_\psi Y$ is smooth. Because X is isometric to $X \oplus_\psi \{0\}$ which is a subspace of $X \oplus_\psi Y$, then X and similarly Y must be smooth. It remains to prove that ψ is smooth, but by Lemma 10, if ψ is not smooth, there exists $(x, y) \in S_{X \oplus_\psi Y}$ such that $\nabla_{(x,y)}$ contains more than one point which can not happen, and the smoothness of ψ is proved

Sufficiency. This follows from Lemma 10. □

Again, since, for every Banach space X, X is uniformly smooth if and only of $\widetilde{X}$ is smooth, we obtain

Corollary 12. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly smooth if and only if X and Y are uniformly smooth and ψ is smooth.*

5. U-spaces and u-spaces

A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in S_X$, we have $\|x + y\| \le 2(1 - \delta)$ whenever $f(y) < 1 - \varepsilon$ for some $f \in \nabla_x$ (see [13]). A Banach space X is called a *u-space* if for any $x, y \in S_X$ with $\|x + y\| = 2$, we have $\nabla_x = \nabla_y$. Obviously, every U-space is a u-space.

Remark 13. Let us collect together some properties of u-spaces and U-spaces:

(1) If X^* is a u-space, then X is a u-space. The converse holds whenever X is reflexive.
(2) If X is a U-space, then X is a u-space. The converse holds whenever $\dim X < \infty$.
(3) $\widetilde{X}$ is a u-space if and only if X is a U-space.

Proof. (1) Let $x, y \in S_X$ be such that $\|x + y\| = 2$. We prove that $\nabla_x = \nabla_y$. Let $f \in \nabla_x$, and $h \in \nabla_{x+y}$. It follows that $h(x) = h(y) = 1$ and $\|f + h\| = 2$. By the assumption that X^* is a u-space and $h(y) = 1$, we have $f(y) = 1$. This implies that $\nabla_x \subset \nabla_y$, and then $\nabla_x = \nabla_y$ as required.

(2) The first assertion is obvious and the latter one follows from the compactness of the unit ball.

(3) It is known that $\widetilde{X}$ is a U-space if and only if X is a U-space (see [6] or [15]). In virtue of (2), it suffices to prove that X is a U-space whenever $\widetilde{X}$ is a u-space. Suppose that X is not a U-space. Then there exist an $\epsilon_0 > 0$ and sequences $\{x_n\}, \{y_n\} \subset S_X$, and $\{f_n\} \subset S_{X^*}$ such that $f_n(x_n) = 1$ and $f_n(x_n - y_n) \ge \epsilon_0$ for all $n \in \mathbb{N}$, and $\|x_n + y_n\| \to 2$ as $n \to \infty$. We put $\widetilde{x} = (x_n)_\mathcal{U}$, $\widetilde{y} = (y_n)_\mathcal{U}$ and $\widetilde{f} = (f_n)_\mathcal{U}$. Thus $\|\widetilde{x} + \widetilde{y}\| = 2$, $\widetilde{f}(\widetilde{x}) = 1$ and $\widetilde{f}(\widetilde{y}) \le 1 - \epsilon_0 < 1$. This means that $\nabla_{\widetilde{x}} \ne \nabla_{\widetilde{y}}$ which implies that $\widetilde{X}$ is not a u-space. □

U-spaces can be considered as the "uniform" version of u-spaces. The following diagram explains this claim as well as it shows how the u-spaces are well-placed (see [1], [4], [6], [14], and [15]):

$$X \text{ is UC} \;\Leftrightarrow\; \widetilde{X} \text{ is UC} \;\Leftrightarrow\; \widetilde{X} \text{ is SC}$$

$$X \text{ is US} \;\Leftrightarrow\; \widetilde{X} \text{ is US} \;\Leftrightarrow\; \widetilde{X} \text{ is S}$$

$$X \text{ is UNC} \quad \Leftrightarrow \ \widetilde{X} \text{ is UNC} \quad \Leftrightarrow \ \widetilde{X} \text{ is NC}$$

$$X \text{ is a U-space} \quad \Leftrightarrow \quad \widetilde{X} \text{ is a U-space} \quad \Leftrightarrow \ \widetilde{X} \text{ is a u-space}$$

$$\begin{array}{ccccccccccc}
C_{\mathrm{NJ}}(1,X) < 2 & \Rightarrow & \text{UNS} & & & & & & & & \\
 & & \Uparrow & & & & & & & & \\
\text{UC} & \Rightarrow & U & \Rightarrow & \text{UNSQ} & & \text{US} & \Rightarrow & U & \Rightarrow & \text{UNSQ} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{SC} & \Rightarrow & u & \Rightarrow & \text{NSQ} & & \text{S} & \Rightarrow & u & \Rightarrow & \text{NSQ}
\end{array}$$

UC $\equiv$ Uniformly Convex, SC $\equiv$ Strictly Convex, US $\equiv$ Uniformly Smooth, S $\equiv$ Smooth, UNC $\equiv$ Uniformly Noncreasy , NC $\equiv$ Noncreasy, $C_{\mathrm{NJ}}(\cdot) \equiv$ a generalized Jordan-von Neumann constant, UNS $\equiv$ Uniform Normal Structure, UNSQ $\equiv$ Uniformly Nonsquare, NSQ $\equiv$ Nonsquare, U $\equiv$ a U-space, $u \equiv$ a u-space

Examples of u-spaces which are not U-spaces can be obtained from the direct product spaces $(\mathbb{R}^2_{p_1} \oplus \mathbb{R}^2_{p_2} \oplus \mathbb{R}^2_{p_3} \oplus \cdots)_2$ where (p_n) is a sequence of positive numbers strictly decreasing to 1, and $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ where each l_n is the l_n-space. Actually, both spaces are strictly convex, but with the James constant and the Jordan-von Neumann constant are both equal to 2, i.e., the spaces are not uniformly nonsquare, and hence can not be U-spaces. Sims and Smith [18] have shown that the space $(l_2 \oplus l_3 \oplus l_4 \oplus \cdots)_2$ has asymptotic property (P) but not property (P).

Examples of infinite dimensional u-spaces that are not strictly convex or smooth are easily established.

Let $\psi \in \Psi$. We say that ψ is a *u-function,* if for any interval $[a,b] \subset (0,1)$, we have ψ is smooth at a and b whenever ψ is affine on $[a,b]$.

Theorem 14. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the Banach space $X \oplus_\psi Y$ is a u-space if and only if X and Y are u-spaces and ψ is a u-function.*

Proof. Necessity . Suppose there exist a and $b \in [0,1]$ such that ψ is affine on $[a,b]$ but $\psi'_-(a) < \psi'_+(a) = \psi'_-(b)$. Fix $x_0 \in S_X$, $f_0 \in \nabla_{x_0}$, $y_0 \in S_Y$, and $g_0 \in \nabla_{y_0}$. Consider $w = \frac{1}{\psi(a)}((1-a)x_0, ay_0)$ and $z = \frac{1}{\psi(b)}((1-b)x_0, by_0)$. We have $w, z \in S_{X\oplus_\psi Y}$ and $\|w+z\|_\psi = 2$. Indeed,

$$\begin{aligned}
\|w+z\|_\psi &= \left\| \left(\frac{1-a}{\psi(a)} x_0 + \frac{1-b}{\psi(b)} x_0, \frac{a}{\psi(a)} y_0 + \frac{b}{\psi(b)} y_0 \right) \right\|_\psi \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(\frac{\frac{a}{\psi(a)} + \frac{b}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \psi \left(a \frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} + b \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \right) \\
&= \left(\frac{1}{\psi(a)} + \frac{1}{\psi(b)} \right) \left(\frac{\frac{1}{\psi(a)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \psi(a) + \frac{\frac{1}{\psi(b)}}{\frac{1}{\psi(a)} + \frac{1}{\psi(b)}} \psi(b) \right) \\
&= 2.
\end{aligned}$$

To obtain a contradiction, it remains to show that $\nabla_z \neq \nabla_w$. Now, for $\gamma \in [\psi'_-(b), \psi'_+(b)]$, we have

$$\psi(b) - b\gamma \leq \psi(b) - b\psi'_-(b) = \psi(a) - a\psi'_+(a) < \psi(a) - a\psi'_-(a).$$

Thus, $((\psi(a) - a\psi'_-(a))f_0, (\psi(a) + (1-a)\psi'_-(a))g_0) \in \nabla_w \setminus \nabla_z$, that is $\nabla_z \neq \nabla_w$.

Sufficiency. Let us prove that $X \oplus_\psi Y$ is a u-space. Let w and z be elements in the unit sphere of $X \oplus_\psi Y$ such that $\|w+z\|_\psi = 2$. Put $w = (x_1, y_1)$ and $z = (x_2, y_2)$. We have $\|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi = 2$ since $2 = \|w+z\|_\psi = \|(x_1+x_2, y_1+y_2)\|_\psi \leq \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\|_\psi \leq \|(\|x_1\|, \|x_2\|)\|_\psi + \|(\|y_1\|, \|y_2\|)\|_\varphi = 2$. By the convexity of ψ, it follows that

$$\begin{aligned}
2 &= (\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|)\psi\left(\frac{\|y_1\| + \|y_2\|}{\|x_1\| + \|y_1\| + \|x_2\| + \|y_2\|}\right) \\
&\leq (\|x_1\| + \|y_1\|)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right) + (\|x_2\| + \|y_2\|)\psi\left(\frac{\|y_2\|}{\|x_2\| + \|y_2\|}\right) \\
&= 2.
\end{aligned}$$

Thus, ψ is affine on $[a \wedge b, a \vee b]$, where $a = \frac{\|y_1\|}{\|x_1\|+\|y_1\|}$ and $b = \frac{\|y_2\|}{\|x_2\|+\|y_2\|}$. Since $\|w + z\| = 2$, there exists $F = (f_1, g_1) \in X^* \oplus_\varphi Y^*$ such that $F \in \nabla_w \cap \nabla_z$. Hence,

$$\begin{aligned}
F(w) &= f_1(x_1) + g_1(y_1) \\
&\leq \|f_1\|\|x_1\| + \|g_1\|\|y_1\| \\
&= \frac{\|f_1\|\|x_1\| + \|g_1\|\|y_1\|}{(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|)}(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&\leq \varphi\left(\frac{\|g_1\|}{\|f_1\| + \|g_1\|}\right)\psi\left(\frac{\|y_1\|}{\|x_1\| + \|y_1\|}\right)(\|f_1\| + \|g_1\|)(\|x_1\| + \|y_1\|) \\
&= \|F\|_\varphi \|w\|_\psi = 1.
\end{aligned}$$

Thus, we have

$$(\alpha) \qquad f_1(x_1) = \|f_1\|\|x_1\| \text{ and } g_1(y_1) = \|g_1\|\|y_1\|.$$

In the same way, we also have

$$(\beta) \qquad f_1(x_2) = \|f_1\|\|x_2\| \text{ and } g_1(y_2) = \|g_1\|\|y_2\|.$$

Now we show that $\nabla_w = \nabla_z$. We consider first the case when all $\|x_1\|$, $\|y_1\|$, $\|x_2\|$, $\|y_2\|$ are positive. In this case, we can assume that $0 < a \leq b < 1$. (α) and (β) give $\frac{f_1}{\|f_1\|} \in \nabla_{\frac{x_1}{\|x_1\|}} \cap \nabla_{\frac{x_2}{\|x_2\|}}$ and $\frac{g_1}{\|g_1\|} \in \nabla_{\frac{y_1}{\|y_1\|}} \cap \nabla_{\frac{y_2}{\|y_2\|}}$. It follows that $\|\frac{x_1}{\|x_1\|} + \frac{x_2}{\|x_2\|}\| = 2$ and $\|\frac{y_1}{\|y_1\|} + \frac{y_2}{\|y_2\|}\| = 2$. Thus, $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$ since both X and Y are u-spaces.

If $a < b$, then, since ψ is affine on $[a, b]$, a and b must be smooth points of ψ. Consequently,

$$(\gamma) \qquad \psi(a) - a\gamma = \psi(b) - b\gamma \text{ and } \psi(a) + (1-a)\gamma = \psi(b) + (1-b)\gamma,$$

where $\gamma = \psi'(a) = \psi'(b)$.

By using (γ) together with Lemma 10 and the equations $\nabla_{\frac{x_1}{\|x_1\|}} = \nabla_{\frac{x_2}{\|x_2\|}}$ and $\nabla_{\frac{y_1}{\|y_1\|}} = \nabla_{\frac{y_2}{\|y_2\|}}$, we have $\nabla_z = \nabla_w$.

If $a = b$, then, by Lemma 10, we have

$$\begin{aligned}
&\nabla_{(x_1,y_1)} \\
&= \{((\psi(a) - a\gamma)f, (\psi(a) + (1-a)\gamma)g) : \gamma \in G(a), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_1/\|x_1\|} \text{ and } g \in \nabla_{y_1/\|y_1\|}\} \\
&= \{((\psi(b) - b\gamma)f, (\psi(b) + (1-b)\gamma)g) : \gamma \in G(b), f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \nabla_{(x_2,y_2)}.
\end{aligned}$$

Thus $\nabla_z = \nabla_w$ as well.

Now we consider the case when exactly one of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ is equal to 0. We assume that $\|y_1\| = 0$, thus $a = 0 < b$ and 0 and b are smooth points. By (α), (β), and by the assumption that X is a u-space, we have $\nabla_{x_1} = \nabla_{\frac{x_2}{\|x_2\|}}$. Since 0 is a smooth point, we have $F = (f_1, 0)$. This in turn implies that $\psi(b) - b\psi'(b) = 1$ and $\psi(b) + (1-b)\psi'(b) = 0$ since $F \in \nabla_w \cap \nabla_z$. Thus, by Lemma 10,

$$\begin{aligned}
&\nabla_{(x_2,y_2)} \\
&= \{((\psi(b) - b\psi'(b))f, (\psi(b) + (1-b)\psi'(b))g) : f \in \nabla_{x_2/\|x_2\|} \text{ and } g \in \nabla_{y_2/\|y_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_2/\|x_2\|}\} \\
&= \{(f, 0) : f \in \nabla_{x_1}\} \\
&= \nabla_{(x_1,y_1)}.
\end{aligned}$$

Finally, suppose two of the numbers $\|x_1\|, \|x_2\|, \|y_1\|, \|y_2\|$ are equal to 0. We can assume that $\|y_1\| = \|y_2\| = 0$, thus $a = b = 0$. The proof of the equality $\nabla_z = \nabla_w$ is similar to the one of the case when $a = b$. □

Corollary 15. *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following statements are equivalent:*

(1) *$X \oplus_\psi Y$ is a U-space;*
(2) *$X^* \oplus_\varphi Y^*$ is a U-space;*
(3) *X and Y are U-spaces and ψ is a u-function;*
(4) *X and Y are U-spaces and φ is a u-function, where φ is the dual function of ψ.*

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